

Analysis of transmission problems on Lipschitz boundaries in stronger norms

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Abstract — We concentrate on a model diffusion equation on a Lipschitz simply connected bounded domain with a small diffusion coefficient in a Lipschitz simply connected subdomain located strictly inside of the original domain. We study asymptotic properties of the solution with respect to the small diffusion coefficient vanishing. It is known that the solution asymptotically turns into a solution of a corresponding diffusion equation with Neumann boundary conditions on a part of the boundary. One typical proof technique of this fact utilizes a reduction of the problem to the interface of the subdomain, using a transmission condition. An analogous approach appears in studying domain decomposition methods without overlap, reducing the investigation to the surface that separates the subdomains and in theoretical foundation of a fictitious domain, also called embedding, method, e.g., to prove a classical estimate that guarantees convergence of the solution of the fictitious domain problem to the solution of the original Neumann boundary value problem.

On a continuous level, this analysis is usually performed in an $H^{1/2}$ norm for second order elliptic equations. This norm appears naturally for Poincaré-Steklov operators, which are convenient to employ to formulate the transmission condition. Using recent advances in regularity theory of Poincaré-Steklov operators for Lipschitz domains, we provide, in the present paper, a similar analysis in an $H^{1/2+\alpha}$ norm with $\alpha > 0$, for a simple model problem. This result leads to a convergence theory of the fictitious domain method for a second order elliptic PDE in an $H^{1+\alpha}$ norm, while the classical result is in an H^1 norm. Here, $\alpha < 1/2$ for the case of Lipschitz domains we consider.

Keywords: Discontinuity, Lipschitz domains, embedding, extension theorems, regularity, diffusion equation, asymptotic properties

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1. INTRODUCTION

Partial differential equations (PDEs) with highly discontinuous coefficients between subdomains often appear in engineering applications to model processes in materials with contrast coefficients, e.g., in typical composite materials. Such problems were historically studied in relation to so-called fictitious domain, or embedding, method, which is a classical technique of approximating a solution of a boundary value problem for a PDE by a solution of a similar a PDE on

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an extended domain, see [1, 6, 18, 19, 29] and some modern treatment, e.g., [2, 5, 9, 10, 13, 14, 21, 22, 31]. The complement of the original domain embedded into the extended domain is usually called a fictitious domain. Typically, the PDE coefficients are kept the same in the original domain. In the fictitious domain, the coefficients are chosen to force the solution on the extended domain to satisfy approximately the boundary conditions on the original domain.

A simple example is a problem for the Laplacian with a cavity in the original domain, i.e. with homogeneous Neumann boundary conditions on the surface of the cavity. We fill the cavity with the fictitious domain and consider the diffusion equations with the diffusion coefficient equal to one in the original domain and equal to a small positive constant value in the fictitious domain. Informally speaking, a small diffusion coefficient forces the fluxes to be small in the fictitious domain and, in the limit, when the small diffusion coefficient becomes zero, leads to the no-flux case, i.e. to the original homogeneous Neumann boundary condition on the surface of the cavity.

Let us highlight that the current state of the art in fictitious domain methods does not require actually constructing explicitly the problem with discontinuous coefficients, neither it needs the fictitious coefficient to be small, but nonzero. Thus, an analysis of behavior of the solution of PDEs with highly discontinuous coefficients is no longer of importance in fictitious domain methods. Nevertheless, it is convenient to utilize the historical terminology of the fictitious domain method to study PDEs with highly discontinuous coefficients between subdomains, and we follow it in the present paper, even though in a way it turns things upside down in terms of what the original problem and domains are.

Asymptotic properties of the solution with respect to the small diffusion coefficient vanishing are described by a well-known estimate that guarantees convergence in an H^1 norm of the solution of the fictitious domain problem to the solution of the original problem in the original domain; e.g., [3]. One typical proof technique of the estimate reduces the problem to the interface of the fictitious domain extension, using a transmission condition. An analogous approach appears in studying domain decomposition methods without overlap, reducing the investigation to the surface that separates the subdomains; e.g., [4, 27].

