University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 9, 2006

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
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1. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, \ldots$. Prove that K is compact from the definition, without the use of the Heine-Borel theorem.

   Solution

   Let $\mathcal{A}$ be an open cover of K. There exists $S_0 \in \mathcal{A}$ such that $0 \in S_0$.Since S_0 is open, there is an $\varepsilon > 0$ such that $|x| < \varepsilon \Longrightarrow x \in S_0$;hence, choosing integer $n_0 > 1/\varepsilon$, we have $1/n \in S_0$ for all $n > n_0$. For each $n = 1, \ldots, n_0$, there is $S_n \in \mathcal{A}$ such that $1/n \in S_n$. Then $K \subset S_0 \cup S_1 \cup \cdots \cup S_{n_0}$.

2. If $s_n \le t_n$ for all $n > N$, where N is fixed, prove from the definition that

$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n$$

Solution

By definition,

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} a_n, \quad a_n = \sup A_n, \quad A_n = \{s_k : k \ge n\}$$

and

$$\limsup_{n\to\infty} t_n = \lim_{n\to\infty} b_n, \quad b_n = \sup B_n, \quad B_n = \{t_k : k \ge n\}\,.$$

Suppose $n > N$. Since $s_k \le t_k$ for all $k \ge n$, every upper bound on B_n is also an upper bound on A_n:

$$(t_k \le u \forall t_k \in B_n) \Longrightarrow (s_k \le u \forall s_k \in A_n)\,.$$

Because supremum is the least upper bound, $a_n = \sup A_n \le \sup B_n = b_n$ for all $n > N$, hence

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n = \limsup_{n\to\infty} t_n.$$

3. Define $\{a_n\}$ recursively by

$$a_1 = \sqrt{k}; \qquad\qquad a_{n+1} = \sqrt{k + a_n}, \quad (n \geq 1),$$

where k is a fixed constant, $k > 1$.

(a) (16 points) Prove the sequence $\{a_n\}$ has a limit.

(b) (4 points) Find the limit.

Justify all steps.

Solution

(a) The idea is to show the sequence is increasing and bounded. Note that all terms are non-negative.

Claim: The sequence is increasing. By induction.

$$\begin{aligned} a_1 &= \sqrt{k} \\ a_2 &= \sqrt{k + \sqrt{k}} > \sqrt{k} = a_1 \end{aligned}$$

So we have $a_2 > a_1$. Now suppose we have $a_n > a_{n-1}$. We need to show that $a_{n+1} > a_n$:

$$\begin{aligned} a_{n+1}^2 &= k + a_n \\ a_{n+1}^2 - a_n^2 &= k + a_n - a_n^2 \\ a_{n+1}^2 - a_n^2 &= k + a_n - (k + a_{n-1}) = a_n - a_{n-1} \geq 0. \end{aligned}$$

Since a_n is positive, we have $a_{n+1} - a_n \geq 0$.

Claim: The sequence is bounded by k. By induction.

$$a_1 = \sqrt{k} < k \qquad \text{since } k > 1.$$

Now assume $a_n < k$. Then

$$a_{n+1} = \sqrt{k + a_n} < \sqrt{2k} \leq \sqrt{k^2} = k.$$

So the sequence is increasing and bounded, hence a limit exists.

(b) Since $\{a_n\}$ converges, it has a limit, say L. The subsequence $\{a_{n+1}\}$ also converges to L. So we have, by the continuity of the square root (which allows us to interchange the square root and limit):

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{k + a_n} \\ L &= \sqrt{k + L} \\ L^2 &= k + L \\ L^2 - L - k &= 0 \\ L &= \frac{1 \pm \sqrt{1 + 4k}}{2}. \end{aligned}$$

Since $L > 0$, we have that $L = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4k}$.

4. Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, with the property that

$$0 < \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \ell < \infty.$$

(a) What is the largest value of r *implied by this condition* such that the power series converges absolutely for $|x-a| < r$?

(b) Let $r_1 < r$. Prove that the power series given above converges uniformly on the set $\{x : |x-a| \le r_1\}$.

Solution

(a) By the ratio test, the series converges absolutely provided

$$\lim_{n\to\infty} \left|\frac{c_{n+1}/(x-a)^{n+1}}{c_n/(x-a)^n}\right| = \left|(x-a)\frac{c_{n+1}}{c_n}\right| < 1,$$

This condition holds if

$$|x-a| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1,$$

or, equivalently,

$$|x-a| < \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \frac{1}{\ell}.$$

Thus $r = 1/\ell$ is the largest value of r such that the power series converges absolutely for $|x-a| < r$.

