University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
July 11, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. All solutions will be graded and your final grade will be based on the total of all of them.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________
2.	________
3.	________
4.	________
5.	________
Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. A function $f : \mathbb{R} \to \mathbb{R}$ is called additive if $f(x+y) = f(x) + f(y)$ for all real x, y. Show that if f is additive and continuous, then $f(x) = cx$ for some real constant c.

2. Find an example of a metric space (V, d) and a set $A \subset V$ such that A is closed and bounded but not compact.

   Make sure you actually formulate the definitions and prove that your set A is closed, bounded, and not compact in your metric space. Simply an example without detailed proofs is insufficient.

3. If $\{a_n\}$ is a convergent sequence of real numbers, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove, or find a counterexample.

4. For a real variable $x \in [-1, 1]$ let $D(x)$ be a function that takes the value 1 if x is rational and the value 0 otherwise. Is the function $F(x) = xD(x)$ Riemann integrable? If so, what is the value of the integral $\int_{-1}^{1} F(x)dx$?

5. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, let

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function f.

(b) Show that the sequence of derivatives $\{f_n'\}$ does not converge uniformly on $\mathbb{R}$ to any function.