On a continuous level, this analysis is usually performed in an $H^{1/2}$ norm for second order elliptic equations. This norm appears naturally for Poincaré-Steklov operators, which are convenient to employ to formulate the transmission condition. A regularity theory of Poincaré-Steklov operators for Lipschitz domains, which is closely related to *regularity of transmission and diffraction problems* and properties of *layer potentials* on nonsmooth interfaces, have been extensively studied recently; see, e.g., papers [24, 30], the book [23], and references therein. In [17], these results have been used to develop a regularity theory of the diffusion equation with highly discontinuous coefficients; see also [25, 26].

Taking advantage of the recent progress, we provide an analysis of the transmission problem in an $H^{1/2+\alpha}$ norm with some $\alpha > 0$. This result leads to a convergence theory of the fictitious domain method for a second order elliptic PDE in an $H^{1+\alpha}$

norm, while the standard result is in an H^1 norm. Here, $\alpha < 1/2$ is required for the case of Lipschitz domains we consider. For smooth domains a similar result is given in [19] using a different technique.

2. A MODEL DIFFUSION EQUATION

Following [17], we consider a boundary value problem in two dimensions

$$\operatorname{div}(k \operatorname{grad} u - \varphi) = 0, \quad u \in H^1(\square), \quad \varphi \in (L_2(\square))^2, \quad (2.1)$$

where \square is a Lipschitz simply connected bounded domain.

Let $\mathcal{D} \subset \square$ be a Lipschitz connected domain and let the open set \mathcal{D}^\perp be defined by the conditions:

$$\mathcal{D} \cap \mathcal{D}^\perp = \emptyset, \quad \bar{\mathcal{D}} \cup \bar{\mathcal{D}}^\perp = \bar{\square}.$$

We assume that \mathcal{D}^\perp is a Lipschitz simply connected domain.

Let us also assume for simplicity that \mathcal{D}^\perp is strictly inside \square . This assumption, in particular, forces the intersection $\partial\mathcal{D} \cap \partial\square = \partial\mathcal{D}^\perp$ to have a positive Lebesgue measure on $\partial\square$, which ensures that any function in $H^1(\square)$, which is constant in \mathcal{D} , actually vanishes there. Informally speaking, these assumptions simply mean that the domain $\mathcal{D} \subset \square$ has a cavity \mathcal{D}^\perp in it.

We define the boundary of the cavity as

$$\Gamma = \partial\mathcal{D} \cap \partial\mathcal{D}^\perp = \partial\mathcal{D}^\perp.$$

We assume that k is a piecewise constant function on \square , and highly discontinuous :

$$k = \omega \text{ on } \mathcal{D}^\perp, \quad k = 1 \text{ on } \mathcal{D}, \quad \text{where } 0 < \omega \leq 1. \quad (2.2)$$

The interface Γ separates subdomains \mathcal{D} and \mathcal{D}^\perp , where the diffusion coefficient takes different values.

For the ‘‘right-hand side’’ φ , we assume that

$$\varphi \in (H^\alpha(\square))^2, \quad 0 \leq \alpha < 1/2$$

and that its restriction on \mathcal{D}^\perp satisfies:

$$\varphi|_{\mathcal{D}^\perp} = 0.$$

We need the later assumption as we want to fix φ and to have the solution u uniformly bounded as a function of $\omega \rightarrow 0$ at the same time.

3. PROPERTIES OF THE SOLUTION AS A FUNCTION OF $\omega \rightarrow 0$

The following several statements are already known, e.g., [3, 17, 20].

First, we examine the H^1 boundedness of the solution u , uniform in ω [3]:

Theorem 3.1. *Here and throughout, let C be a generic constant, independent of ω , i.e. $C \neq C(\omega)$. Under the assumptions of Section 2, we have*

$$\|u\|_{H^1(\square)} \leq C \|\varphi\|_{(L_2(\square))^2}. \quad (3.1)$$

Second, we present an estimate [3, 20] that guarantees convergence in an H^1 norm of the solution u of the fictitious domain problem (2.1) to the solution of the “original” problem, defined below, in the domain \mathcal{D} when $\omega \rightarrow 0$.

