IRREGULARITY STRENGTH OF DIGRAPHS

by

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Thesis directed by Professor Michael Jacobson

ABSTRACT

An irregular digraph is a digraph whose vertex degree pairs are all distinct. Irregular digraphs and irregular multi-digraphs have a very nice property that they have an error-correcting degree matrix. In order to add to the study of irregular multi-digraphs, we introduce the term irregularity strength of digraphs.

Define a pair of maps f, g where f is from the arc set of a digraph into the positive integers and where g induces vertex weights on the vertices subject to

$$g(v) = (\sum_{vx \in A(D)} f(vx), \sum_{uv \in A(D)} f(uv)).$$

If all the degree weights induced by g are distinct across the vertex set of D, then we say f is an irregular labeling of the digraph D and g is an irregular vertex weighting of the digraph D. Let $I(D)$ be the set of irregular labelings of D. If s is the maximum value of $f(e)$ for $e \in A(D)$, we say f is an irregular s-labeling, g an irregular s-weighting if f and g satisfy the above conditions. The irregularity strength of a digraph D is the minimum such s used for $f \in I(D)$. We define the irregularity strength $\vec{s}$ of a digraph $\vec{D}$ to

$$\vec{s}(\vec{D}) = \min_{f \in I(\vec{D})} \max_{e \in A(D)} f(e).$$

We give various techniques for determining digraph irregularity strength and constructing irregular labelings of digraphs using the minimum possible maximum value as a label. In particular, if we know that $\vec{s}$ is the irregularity strength of $\vec{D}$, then we write $\vec{s}(\vec{D}) = \vec{s}$. An irregular labeling of a digraph with the minimal number of labels is an irregular $\vec{s}$-labeling of $\vec{D}$. That is, we give techniques for irregular $\vec{s}$-labelings of tournaments, some orientations of paths, every orientation of $K_{3,3}$, $K_{4,4}$, and show that for various orientations of star forests, trees, and cross-product graphs there exist irregular $\vec{s}$-labelings analogous to the irregular labelings of their underlying graphs.

We develop the topic of irregular orientations of a graph G. An irregular orientation of a graph G, $\vec{O}$, is an orientation of a graph which has the error-correcting degree matrix typical of an irregular digraph. We show a class of graphs that have irregular orientations.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________
Michael Jacobson

DEDICATION

We would like to dedicate the thesis to the numerous ski resorts that can be found in Colorado including Copper Mountain and Winter Park. Having learned to ski at Copper Mountain, the author cannot help but attribute some of his appreciation for learning to the ski industry and to Copper Mountain in particular. While skiing has been subsumed by snowboarding throughout most of the country, we raise a toast to those skiing purists who manage to set the snowboard aside a few times each year and strap on their ski-boots.

ACKNOWLEDGMENT

This thesis continues the work of several notable mathematicians and in particular is modeled after a former student of my advisor, Hal Hackett. His thesis served as the foundation for at least two major pieces of this thesis. Without his work, the current thesis would probably not have seemed feasible from the outset. In addition, we must acknowledge the work of Z. Skupień and a number of other Polish mathematicians including but not limited to J. Górska, Z. Dziechcińska-Halamoda, Z. Majcher, J. Michael. Of course, the entire topic was modeled after work by Chartrand, Jacobson, Lesniak, Oellerman, Ruiz, and Saba. The topic of irregularity strength for graphs is storied and we would also be remiss if we did not mention the paper by Gargano, Kennedy and Quintas. Closer to home there are a number of people in the department we would like to thank. Of course, all the members of the committee helped out. Among the graduate students, we would like to thank Arthur Busch, Angela Harris, Nathan Kurtz, Shelley Speiss, and Craig Tennenhouse for their cameraderie. A special thanks goes to Michael Ferrara whose meticulous criticism and support were essential for getting the project off of the ground and keeping it in the air.

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1. Introduction to Graph Theory

Graph theory is the study of graphs and graph parameters. Sometimes a thorough study of two graph parameters will lead to a better understanding of the relationship of the two parameters or a discovery of a third important parameter which is a function of the first two parameters.

