Name: ________________________________

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. All solutions will be graded and your final grade will be based on the total of all of them.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: cite theorems that you use, provide counterexamples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1. __________
2. __________
3. __________
4. __________
5. __________

Total __________

COMMITTEE
Andrew Knyazev
Weldon A. Lodwick
Jan Mandel

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.
1. A function $f: \mathbb{R} \to \mathbb{R}$ is called additive if $f(x + y) = f(x) + f(y)$ for all real $x, y$. Show that if $f$ is additive and continuous, then $f(x) = cx$ for some real constant $c$. 
2. Find an example of a metric space \((V; d)\) and a set \(A \subset V\) such that \(A\) is closed and bounded but not compact.

Make sure you actually formulate the definitions and prove that your set \(A\) is closed, bounded, and not compact in your metric space. Simply an example without detailed proofs is insufficient.

The unit ball \(B\) in \(l_2\)

\[ B = \left\{ (x_i) : \sum_{i=1}^{\infty} x_i^2 \leq 1 \right\} \]

is closed and bounded: \(B\) is closed because \(B = f^{-1}([0,1])\) where \([0,1] \subset \mathbb{R}\) is closed and \(f : l_2 \to \mathbb{R}\), \(f : (x_i) \mapsto \|x_i\|_{l_2}^2\) is continuous (because the norm on a normed space is a continuous function). Since \(\|u - 0\| \leq 1\) for all \(u \in B\), \(B\) is bounded. Let \(u^1 = (1,0,0,0,...)\), \(u^2 = (0,1,0,0,...)\), \(u^3 = (0,0,1,0,...)\)....Then \(u^k \in B\) and \(\|u^k - u^j\| = 1\) when \(k \neq j\). Consequently, no subsequence of the sequence \(\{u^k\}\) in \(l_2\) can be Cauchy because the definition of Cauchy sequence \(\forall \varepsilon > 0 \exists M \forall k,j > M : \|u^k - u^j\| < \varepsilon\) is falsified for \(\varepsilon = 1/2\). Thus, this sequence \(\{u^i\}_{i=1}^{\infty}\) is one in which each component belongs to the unit ball \(B\), but has no convergent subsequence.
3. If \( \{a_n\} \) is a convergent sequence of real numbers, then

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.
\]

Prove, or find a counterexample.
4. For a real variable $x \in [-1, 1]$ let $D(x)$ be a function that takes the value 1 if $x$ is rational and the value 0 otherwise. Is the function $F(x) = xD(x)$ Riemann integrable? If so, what is the value of the integral $\int_{-1}^{1} F(x)\,dx$?

If $x_i \geq 0$ the contribution from this interval to the upper Riemann sum is $x_{i+1}(x_{i+1} - x_i)$, since the $\sup_{x \in [x_i, x_{i+1}]} F(x) = x_{i+1}$. Indeed, first, we observe that $F(x) \leq x_{i+1}$ for $x \in [x_i, x_{i+1}]$. Second, if $x_{i+1}$ is rational, than simply $F(x_{i+1}) = x_{i+1}$. If $x_{i+1}$ is irrational, than there exists a sequence $y^{(j)}$, $j = 1, \ldots, \infty$ of rational numbers such that $y^{(j)} \in [x_i, x_{i+1}]$ and $y^{(j)} \to x_{i+1}$ as $j \to \infty$, because of the density of rationals on the real line. E.g., one can use a decimal representation of $x_{i+1}$ chopped at the $j$-th decimal digit to construct a specific example of the sequence $y^{(j)}$, $j = 1, \ldots, \infty$. Since every $y^{(j)}$ is rational, we have $F(y^{(j)}) = y^{(j)}$, but $y^{(j)} \to x_{i+1}$, so $F(y^{(j)}) \to x_{i+1}$.

At the same time, if still $x_i \geq 0$, the contribution from this interval to the lower Riemann sum is zero. Indeed, first, we observe that $F(x) \geq 0$ for $x \in [x_i, x_{i+1}]$. But any nonempty interval on a real line contains at least one irrational point, by the density of irrationals on the real line. Thus, $\min_{x \in [x_i, x_{i+1}]} F(x) = 0$.