Theorem 3.2. *Under the assumptions of Section 2, let u_0 be defined in domain \mathcal{D} as satisfying the same differential equation as u , i.e.*

$$\operatorname{div}(\operatorname{grad} u_0 - \varphi) = 0 \text{ in } \mathcal{D}, \quad (3.2)$$

and taking homogeneous Dirichlet boundary conditions on the external boundary $\partial\mathcal{D} \cap \partial\square = \partial\square$ of domain \mathcal{D} and homogeneous Neumann boundary conditions on its internal boundary Γ . Then

$$\|u|_{\mathcal{D}} - u_0\|_{H^1(\mathcal{D})} \leq C\omega \|\varphi|_{\mathcal{D}}\|_{(L_2(\mathcal{D}))^2}. \quad (3.3)$$

Now we turn our attention to regularity. It is well-known that under our assumptions the solution u is regular strictly inside of subdomains \mathcal{D} and \mathcal{D}^\perp for a sufficiently smooth right-hand side φ , but may have singularities near the boundary $\partial\square$ and near the interface Γ . Thus, regularity of the solution u is closely related to *regularity of transmission and diffraction problems*, properties of *layer potentials* and *Steklov-Poincaré operators* on nonsmooth interfaces, which have been extensively studied; see, e.g., recent papers [24, 30], the book [23], and references therein. Under our assumption that domain \mathcal{D}^\perp is strictly inside (SI) of \square , the question of regularity can be reduced to studying layer potentials on Γ without boundary conditions, because in the SI case the boundary Γ is a closed Lipschitz curve without self-intersection and it does not have any common points, also called *junction* points, with the boundary $\partial\square$, where the homogeneous Dirichlet boundary condition is enforced.

For a fixed ω , a regularity result is known to hold, according to [8, 30]. A similar regularity of the solution u , but *uniform* in ω , under additional assumptions is established in [17]:

Theorem 3.3. *Under the assumptions of Section 2, $\|u|_{\mathcal{D}}\|_{H^{1+\alpha}(\mathcal{D})}$ is uniformly bounded in ω , with a fixed positive α up to a certain value α_{\max} , in the following*

sense:

$$\|u|_{\mathcal{D}}\|_{H^{1+\alpha}(\mathcal{D})} \leq C \|\varphi\|_{(H^\alpha(\square))^2}, \quad \alpha \in (0, \alpha_{\max}), \quad (3.4)$$

where $\alpha_{\max} = 1/2$.

We note that in the limit $\omega = 0$, which corresponds to a Neumann boundary value problem on \mathcal{D} , according to Theorem 3.2, the regularity result also holds with $\alpha_{\max} = 1/2$; see, e.g., [16]:

Theorem 3.4. *Under the assumptions of Theorem 3.2, $u_0 \in H^{1+\alpha}(\mathcal{D})$ with a fixed positive α up to a certain value α_{\max} , in the following sense:*

$$\|u_0\|_{H^{1+\alpha}(\mathcal{D})} \leq C \|\varphi\|_{(H^\alpha(\square))^2}, \quad \alpha \in (0, \alpha_{\max}), \quad (3.5)$$

where $\alpha_{\max} = 1/2$.

The following statement is new.

Theorem 3.5. *Under the assumptions of Theorem 3.2, for a small enough non-negative ω , we have*

$$\|u|_{\mathcal{D}} - u_0\|_{H^{1+\alpha}(\mathcal{D})} \leq C \omega \|\varphi\|_{(H^\alpha(\mathcal{D}))^2}, \quad \alpha \in (0, \alpha_{\max}), \quad (3.6)$$

where $\alpha_{\max} = 1/2$.

We prove Theorem 3.5 in the next section. In the course of the proof, we will also establish Theorems 3.1, 3.2, and 3.3.

