University of Colorado Denver  
Department of Mathematical and Statistical Sciences  
Applied Analysis Ph.D. Preliminary Exam  
January 11, 2010

Name: __________________________________________

Exam Rules:

• This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
• Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
• Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
• If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
• Begin each solution on a new page and use additional paper, if necessary.
• Write only on one side of paper.
• Write legibly using a dark pencil or pen.
• Ask the proctor if you have any questions.

Good luck!

1. _________  5. _________
2. _________  6. _________
3. _________  7. _________
4. _________  8. _________

Total _________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:  
Julien Langou, Weldon Lodwick, Jan Mandel (Chair)
1. Prove that every sequence of real numbers contains a monotone subsequence.
2. Prove or disprove that if $M$ is infinite compact subset of $\mathbb{R}$, then $M$ contains a nondegenerate interval. (A nondegenerate interval is of the form $(a, b], [a, b], (a, b), [a, b)$ with $a < b$.)
3. Show that in a neighborhood of $(0, 0, 0, 0)$ the system of equations

\[3w + x - y + z^2 = 0\]
\[w - x + 2y + z = 0\]
\[2w + 2x - 3y + 2z = 0\]

\[\text{can be solved for } w, x, z \text{ in terms of } y; \text{ for } w, y, z \text{ in terms of } x; \text{ for } x, y, z \text{ in terms of } w; \text{ but not for } w, x, y \text{ in terms of } z.\]
4. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ defined by

$$
\forall y \in \mathbb{R}, \quad g(y) = \int_0^1 f(x) e^{xy} \, dx.
$$

(a) Show that $g$ is continuous.
(b) Show that $\lim_{y \to -\infty} g(y) = 0$. 
5. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $\alpha$ in $\mathbb{R}$ and $\beta$ in $\mathbb{R}$ such that $\lim_{x \to -\infty} f(x) = \alpha$ and $\lim_{x \to +\infty} f(x) = \beta$. Show that $f$ is uniformly continuous on $\mathbb{R}$, and bounded.
6. Let $X \subset \mathbb{R}$, and $(f_n : X \to \mathbb{R})_{n \in \mathbb{N}}$ be a sequence of functions uniformly continuous on $X$ and uniformly converging on $X$ to the function $f : X \to \mathbb{R}$. Show that $f$ is uniformly continuous on $X$. 
7. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^n$ when

(a) $c_n = \ln \left(1 + \frac{1}{n}\right)$

(b) $c_n$ is the $n$-th decimal digit of $\pi$. 
8. Let $E_n$ be subsets of a metric space and $E = \bigcup_{n=1}^{N} E_n$. Prove that $E' = \bigcup_{n=1}^{N} E'_n$, where $A'$ denotes the set of all limit points of $A$. 