There are various ways of recording the adjacency matrix of the graph, i.e., efficient ways of describing the edge set and vertex set of a graph. By defining a digraph and the degree sequence of a digraph, the density of a graph or digraph is given, but the degree sequence communicates great deal more information. Because a degree sequence is not unique up to graph isomorphism, the degree sequence of a graph does not give all the information about a graph. After making the cursory definitions, we define the powers of a graph $G, G^2, ..., G^k$ and distinguish the power operation from the cross-product operation as well as the lexicographic product operation.

1.1 Terminology

First define a graph and some substructures of the complete graph including paths, trees, cycles, circuits, walks, multipartite and bipartite graphs in the following way.

Definition 1.1 ([36]) *A simple graph G is a set of two sets V and E where E is a set of unordered pairs of elements from the vertex set. The order $n(G)$ of a graph is the cardinality of V. The size of a graph, $e(G)$, is the cardinality of E.*

There appears to be one way of defining a graph, and this description or definition seems to recur through the literature of graph theory, though no one author seems to be the originator or inventor of the concept of a graph.

Theorem 1.2 ([36]) *The maximum size of a simple graph where* $|V| = n$ *is* $\binom{n}{2}$.

This theorem is easy to see because each vertex in a simple graph (graph where no double edges appear) is $n-1$ where n is the order of the graph. Now, because each edge has two endpoints, we get that the maximum number of edges in a simple graph is $\frac{n(n-1)}{2}$. This edge count is clear by what we call the First Theorem of Graph Theory [10]: every simple graph has the property that the sum of the degrees in the degree sequence of a graph is twice the number of edges in the graph.

Definition 1.3 ([36]) *A clique or complete graph* K_n *is a simple graph on* n *vertices of maximal size.*

Definition 1.4 ([36]) *An isomorphism from a simple graph* G *to a simple graph* H *is a bijection* $f : V(G) \longrightarrow V(H)$ *such that* $uv \in E(G)$ *if and only if* $f(u)f(v) \in E(H)$*. We say* G *is isomorphic to* H *if there exists such a function* f*.*

The concept of a subgraph can be used to understand the whole of graph theory as the study of subgraphs of the complete graph.

Definition 1.5 ([36]) *A simple graph* H *is a subgraph of* G *when there exists a subset of* $V(G)$ *and a subset of* $E(G)$ *that when taken together as a graph* G', *gives G' isomorphic to H. We write $H \subset G$ if H is a subgraph of G.*

Definition 1.6 ([36]) *The neighborhood of a vertex v, $N(v) = \{U : u \in U \text{ if } uv \in E(G)\}$. The closed neighborhood of v, denoted $N[v], = N(v) \cup \{v\}$.*

Definition 1.7 ([36]) *The neighborhood of a subset of vertices U of $V(G)$ is the union of the neighborhoods of the vertices of U.*

The degree of a vertex is the fundamental idea we will be discussing. The degree sequence of a digraph or graph appears to be the next step along the sets of information from the density of a graph or digraph that create an ascending tower of information, and which terminate with the full incidence or adjacency matrix of the desired graph or digraph.

Definition 1.8 ([36]) *The degree of a vertex v written $deg(v)$ is $= |U : u \in U \text{ if } uv \in E(G)|$. We call $N(v)$ the neighborhood of v and define it to be the set of vertices adjacent with v. Notice $|N(v)| = deg(v)$.*

Definition 1.9 ([36]) *The minimum degree of G written $\delta(G)$ is the*

$$\min\{deg(v) : v \in V(G)\}.$$

The maximum degree of G written $\Delta(G)$ is $\max\{deg(v) : v \in V(G)\}$.

Definition 1.10 ([36]) *Given a graph G of order n, an n-term list of the degrees of G is the degree sequence of G. We denote the degree sequence of G by $\pi(G)$. Given a sequence S of n integers, we say that the sequence S is graphic if there exists a graph G of order n such that $\pi(G) = S$.*

To shorten notation, if the degree i has multiplicity k in a degree sequence, we sometimes write i^k to record those terms of degree i in the degree sequence.

The Erdös-Gallai theorem, presented as a basic theorem in most courses, relies on very fundamental, though quite difficult proof techniques and establishes a very deep, though not quite intuitive idea.

