

University of Colorado Denver  
Department of Mathematical and Statistical Sciences  
Applied Analysis Ph.D. Preliminary Exam  
January 12, 2009

Name: \_\_\_\_\_

**Exam Rules:**

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1. _____	5. _____
2. _____	6. _____
3. _____	7. _____
4. _____	8. _____

Total \_\_\_\_\_

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**Analysis Preliminary Exam Committee:**  
Andrew Knyazev, Julien Langou, Jan Mandel (Chair)

1. Let  $f$  and  $g$  be Riemann integrable on  $[a, b]$  and  $f \leq g$ . Using the definition of Riemann integral, show that

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

**Solution**

The Riemann integral satisfies

$$\int_a^b f(x) dx = \inf_P U(f, P)$$

where

$$U(f, P) = \sum_{i=1}^n \sup_{t \in [x_{i-1}, x_i]} f(t) (x_i - x_{i-1})$$

and the infimum is taken over all partitions  $P = \{x_0, \dots, x_n\}$  such that

$$0 = x_0 < x_1 < x_2 < \dots < x_n = 1.$$

Let  $P$  be a partition. Then

$$f(t) \leq g(t), \quad \forall t \in [x_{i-1}, x_i].$$

Thus

$$U(f, P) \leq U(g, P) \text{ for any partition } P.$$

Consequently,

$$\int_a^b f(x) dx = \inf_P U(f, P) \leq \inf_P U(g, P) = \int_a^b g(x) dx.$$

2. Suppose that  $f_n$  and  $g_n$  are real functions on a set  $S$ ,  $f_n \rightarrow 0$  uniformly on  $S$  and  $\{g_n\}$  are uniformly bounded on  $S$ . Show that  $f_n g_n \rightarrow 0$  uniformly on  $S$ .

**Solution**

Let us denote the independent variable by  $x \in S$ . The definition of the uniform convergence  $f_n \rightarrow 0$  states that

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n > N \quad \forall x \in S : \quad |f_n(x)| < \varepsilon.$$

The definition of the uniform boundedness of  $\{g_n\}$  states that

$$\exists C > 0 \quad \forall x \in S \quad \forall n : \quad |g_n(x)| < C.$$

Thus,  $\forall x \in S$  and  $\forall n$  we have

$$|g_n(x)f_n(x)| \leq C |f_n(x)|.$$

We need to prove that

$$\forall \varepsilon_1 > 0 \quad \exists N_1 \quad \forall n > N_1 \quad \forall x \in S : \quad |g_n f_n(x)| < \varepsilon_1.$$

Let us set  $\varepsilon = \varepsilon_1/C$  then taking  $N_1 = N$  gives  $\forall n > N_1 = N \quad \forall x \in S$ :

$$|g_n(x)f_n(x)| < C |f_n(x)| < C\varepsilon = \varepsilon_1,$$

which is the required claim.

3. Determine if  $d(x, y) = \sqrt{|x - y|}$  is a metric on  $\mathbb{R}$  or not.

**Solution**

We will prove that  $d$  is a metric on  $\mathbb{R}$  by proving that  $d$  is nonnegative, symmetric and satisfies the triangle inequality. Nonnegativity ( $d(x, y) > 0$  if  $x \neq y$  and  $d(x, x) = 0$ ) and symmetry ( $d(x, y) = d(y, x)$ ) are trivially true. We focus on the triangle inequality. Take any  $x, y$  and  $z$  in  $\mathbb{R}$ . We want to prove that  $d(x, y) \leq d(x, z) + d(z, y)$ .

$$\begin{aligned} d(x, y)^2 &= |x - y| = |x - z + z - y| \leq |x - z| + |z - y| \\ &= d(x, z)^2 + d(z, y)^2 \leq d(x, z)^2 + d(z, y)^2 + 2d(x, z)d(z, y) \\ &= (d(x, z) + d(z, y))^2 \end{aligned}$$

Therefore, for any  $x, y$  and  $z$  in  $\mathbb{R}$ ,  $d(x, y) \leq d(x, z) + d(z, y)$ .

4. Suppose  $f$  is a real function on  $\mathbb{R}$  with bounded first derivative. For a given  $\varepsilon$ , define  $g_\varepsilon(x) = x + \varepsilon f(x)$ . Prove that  $g_\varepsilon$  is one to one if  $\varepsilon$  is small enough.

**Solution**

From the assumptions, there exists  $M$  such that for any  $x$  in  $\mathbb{R}$ ,  $|f'(x)| \leq M$ . We will prove that for sufficiently small  $\varepsilon$ ,  $g_\varepsilon$  is strictly increasing, therefore one-to-one. Indeed,

$$g'_\varepsilon(x) = (x + \varepsilon f(x))' = 1 + \varepsilon f'(x) \geq 1 - \varepsilon M,$$

so

$$g'_\varepsilon(x) > 0$$

if

$$0 < \varepsilon < 1/M.$$

5. Suppose  $X$  and  $Y$  are metric spaces,  $f : X \rightarrow Y$  uniformly continuous, and  $\{x_n\}$  is a Cauchy sequence in  $X$ . Prove that  $\{f(x_n)\}$  is Cauchy in  $Y$ .

