University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 22, 2005

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	______	5.	______
2.	______	6.	______
3.	______	7.	______
4.	______	8.	______
		Total	______

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
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1. Let X and Y be metric spaces, $f : X \to Y$ continuous, and $E \subset X$. Show that

$$f(\overline{E}) \subset \overline{f(E)}$$

where $\overline{E}$ means the closure of E. Give an example where $f(\overline{E}) \neq \overline{f(E)}$.

2. Prove that for $n \geq 2$,

$$\ln(n) - \sum_{k=2}^{n} \frac{1}{k} < 1/2.$$

3. Let f be differentiable on $[a, b]$ with $|f'(x)| < \beta$. Let P be a partition of $[a, b]$, and let $U(f, P)$ and $L(f, P)$ be the upper and lower Riemann sums. Prove that

$$U(f, P) - L(f, P) \leq \beta |P| (b - a)$$

4. Construct a function f on $[0, \infty]$ satisfying

   (a) f continuous

   (b) $0 < f < 1$

   (c) $\int_0^\infty f(x)dx < 1$

   (d) $\limsup_{x\to\infty} f(x) = 1$

   Note: The items in the above list are *not* parts of the problem with equal value as mentioned on the front page. This is a single problem to construct a function that satisfies *all* of the above conditions.

5. Let I be a closed and bounded interval in $\mathbb{R}$ and suppose that $g(x) : I \to I$ is a continuous function on I with a continuous first derivative on I such that $|g'(x)| < 1$ $\forall x \in I$. Show that the sequence defined by

$$x_0 \in I, \qquad x_{n+1} = g(x_n)$$

converges to the unique fixed point of g, i.e. $\lim_{n\to\infty} x_n = x^*$ where $g(x^*) = x^*$.

6. Let $F(x) = f_1(x) + f_2(x) + f_3(x) + ...$ be a series that is uniformly convergent over (a, b) and for which each $f_k(x)$ is continuous on $[a, b]$. Assume that we already know that $f_k(b)$ converges to a value, which we will define to be $F(b)$. Prove that

$$F(b) = \lim_{x \to b^-} F(x).$$

7. Prove the *Mean-Value Theorem for Integrals*: Let u and v be continuous real-valued functions on an interval $[a, b]$, and suppose that $v \geq 0$ on $[a, b]$. Then there exists a point ξ in $[a, b]$ such that

$$\int_a^b u(x)\, v(x)\, dx = u(\xi) \int_a^b v(x)\, dx.$$

8. Give an example of a function $f(x, y)$ such that

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) \neq \lim_{y \to 0} \lim_{x \to 0} f(x, y)$$