Theorem 1.11 (Erdös-Gallai) *A sequence* π*:* $d_p \leq d_{p-1} \leq \cdots \leq d_1$ *of non-negative integers whose sum (say s) is even is graphic if and only if*

$\sum_{i=1}^{i=k} d_i \leq k(k-1) + \sum_{j=k+1}^{j=p} \min\{d_j, k\}$ *for every* k *such that* $1 \leq k \leq p$*.*

Theorem 1.12 (Havel-Hakimi) *For* $n > 1$ *a non-increasing integer list* $\pi = d_1, d_2, ..., d_p$ *with* p *terms is graphic if and only if* π' *is graphic where* π' *is obtained from* π *by deleting its largest element* d_1 *and subtracting 1 from its* d_1 *next largest terms. The only 1-term graphic sequence is* $d_1 = 0$*.*

As a first step toward understanding the basic structural aspects of a graph, consider the following concept of a walk.

Definition 1.13 ([10]) *A walk is a list* $v_1, e_1, v_2, ..., e_k, v_k$ *of vertices and edges such that for all* $1 \leq i \leq k$*, the edge* e_i *is the unordered pair* $v_{i-1}v_i$*. The length of a walk is the number of edges in the walk. Walks are often abbreviated by just listing the order of vertices.*

Definition 1.14 ([10]) *A trail is a walk with no repeated edge.*

Definition 1.15 ([10]) *A circuit is a closed trail; that is, a circuit is a trail whose initial and terminal vertices are identical.*

Definition 1.16 ([10]) *The adjacency matrix of* G*,* $A(G)$ *is an* n *by* n *square matrix which has an entry at the* uv *and* vu *coordinate if and only if* $u \sim v$ *in* G*; that is, if* $uv \in E(G)$*. The powers of* $A(G) : A(G), A^2(G), A^3(G), ...,$ *are just the powers of* $A(G)$ *under standard matrix multiplication.*

Theorem 1.17 ([10]) *The number of* uv*-walks of length* j *in* G *is given by the* uv*-coordinate of* $A^j(G)$*.*

Definition 1.18 ([10]) *A* uv*-path is a walk from* u *to* v *which is a subgraph of* G *such that no vertex or subsequence of two vertices repeats.*

Definition 1.19 ([10]) *A path of order* n*,* P_n*, is a tree with degree sequence* $(1)^2, (2)^{n-2}$*.*

Definition 1.20 ([10]) *A graph* G *is connected if there is a* uv*-path in* G *for every pair of vertices* $u, v \in V(G)$*.*

Definition 1.21 ([10]) *A component of a graph* G *is a connected subgraph* H *of* G *such that for no* $v \in V(H)$*,* $u \in V(G - H)$*, does* G *contain a* uv*-path.*

Definition 1.22 ([10]) *A cycle of order* n*,* C_n*, is the unique connected simple graph with degree sequence* $(2)^n$*.*

Definition 1.23 ([10]) *A tree* T_n *is a simple graph which has no cycles as subgraphs, but which has* $n(G) - 1 = e(G)$*.*

A tree is an example of a bipartite graph.

Definition 1.24 ([36]) *A bipartite graph has a bipartition of its vertex set such that no edge has both endpoints in the same partite set of the bipartition.*

Theorem 1.25 ([36]) *Bipartite graphs have no odd cycles.*

Definition 1.26 ([36]) *A biclique $K_{m,n}$ is a bipartite simple graph with m vertices in one partite set, n vertices in the other partite set and maximum density.*

Now we develop the idea of circuits.

Definition 1.27 ([10]) *An Eulerian circuit C is a circuit in G such that $E(G) \subset E(C)$.*

Theorem 1.28 ([10]) *A connected graph G is Eulerian if and only if the degree of every vertex in G is even.*

Definition 1.29 ([10]) *A cycle in G is a closed uv-walk in G such that no vertex repeats.*

Eulerian graphs and multi-graphs will be especially important in our study of irregular labelings of digraphs. Specifically, we will use rooted circuits which we pack in a special directed multi-graph called a complete symmetric digraph with loops. The complete symmetric digraph with loops has applications in Chapter 7, but notice here that a closed circuit is Eulerian and that any connected graph that decomposes into Eulerian circuits is necessarily itself Eulerian.