Similarly, if now $x_{i+1} \leq 0$, the contribution from this interval to the upper Riemann sum is zero, while the contribution from this interval to the lower Riemann sum is $x_i(x_{i+1} - x_i)$. If there is an interval such that $x_i < 0 < x_{i+1}$, its contribution to either sum is negligible, since $x_{i+1} - x_i \to 0$.

The Riemann sums on $[-1, 1]$ can be found by computing separately the Riemann sums on $[-1, 0]$ and $[0, 1]$ and adding them, by additivity of Riemann sums.

We first deal with the upper Riemann sum. Let us notice that the upper Riemann sum of $F(x) = xD(x)$ for $x \in [0, 1]$ is exactly the same as the upper Riemann sum of the Riemann-integrable function $f(x) = x$. Thus, summing up, and taking the limit of the upper Riemann sum of $F(x)$ for $x \in [0, 1]$ gives us the same number as simply $\int_{0}^{1} f(x)\,dx = \int_{0}^{1} x\,dx = 1/2$. The other term, of the upper Riemann sum of $F(x)$ for $x \in [-1, 0]$ is zero, therefore the upper Riemann sum on $[-1, 1]$ for $F(x)$ is equal to $1/2 + 0 = 1/2$.

Similar arguments show that the lower Riemann sum is $\int_{-1}^{0} x\,dx = -1/2$. Since the upper and lower Riemann sums are different, the function $F(x)$ is not Riemann integrable.
5. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, let

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function $f$.

(b) Show that the sequence of derivatives $\{f'_n\}$ does not converge uniformly on $\mathbb{R}$ to any function.

(a) Let $\varepsilon > 0$. Clearly,

$$f(x) = \lim_{n \to \infty} \frac{x}{1 + nx^2} = 0.$$

To prove that $f_n \to 0$ on $\mathbb{R}$, we need to show that there is $M$ such that $|f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$ and all $n > M$. First,

$$\left| \frac{x}{1 + nx^2} \right| \leq |x| < \varepsilon \quad \text{if } |x| < \varepsilon.$$

and

$$\left| \frac{x}{1 + nx^2} \right| \leq \left| \frac{x}{nx^2} \right| \leq \frac{1}{n\varepsilon} \quad \text{if } |x| \geq \varepsilon.$$

Thus,

$$\left| \frac{x}{1 + nx^2} \right| < \varepsilon \quad \forall n > \frac{1}{\varepsilon^2}.$$

and we can take $M = \left\lceil \frac{1}{\varepsilon^2} \right\rceil$.

(b) The theorem about differentiation of functional sequences states that if $\{f'_n\}$ exists and converges uniformly on an interval $[a, b]$ and $f_n(x_0)$ converges at some $x_0 \in [a, b]$, then there exists a differentiable function $\phi$ such that $f_n \to \phi$ uniformly on $[a, b]$ and $f'_n \to \phi'$ uniformly on $[a, b]$. Let us take $[a, b] = [-1, 1]$. We already know from part (a) that $f_n(x_0)$ converges (to zero), e.g., at $x_0 = 0$.

We prove by contradiction. Let us assume that $\{f'_n\}$ converges uniformly on $\mathbb{R}$, thus it also converges uniformly on $[a, b] = [-1, 1]$. (It would be an error to use such a contradiction on $\mathbb{R}$ directly since the theorem is formulated for a bounded closed interval only.)

The assumptions of the theorem above are satisfied, so there exists a differentiable function $\phi$ such that $f_n \to \phi$ uniformly on $[a, b]$ and $f'_n \to \phi'$ uniformly on $[a, b]$. On the one hand, from part (a), $f_n \to f \equiv 0$ uniformly on $\mathbb{R}$ and thus on $[a, b]$, so $\phi = f \equiv 0$ by the uniqueness of the limit, and clearly $\phi' \equiv 0$. On the other hand, by direct calculation,

$$f'_n(x) = \left( \frac{x}{1 + nx^2} \right)' = \frac{(1 + nx^2) x' - (1 + nx^2)' x}{(1 + nx^2)^2} = 1 + nx^2 - 2nx^2 = \frac{1 - nx^2}{(1 + nx^2)^2}.$$
Thus,
\[ \lim_{n \to \infty} f'_n(0) = \lim_{n \to \infty} 1 = 1 \neq 0, \]
which is a contradiction.