(b) By the Weierstrass M-test, the power series converges uniformly on the set $E = \{x : |x-a| \le r_1\}$ provided $\sum_{n=0}^{\infty} M_n$ converges, where $|c_n(x-a)^n| \le M_n$ for all $x \in E$. Let $M_n = |c_n| r_1^n$, which satisfies $|c_n(x-a)^n| \le M_n$ for all $x \in E$. From part (a), $\sum_{n=0}^{\infty} c_n y^n$ converges absolutely provided $|y| < r$. Thus $\sum_n M_n$ converges because $0 < r_1 < r$. It follows that the power series converges uniformly on the set $\{x : |x-a| \le r_1\}$.

5. Consider the sequence of functions $f_n(x) = 2(n+1)x(1-x^2)^n$ for $x \in [0,1]$.

   (a) (10 points) What is the pointwise limit of $f_n(x)$?

   (b) (5 points) What is $\int_0^1 f_n(x)dx$?

   (c) (5 points) Does the sequence of functions converge uniformly to its pointwise limit on the interval $[0,1]$? Justify all steps.

Solution

(a) Claim: $\lim\limits_{n\to\infty} f_n(x) = 0$ for $0 \le x \le 1$. Consider 3 cases:
Case 1: $x = 0$. Then $f_n(0) = 0$ and $\lim\limits_{n\to\infty} f_n(0) = 0$.
Case 2: $x = 1$. Then $f_n(0) = 0$ and $\lim\limits_{n\to\infty} f_n(0) = 0$.
Case 3: Fix x, $0 < x < 1$. Let $a = \left(1-x^2\right)$. Then $0 < a < 1$, so $\ln a < 0$ and

$$\lim_{n\to\infty} 2\,(n+1)\,xa^n = \lim_{n\to\infty} 2\,(n+1)\,xe^{n\ln a} = 0$$

by the properties of the exponential function,

$$\lim_{x\to+\infty} p\,(x)\,e^{-x} = 0$$

for any polynomial p. Hence, $\lim\limits_{n\to\infty} f_n(x) = 0$.
So $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $0 \le x \le 1$.

(b) For each fixed n, $f_n(x)$ is bounded and continuous. So we calculate directly, using u-substitution and letting $u = 1 - x^2$.

$$\int_0^1 f_n(x)\;dx = -(1-x^2)^{n+1}\Big|_0^1 = -(0-1) = 1.$$

(c) Since

$$\int_0^1 \lim_{n\to\infty} f_n(x)\;dx = 0 \ne \lim_{n\to\infty}\int_0^2 f_n(x)\;dx = 1,$$

the convergence cannot be uniform over $[0,1]$.

6. (a) (5 points) Suppose f is continuous on the interval $I = [a, b]$. Does f necessarily have a fixed point on I if $|f'(x)| < 1$ for all $x \in I$? Explain.

   (b) (3 points) Find all real fixed points x^* of the function $f(x) = x^3 + x - 8$.

   (c) (12 points) With f given above, find all values of x_0 for which the iteration $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$ converges. Justify the answer. If any theorems are used, identify them clearly; if convenient theorems are not available, work out a detailed proof.

Solution

(a) The condition $|f'(x)| < 1$ on an interval I does not guarantee a fixed point on I. For example, the function $f(x) = 2$ does not have a fixed point on $I = [0, 1]$ even though $|f'(x)| = 0 < 1$ on I. (The missing condition for a fixed point is that f maps I into itself.)

(b) Solving $f(x) = x$, we find that there are two fixed points: $x_1^* = 2$, $x_2^* = -2$.

(c) Note that $f'(x) = 3x^2 + 1$, so $f'(x^*) = 13 > 1$; hence, we would expect that for x_0 different from x_1^* and x_2^*, x_{n+1} does not converge. However, this needs to be proved, unless we can quote and apply a convenient theorem.