Definition 1.30 ([10]) *We say a subgraph H of G spans G if $V(G) \subset V(H)$.*

Again, subgraph containment is vital to one of the techniques we will apply to the study of digraph irregularity strength. First, notice any simple graph is a subgraph of K_n and similarly that any simple digraph with loops is a subdigraph of the complete symmetric digraph with loops. Meanwhile, every oriented graph can be taken as a subdigraph of some tournament even though this is not the case for simple digraphs. This is because simple digraphs are a super-set of oriented simple graphs; simple digraphs can have two arcs between a pair of vertices which are not identical, 2 copies of the same edge directed in opposite directions.

Definition 1.31 ([10]) *A Hamiltonian cycle is a spanning cycle of G.*

Definition 1.32 ([10]) *A spanning tree is a simple spanning subgraph H of G which is a tree.*

Theorem 1.33 ([10]) *All connected graphs have a spanning tree.*

Definition 1.34 ([10]) *A factor is a spanning subgraph of G which is regular.*

Definition 1.35 ([10]) *The complement of a graph G is the ordered pair*

$$< V(G), (E(H) - E(G)) >$$

where H is the complete graph on the vertex set of G.

Digraphs come in many types. There are simple oriented graphs, oriented graphs with loops, and looped and loopless directed multigraphs. The fundamental theorem of digraph theory is that the sum of out-degrees are equal to the sum of in-degrees in any digraph.

1.2 Degree sequences and Digraphs

We begin by defining an oriented graph.

Definition 1.36 ([5]) *An orientation $\vec{G}$ of the graph G is an ordered pair of sets $V(\vec{G}), A(\vec{G})$ where we have $V(\vec{G}) = V(G)$ and the arc set of $\vec{G}$ is just a* *set formed from the edge set of G by adding some direction to each edge, that is we replace each unordered pair with exactly one ordered pair of vertices. If the ordered pair $(u, v) \in A(\vec{G})$ then we say that $\vec{uv}$ is directed from u to v and that u beats v. We often abuse notation by saying $\vec{uv} \in A(\vec{G})$.*

Definition 1.37 ([5]) *The degree of a vertex in an oriented graph $\vec{G}$ is an ordered pair $deg(v) = (a, b)$ such that the cardinality of the set of arcs $\vec{vu} \in A(\vec{G})$ is a, and the cardinality of the set of arcs $\vec{uv} \in A(\vec{G})$ is b. We also say $deg^+(v) = a$, $deg^-(v) = b$ and $deg^{tot}(v) = a + b$. The term*

$$\delta^0(\vec{G}) = \min\{x : x = \min\{deg^+(v), deg^-(v)\} \text{ where } v \in V(\vec{G})\}.$$

Similarly,

$$\delta^+(\vec{G}) = \min\{deg^+(v) : v \in V(\vec{G})\},$$

$$\delta^-(\vec{G}) = \min\{deg^-(v) : v \in V(\vec{G})\},$$

and

$$\delta^{tot}(\vec{G}) = \min\{deg^+(v) + deg^-(v) : v \in V(\vec{G})\}.$$

Terms $\Delta^0, \Delta^+, \Delta^-, \Delta^{tot}$ are defined analogously.

Definition 1.38 ([5]) *A source is a vertex with $deg^- = 0$. A sink is a vertex with $deg^+ = 0$. An out-directed orientation of a graph is an orientation of a graph with one source w and $deg^-(v) \geq 1$ for all $v \neq w$. An in-directed orientation of a graph is an orientation of a graph with one sink w and $deg^+(v) \geq 1$ for all $v \neq w$. An anti-graph is an orientation of a graph where every vertex is a source or a sink.*

Tournaments have many special properties: among them we find that every tournament has at least one king, [10], and every tournament is traceable [9]. Only two complete graphs can be oriented so that they are anti-graphs: K_1 and K_2.