4. PROOF OF THEOREM 3.5

Let function $\hat{u} \in H^1(\square)$ satisfies the same equation as function u , i.e.

$$\int_{\mathcal{D}} (\text{grad } \hat{u} - \varphi) \cdot \text{grad } v \, d\mathcal{D} = 0, \quad \forall v \in H^1(\mathcal{D}), \quad (4.1)$$

in \mathcal{D} and vanishes in \mathcal{D}^\perp , so that $\hat{u}|_{\mathcal{D}} \in H^1(\mathcal{D})$.

We note that equation (4.1) is a uniformly elliptic homogeneous Dirichlet boundary value problem on a Lipschitz domain and is independent of ω . Therefore,

$$\|\hat{u}|_{\mathcal{D}}\|_{H^{1+\alpha}(\mathcal{D})} \leq C \|\varphi|_{\mathcal{D}}\|_{(H^\alpha(\mathcal{D}))^2}, \quad \alpha \in (0, \alpha_{\max}) \quad (4.2)$$

by well-known regularity results; see, e.g., [11, 12].

We now simplify equation (2.1) by introducing a new function

$$w = u - \hat{u} \in H^1(\square).$$

By construction, the function w is piece-wise harmonic, i.e. the restriction $w|_{\mathcal{D}}$ is harmonic in \mathcal{D} and the restriction $w|_{\mathcal{D}^\perp}$ is harmonic in \mathcal{D}^\perp . On the interface of the jump in the coefficients, Γ , the functions $w|_{\mathcal{D}}$ and $w|_{\mathcal{D}^\perp}$ have the same trace $w|_\Gamma$ from \mathcal{D} and \mathcal{D}^\perp and satisfy the following standard transmission condition on Γ between \mathcal{D} and \mathcal{D}^\perp for the normal components:

$$\mathbf{n} \cdot (\text{grad}(w|_{\mathcal{D}} + \hat{u}|_{\mathcal{D}}) - \boldsymbol{\varphi}|_{\mathcal{D}})|_\Gamma = \mathbf{n} \cdot (\boldsymbol{\omega} \text{grad}(w|_{\mathcal{D}^\perp}))|_\Gamma, \quad (4.3)$$

where the left side corresponds to \mathcal{D} and the right side to \mathcal{D}^\perp , and \mathbf{n} is the normal direction on Γ , oriented outward \mathcal{D} .

Let us rewrite equation (4.3) by collecting all terms with w in the left-hand side:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}} w|_{\mathcal{D}}|_\Gamma - \boldsymbol{\omega} \frac{\partial}{\partial \mathbf{n}} w|_{\mathcal{D}^\perp}|_\Gamma \\ = \mathbf{n} \cdot (\boldsymbol{\varphi}|_{\mathcal{D}} - \text{grad} \hat{u}|_{\mathcal{D}})|_\Gamma. \end{aligned} \quad (4.4)$$

Let us now define a Steklov-Poincaré operator $S_{\mathcal{D}}$ that, for a given harmonic function $w|_{\mathcal{D}}$, where $w \in H^1(\square)$, maps its trace on Γ into its Neumann datum on Γ :

$$S_{\mathcal{D}} : w|_{\mathcal{D}}|_\Gamma \mapsto \frac{\partial}{\partial \mathbf{n}} w|_{\mathcal{D}}|_\Gamma.$$

Let us similarly define the Steklov-Poincaré operator $S_{\mathcal{D}^\perp}$ for the other domain \mathcal{D}^\perp using the normal direction on Γ , oriented outwards from \mathcal{D}^\perp . Steklov-Poincaré operators act between the corresponding space of trace functions

$$\Lambda = \{w|_\Gamma, \forall w \in H^1(\square)\}.$$

and its dual Λ' . These operators are symmetric and positive semi-definite on Λ and bounded as mappings $\Lambda \rightarrow \Lambda'$; see, e.g., [28]. Moreover, the operator $S_{\mathcal{D}}$ is positive definite, because of the homogeneous Dirichlet boundary conditions on $\partial \square$. As the domain \mathcal{D}^\perp is strictly inside, i.e., Γ is a closed curve, the space Λ is simply $H^{1/2}(\Gamma)$ and Λ' is $H^{-1/2}(\Gamma)$.