**Solution**

Let  $\varepsilon > 0$ . Since  $f$  is uniformly continuous,  $\exists \delta \geq 0$  such that, for all points  $u \in X$  and  $v \in X$  such that  $d_X(u, v) \leq \delta$ , we have  $d_Y(f(u), f(v)) \leq \varepsilon$ . Since  $\{x_n\}$  is a Cauchy sequence in  $X$ ,  $\exists N$  natural number such that  $d_X(x_k, x_\ell) \leq \delta$  if  $k \geq N$  and  $\ell \geq N$ . Therefore,

$$d_Y(f(x_k), f(x_\ell)) \leq \varepsilon \text{ if } k \geq N \text{ and } \ell \geq N.$$

6. Consider the set  $\mathbb{Q}$  of all rational numbers as a metric space with  $d(x, y) = |x - y|$ , and  $E = \{x \in \mathbb{Q} : x > 0 \text{ and } 2 < x^2 < 3\}$ . Show that  $E$  is closed. (You can assume known that there is no  $x \in \mathbb{Q}$  such that  $x^2 = 2$  or  $x^2 = 3$ .)

**Solution**

Let  $z \in \mathbb{Q}$  be a limit point of the set  $E$ . By definition of limit point,  $\exists \{q_n\}$  such that  $\forall n, q_n \in E$ , and  $q_n \rightarrow z$  in  $\mathbb{Q}$ . Since  $\mathbb{Q} \subset \mathbb{R}$  with the same metric, it also holds  $q_n \rightarrow z$  in  $\mathbb{R}$ . Consequently,  $q_n^2 \rightarrow z^2$  in  $\mathbb{R}$ . Since  $q_n \in E$ , thus  $2 < q_n^2 < 3$  for all  $n$ , it follows that  $2 \leq z^2 \leq 3$ . Since there is no  $z \in \mathbb{Q}$  such that  $z^2 = 2$  or  $z^2 = 3$ , it follows that  $2 < z^2 < 3$ . From  $q_n \geq 0$ , we have that  $z \geq 0$  but  $z = 0$  is not possible because  $z^2 \geq 2$ , so  $z > 0$ . Thus  $z \in E$  by the definition of  $E$ .

7. Give an example of a function  $f$  on  $\mathbb{R}^2$  such that  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist at every point of  $\mathbb{R}^2$  but  $f$  is not continuous at  $(0,0)$ .

**Solution**

Let

$$f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Since the denominator is nonzero for  $(x, y) \neq (0, 0)$ , the partial derivatives exist except at  $(0, 0)$  by elementary rules of differentiation. At  $(0, 0)$ ,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

since  $f(h, 0) = 0$  for all  $h$ . Similarly,  $\frac{\partial f}{\partial x}(0, 0) = 0$ . Thus  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist at every point of  $\mathbb{R}^2$ . Now

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} 0 = 0 \neq \lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2},$$

so  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist, because if  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = A$  then both limits above would have to equal to  $A$ . (Note that the statement of the problem does not say “show from the definition of limit”, so we can leave it at that.)

8. Find all intervals where the series  $\sum_{n=1}^{\infty} xe^{-nx}$  converges, converges absolutely, and converges uniformly.

**Solution**

Note that  $e^{-nx} = (e^{-x})^n$ , thus, for any  $x > 0$ , the series is a geometric series with quotient  $q = e^{-x} \in (0, 1)$ , and so it converges absolutely,

$$\sum_{n=1}^{\infty} xe^{-nx} = \frac{x}{1 - e^{-x}}, \quad x > 0.$$

For  $x = 0$ , all terms of the series are zero, so the series converges absolutely also. Thus the series converges absolutely for all  $x \geq 0$ . For  $x < 0$ ,  $e^{-nx} \rightarrow \infty$ , so the series diverges because its terms do not converge to zero. We have for  $x \geq 0$ ,

$$0 \leq xe^{-nx} = xe^{-x} e^{-(n-1)x}$$

The function  $f(x) = xe^{-x}$  is bounded on  $[0, \infty)$ : since  $\lim_{x \rightarrow \infty} f(x) = 0$ , there exists  $A$  such that  $f(x) \leq 1$  for all  $x > A$ , and  $f$  is continuous on the compact interval  $[0, A]$ , thus bounded on  $[0, A]$ . So, let  $M \geq xe^{-x}$  for all  $x \in [0, \infty)$ . Then for any  $a > 0$ ,

$$|xe^{-nx}| \leq M (e^{-a})^{n-1}, \quad \forall x \in x \in (a, \infty).$$

So by comparison with the geometric series  $\sum_{n=1}^{\infty} (e^{-a})^{n-1}$ , convergence is uniform in any interval of the form  $(a, +\infty)$ , for any  $a > 0$ .

However as  $a$  gets close to 0, the quotient  $e^{-a}$  gets close to 1 so the convergence gets slower and slower. We will prove that convergence is not uniform on  $[0, \infty)$ . Suppose  $s(x) = \sum_{n=1}^{\infty} xe^{-nx}$  uniformly on  $[0, \infty)$ . Since the functions  $g_n(x) = xe^{-nx}$  are continuous on  $[0, \infty)$ ,  $s(x)$  is also continuous on  $[0, \infty)$ . But

$$\lim_{x \rightarrow 0^+} s(x) = \lim_{x \rightarrow 0^+} \frac{x}{1 - e^{-x}} = \lim_{x \rightarrow 0^+} \frac{x}{1 - (1 - x + o(x))} = 1 \neq s(0) = 0,$$

which is a contradiction.