Definition 1.39 ([5]) *A tournament is an orientation of* K_n*.*

We go over the structures which are analogous to the basic structures of path and cycle in a graph.

Definition 1.40 ([5]) *A directed path is an orientation of a path such that only the terminal and initial vertices do not have degree* $(1, 1)$*.*

Definition 1.41 ([5]) *A directed cycle is an orientation of a cycle such that all the vertices have degree* $(1, 1)$*.*

Definition 1.42 ([5]) *We say an orientation of a graph is traceable if there is a directed Hamiltonian path contained as a subdigraph.*

Theorem 1.43 ([5]) *Every tournament is traceable.*

Definition 1.44 ([5]) *A transitive tournament is a tournament with the property that if* $\vec{uv}, \vec{vw} \in A(T_n)$ *then* $\vec{uw} \in A(T_n)$*.*

Theorem 1.45 ([5]) *A transitive tournament is acyclic.*

A valuable way of viewing a tournament is in terms of its vertex degrees.

Definition 1.46 *Let* N_0 *be the set of nonnegative integers,* $p \in N_0$ *and let* $B_p = \{(a, b) : a + b = p,\ a, b \in N_0\}$*.*

Definition 1.47 *A block is a set of entries from B_p. A complete block is B_p for some value p.*

Notice that if we have a complete block then this block can be realized by a digraph: a transitive tournament of order p realizes the block B_{p-1} as a digraph.

Theorem 1.48 *The transitive tournament T_n is an irregular oriented graph and its sets of degree pairs has deg $(V(T_n)) = B_{n-1}$.*

Furthermore, we can describe a sub-block of a block B_p as a subset of the elements of the block B_p. Of special interest is the set of balanced sub-blocks of a block B_p, that is, the sub-blocks whose set of in-arguments sum to the same value as its set of out-arguments. More precisely, a balanced sub-block, [12], $B_p[m_n]$ is a sub-block of B_p with $m_n \leq p$ elements $\{(a_1, b_1), (a_2, b_2), ..., (a_{m_n}, b_{m_n})\}$ where we have

$$\sum_{i=1}^{i=m_n} a_i = \sum_{i=1}^{i=m_n} b_i.$$

We denote the transitive tournament of order n by T_n; there should be no confusion with a tree of order n. We sometimes abuse notation by using T_n to denote an arbitrary tournament of order n (see Chapter 5). An important generalization of a tournament is called a team tournament.

Definition 1.49 ([5]) *A k-team tournament is an orientation of a complete k-partite graph.*

Notice that by the pigeonhole principle every graph has two vertices of the same degree. This is not true of digraphs, with an obvious counterexample being the transitive tournaments.

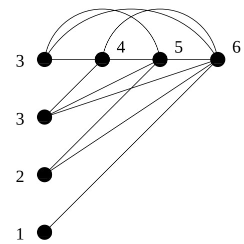


Figure 1.1: A Maximally Near-Irregular Graph has 2 vertices of the same degree.

Theorem 1.50 (Landau) *A non-increasing sequence* $\{s_i\}_{i=1}^{i=n}$ *is a score sequence of an n-vertex tournament if and only if* $\binom{k}{2} \leq \sum_{i\in I,|I|=k} s_i$.

A fact we will use later is that if two vertices have the same degree in a tournament, then those two vertices lie on a directed triangle.

Definition 1.51 *A multigraph* M *is an ordered triple*

$$< V(M), E(M), f >$$

where f *is a map from the elements of* $E(M)$ *into the positive integers. We interpret the image of an edge* e *under* f, $f(e)$ *to be the edge multiplicity of an edge. A multi-digraph is an ordered triple* $< V(M), A(M), f >$ *where we allow both* $\vec{uv}$ *and* $\vec{vu} \in A(M)$. *A simple digraph is a multi-digraph where* $f(a) = 1$ *for all* $a \in A(M)$. *We say a graph* G *underlies a digraph* D *if* $V(D) = V(G)$ *and for all* $uv \in E(G)$, $\vec{uv}$ *or* $\vec{vu} \in A(D)$.

Definition 1.52 ([5]) *A semi-complete digraph* D *is a simple digraph such that* K_n *underlies* D.

