

University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

June 6, 2008

Name: _____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: \mathbb{R} denotes the set of real numbers; \mathbb{Z} denotes the set of integers; and, \mathbb{C} denotes the set of complex numbers. These extend to vector spaces as \mathbb{R}^n , \mathbb{Z}^n , and \mathbb{C}^n , respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

- | | |
|----------|----------|
| 1. _____ | 5. _____ |
| 2. _____ | 6. _____ |
| 3. _____ | 7. _____ |
| 4. _____ | 8. _____ |

Total _____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:

Lynn Bennethum, Julien Langou, Jan Mandel (Chair)

1. Let X be a metric space. Prove

- (a) (8 pts) If $\{V_\alpha\}_{\alpha \in A}$ is any collection of open sets in X , then $\cup_{\alpha \in A} V_\alpha$ is open.
- (b) (8 pts) If $\{V_k : k = 1, 2, \dots, n\}$ is a finite collection of open sets in X , then $\cap_{k=1}^n V_k$ is open.
- (c) (4 pts) Give an example of a metric space and a collection $\{V_\alpha\}_{\alpha \in A}$ of open sets such that $\cap_{\alpha \in A} V_\alpha$ is not open.

2. If X is a complete metric space and also a subspace of metric space Y , prove that X is a closed subset of Y .

3. Let $f : (0, +\infty) \rightarrow \mathbb{R}$, differentiable on $(0, +\infty)$ and such that $\lim_{x \rightarrow +\infty} f' = \ell$.
Show that

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \ell.$$

4. Find all intervals where the series $\sum_{n=1}^{\infty} x e^{-nx^2}$ converges, converges absolutely, and converges uniformly.

5. The space l^∞ is the space of all bounded real sequences $a = (a_1, a_2, \dots)$, equipped with the norm $\|a\|_\infty = \sup \{a_1, a_2, \dots\}$. Show that the set $C = \{a \in l^\infty : \|a\|_\infty \leq 1\}$ is closed and bounded in l^∞ , but it is not compact.

6. Let $f(x) = 1$ if x is rational, 0 if x is irrational. Show that f is not Riemann integrable on the interval $[0, 1]$.

7. On the space $C([0, 1])$ of all continuous functions on the interval $[0, 1]$, consider the functions

$$\|f\|_1 = \int_0^1 |f(t)| dt, \quad \|f\|_0 = \int_0^1 t|f(t)| dt.$$

- (a) (8 pts) You can assume to be known that $\|\cdot\|_1$ is a norm on $C([0, 1])$. Show that $\|\cdot\|_0$ is a norm on $C([0, 1])$.
- (b) (12 pts) Are $\|\cdot\|_1$ and $\|\cdot\|_0$ equivalent?

8. Use the implicit function theorem to prove there exists functions $u(x, y)$, $v(x, y)$, and $w(x, y)$, and an $r > 0$ such that u , v , w are continuously differentiable and satisfy the equations

$$u^5 + xv^2 - y + w = 0$$

$$v^5 + yu^2 - x + w = 0$$

$$w^4 + y^5 - x^4 = 1$$

on $B_r(1, 1)$, and $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$.