Using the Steklov-Poincaré operators just defined, we rewrite equation (4.4) as

$$(S_{\mathcal{D}} + \boldsymbol{\omega} S_{\mathcal{D}^\perp}) \lambda = \chi, \quad (4.5)$$

where we introduce new notations: λ for the common trace $w|_\Gamma$ of $w|_{\mathcal{D}}$ and $w|_{\mathcal{D}^\perp}$ on Γ and χ for the right-hand side of (4.4). The operator $S_{\mathcal{D}} + \boldsymbol{\omega} S_{\mathcal{D}^\perp} \geq S_{\mathcal{D}}$ is symmetric,

positive definite and bounded, *uniformly* in ω , $0 \leq \omega \leq 1$, e.g., [4]. The right-hand side χ does not depend on ω and is bounded, i.e.

$$\|\chi\|_{H^{-1/2}(\Gamma)} \leq C\|\varphi\|_{(L_2(\square))^2} \quad (4.6)$$

as follows from its definition; see, e.g., [23]. Therefore, the solution λ is bounded uniformly in $H^{1/2}(\Gamma)$:

$$\|\lambda\|_{H^{1/2}(\Gamma)} \leq C\|\varphi\|_{(L_2(\square))^2}, \quad (4.7)$$

which leads to uniform boundedness of w and, therefore, of u in $H^1(\square)$, which proves Theorems 3.1.

Let us now consider similar arguments applied to u_0 instead of u . We replace w with w_0 defined only on \mathcal{D} as

$$w_0 = u_0 - \hat{u}|_{\mathcal{D}} \in H^1(\mathcal{D}).$$

As a substitute for (4.3), we derive

$$\mathbf{n} \cdot (\text{grad}(w_0 + \hat{u}|_{\mathcal{D}}) - \varphi|_{\mathcal{D}})|_{\Gamma} = 0, \quad (4.8)$$

where \mathbf{n} is the normal direction on Γ , oriented outward \mathcal{D} . Calling λ_0 to be the trace $w_0|_{\Gamma}$ of w_0 on Γ and using the same χ as for the right-hand side of (4.4), we obtain

$$S_{\mathcal{D}}\lambda_0 = \chi. \quad (4.9)$$

We observe exactly as we expect that equation (4.5) turns into equation (4.9) when $\omega \rightarrow 0$.

To compare λ with λ_0 , we use the equation

$$(S_{\mathcal{D}} + \omega S_{\mathcal{D}^\perp})(\lambda - \lambda_0) = -\omega S_{\mathcal{D}^\perp}\lambda_0 = -\omega S_{\mathcal{D}^\perp}S_{\mathcal{D}}^{-1}\chi, \quad (4.10)$$

derived directly from (4.5) and (4.9). Using again (4.6) and the same arguments as those leading to (4.7), we obtain

$$\|\lambda - \lambda_0\|_{H^{1/2}(\Gamma)} \leq \omega C\|\varphi\|_{(L_2(\square))^2}, \quad (4.11)$$

that leads to a similar bound for $w - w_0$ and, thus for $u|_{\mathcal{D}} - u_0$, in H^1 norm on domain \mathcal{D} , which is the statement of Theorem 3.2.

At this point, we are prepared to study the uniform regularity of λ , which shall demonstrate uniform regularity of u in \mathcal{D} .

We first notice that $\chi \in H^{-1/2+\alpha}(\Gamma)$ because of extra smoothness of our functions φ and \hat{u} that determine χ according to (4.2) and (4.4):

$$\|\chi\|_{H^{-1/2+\alpha}(\Gamma)} \leq C\|\varphi\|_{(H^\alpha(\square))^2}.$$