If the iterations converge, $\lim_{n\to\infty} x_n = x^*$, then, because f is continuous, $x^* = f(x^*)$ so $x^* = x_1^*$ or $x^* = x_2^*$ by part (b). Suppose that $x_0 \neq x_1^*$ and $\lim_{n\to\infty} x_n = x_1^*$. First we need to exclude the case that $x_n = x_1^*$ for some n: Since $f'(x) > 0$ for all x, f is increasing and thus one-to-one, consequently, $x_n \neq x_1^*$ for all n by induction. Now, from the mean value theorem,

$$x_{n+1} - x_1^* = f(x_n) - f(x_1^*) = f'(\xi_n)(x_n - x_1^*)$$

where ξ_n is between x_n and x_1^*; in particular, $\lim_{n\to\infty}\xi_n = x_1^*$. Since $f'(x^*) = 13 > 1$ and f' is continuous, it holds that $f'(\xi_n) \geq 1$ for all $n > n_0$ for some n_0; then $|x_{n+1} - x_1^*| \geq |x_n - x_1^*|$ for all $n > n_0$, hence $|x_n - x_1^*| \geq |x_{n_0} - x_1^*| \neq 0$ for all $n > n_0$. This contradicts with $\lim_{n\to\infty} x_n = x_1^*$. The same contradiction is obtained for x_2^*. Consequently, the iteration $x_{n+1} = f(x_n)$ converges only for $x_0 = x_1^*$ or x_2^*.

7. Suppose $|f(x)| \le 1$ for all $x \in [a,b]$, f is continuous on $[a,b]$, and $g(x)$ is monotonically increasing on $[a,b]$. Show that

$$\left| \int_a^b f dg \right| \le g(b) - g(a).$$

Solution

Let $P = [a = x_0, x_1, \ldots, x_N = b]$ be any partition of the interval $[a,b]$. Then the upper and lower Riemann-Stiltjes sums are

$$\begin{aligned} U(P,f,g) &= \sum_{k=1}^{N} M_k \left[g(x_k) - g\left(x_{k-1}\right)\right], \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) \\ L(P,f,g) &= \sum_{k=1}^{N} m_k \left[g(x_k) - g\left(x_{k-1}\right)\right], \quad m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) \end{aligned}$$

Because $-1 \le m_k \le M_k \le 1$, we have

$$\begin{aligned} U(P,f,g) &\le \sum_{k=1}^{N} \left[g(x_k) - g\left(x_{k-1}\right)\right] = g(b) - g(a), \\ L(P,f,g) &\ge -\sum_{k=1}^{N} \left[g(x_k) - g\left(x_{k-1}\right)\right] = -\left[g(b) - g(a)\right]. \end{aligned}$$

Since the Riemann-Stiltjes integral exists by the assumptions on f and g, and it equals to the infimum of the upper sums and the supremum of the lower sums, we have

$$-\left[g(b) - g(a)\right] \le \int_a^b f dg \le g(b) - g(a).$$

Finally, since g is monotonically increasing, $g(b) \ge g(a)$. (This is a special case of Rudin 6.12d with $M = 1$)

8. Let $x = f(u,v) = u^2 - v^2$ and $y = g(u,v) = 2uv$.

   (a) Find the Jacobian of the transformation $T : (u,v) \mapsto (x,y)$.

   (b) Express $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in terms of u and v.

   (c) Does the transformation $T : (u,v) \mapsto (x,y)$ have an inverse in the neighborhood of the point $(u,v) = (2,-1)$? Prove your answer.

Solution

(a) The Jacobian is $J = \det T'$, where $T^{'}$ is the matrix of the first order partial derivatives,

$$T' = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

hence

$$J = \det \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4(u^2 + v^2).$$

(b) The derivative of the inverse transformation $T^{-1} : (x,y) \mapsto (u,v)$ at (x,y) is $(T')^{-1}$ at (u,v) such that $(x,y) = T(u,v)$. This follows, e.g., from the chain rule: $T \circ T^{-1} = I \Longrightarrow T'\left(T^{-1}\right)' = I \Longrightarrow \left(T^{-1}\right)' = (T')^{-1}$. Hence,

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}^{-1} = \frac{1}{J} \begin{pmatrix} 2u & 2v \\ -2v & 2u \end{pmatrix},$$

which gives

$$\frac{\partial u}{\partial x} = \frac{2u}{4(u^2+v^2)},$$
$$\frac{\partial u}{\partial y} = \frac{2v}{4(u^2+v^2)},$$
$$\frac{\partial v}{\partial x} = \frac{-2v}{4(u^2+v^2)},$$
$$\frac{\partial v}{\partial y} = \frac{2u}{4(u^2+v^2)}.$$

(c) Yes, because $J(2,-1) \neq 0$ and T' is continous, so the inverse function theorem (Rudin 9.24) applies.