In recent works [7, 15, 23], a regularity theory of Steklov-Poincaré operators is established for Lipschitz domains. Using these results for our situation, we have that for all $\alpha \in [0, 1/2)$ our both Steklov-Poincaré operators are bounded:

$$S_{\mathcal{D}} : H^{1/2+\alpha}(\Gamma) \mapsto H^{-1/2+\alpha}(\Gamma), \quad S_{\mathcal{D}^\perp} : H^{1/2+\alpha}(\Gamma) \mapsto H^{-1/2+\alpha}(\Gamma),$$

and the Steklov-Poincaré operator $S_{\mathcal{D}}$ is coercive:

$$\|S_{\mathcal{D}}^{-1}\psi\|_{H^{1/2+\alpha}(\Gamma)} \leq C \left(\|\psi\|_{H^{-1/2+\alpha}(\Gamma)} + \|S_{\mathcal{D}}^{-1}\psi\|_{H^{1/2}(\Gamma)} \right). \quad (4.12)$$

The coerciveness (modulo constants) of $S_{\mathcal{D}^\perp}$ is not important for our further arguments.

Let us rewrite equation (4.5) as

$$(I + \omega S_{\mathcal{D}}^{-1} S_{\mathcal{D}^\perp}) \lambda = S_{\mathcal{D}}^{-1} \chi. \quad (4.13)$$

and equation (4.10) as

$$(I + \omega S_{\mathcal{D}}^{-1} S_{\mathcal{D}^\perp}) (\lambda - \lambda_0) = -\omega S_{\mathcal{D}}^{-1} S_{\mathcal{D}^\perp} S_{\mathcal{D}}^{-1} \chi. \quad (4.14)$$

The operator $S_{\mathcal{D}}^{-1} S_{\mathcal{D}^\perp}$ is bounded in $H^{1/2+\alpha}(\Gamma)$, thus, the operator of equations (4.13) and (4.14) is a small perturbation of the identity and, therefore, it has a bounded inverse in $H^{1/2+\alpha}(\Gamma)$, for nonnegative ω small enough, e.g., for

$$0 \leq \omega \leq \frac{1}{2} \|S_{\mathcal{D}}^{-1} S_{\mathcal{D}^\perp}\|_{H^{1/2+\alpha}(\Gamma)}^{-1}. \quad (4.15)$$

Thus, the regularity estimate holds

$$\|\lambda\|_{H^{1/2+\alpha}(\Gamma)} \leq C \|\varphi\|_{(H^\alpha(\square))^2},$$

uniformly in ω satisfying (4.15). As the function $w|_{\mathcal{D}}$ is a harmonic extension of λ to \mathcal{D} , it is uniformly bounded with respect to the $H^{1+\alpha}(\mathcal{D})$ norm:

$$\|w|_{\mathcal{D}}\|_{H^{1/2+\alpha}(\mathcal{D})} \leq C \|\varphi\|_{(H^\alpha(\square))^2}.$$

Indeed, $w|_{\mathcal{D}}$ solves the Laplace equation in the Lipschitz domain \mathcal{D} with homogeneous the Dirichlet boundary conditions on $\partial\mathcal{D}$ and nonhomogeneous the Dirichlet boundary conditions, given by the function $\lambda \in H^{1/2+\alpha}(\Gamma)$, on Γ . Thus, known regularity results (e.g., [16, 23, 30]) can be applied. Using $u = w + \hat{u}$, we combine the previous inequality with (4.2) to get (3.4) uniformly in ω , satisfying (4.15), which proves Theorem 3.3.

Similarly, from (4.14) we obtain

$$\|\lambda - \lambda_0\|_{H^{1/2+\alpha}(\Gamma)} \leq C\omega \|\varphi\|_{(H^\alpha(\square))^2}.$$

The function w_0 is a harmonic extension of λ_0 to \mathcal{D} , therefore, function $w|_{\mathcal{D}} - w_0$ is a harmonic extension of $\lambda - \lambda_0$ to \mathcal{D} and, as such, satisfies

$$\|w|_{\mathcal{D}} - w_0\|_{H^{1/2+\alpha}(\mathcal{D})} \leq C\omega \|\varphi\|_{(H^\alpha(\square))^2}.$$

Finally, $w|_{\mathcal{D}} - w_0 = u|_{\mathcal{D}} - u_0$, by definition of w and w_0 , which completes the proof of Theorem 3.5.

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