Definition 1.53 ([3]) *If a graph has at most k vertices of the same degree then we say it is k-irregular.*

Chapter 9 concerns itself with irregular orientations of k-irregular graphs. To explain what this means we will need some definitions.

Definition 1.54 ([5]) *An orientation of a multigraph is a 1-1 onto function from the edge set of a multigraph to the arc set of a digraph with the same vertex set. We simply put a direction on each edge.*

Definition 1.55 ([20]) *A digraph D such that every vertex has distinct degree is called an irregular digraph.*

Interest in irregular digraphs is at least two years old and we can trace the origins of this interest back to at least two distinct sources [20], [12]. In [20], the idea of embedding irregular digraphs in regular digraphs is developed as are constructions wherein regular digraphs are built from blocks of irregular digraphs that can be recovered by taking sets of vertices and looking at the induced digraphs on those sets of vertices.

Definition 1.56 ([20]) *If a graph has an orientation which corresponds to an irregular digraph then we say it has an irregular orientation.*

The problem of finding an irregular orientation of a graph seems to be a completely new problem. A graph, with its degree set partitioned into classes by degree, can be oriented in a specific way to partition those degree sets even further. Notice that irregular orientations have obvious necessary conditions on which they depend in order to exist. There do not appear to be any obvious sufficient conditions, though we will certainly explore this problem later (Chapter 9). The idea of labeling a digraph by its degree set is a powerful one because it guarantees that the resultant digraph has automorphism group of order 1. Given a graph there may be more than one irregular orientation, or all irregular orientations may be isomorphic up to digraph isomorphism. This latter situation would mean than the original graph would have a relatively large automorphism group compared to that of the irregularly oriented graph which has automorphism group of order 1. Notice that if we give a graph an irregular orientation the result is an irregular digraph or rather a digraph of irregularity strength 1 (see Chapter 3).

Theorem 1.57 *Every complete graph has a unique irregular orientation up to digraph isomorphism.*

Theorem 1.58 *Every* $K_{m,m+1}$ *has a unique irregular orientation up to digraph isomorphism.*

Definition 1.59 *We say a graphic sequence* π *is uniquely realizable as a simple graph, multigraph, oriented graph, digraph, or multidigraph respectively if there is exactly one simple graph, multigraph, oriented graph, digraph, or multidigraph that realizes that sequence.*

Next consider the theorems which concern themselves with realizations of di-sequences. There are many ways of building up this foundation; we do so here with little proof included, in the fashion followed by many of the texts that cover the same material.

The following theorem by Berge tells us which sequences are di-graphic.

Theorem 1.60 (Berge [6]) *Let* $(r_1, s_1),\ (r_2, s_2), ..., (r_n, s_n)$ *be pairs of integers with*

$$s_n \leq s_{n-1} \leq \cdots \leq s_1.$$

The pairs $\{(r_k, s_k)\}_{i=1}^{i=n}$ *constitute the degree pairs of a digraph with maximum arc multiplicity* m *if and only if,*

$$\sum_{j=1}^{j=k} s_j \leq \sum_{i=1}^{i=n} \min\{r_i, mk\} \quad k \in [n] \tag{1.1}$$

$$\sum_{j=1}^{j=n} s_j = \sum_{i=1}^{i=n} r_i. \tag{1.2}$$

The following definitions are useful when we study realizations of degree sequences for digraphs.

Definition 1.61 *When we replace an arc* $\vec{uv}$ *with the pair of arcs,* $u\vec{x}_2$ *and* $x_1\vec{v}$ *we refer to this construction as an arc-swap.*

Notice this definition is not necessarily standard in the literature [17].

Definition 1.62 *When we replace two arcs* $\vec{uv}$*,* $x_1\vec{x}_2$ *with the pair of arcs* $u\vec{x}_2$ *and* $x_1\vec{v}$ *we refer to this construction as a double arc-swap.*

Definition 1.63 *A null vertex is a vertex of degree zero. It is also called an isolated vertex.*

Definition 1.64 *Selecting a hamiltonian path from a tournament is called tracing a tournament.*

Definition 1.65 *One sequence is majorized by another if the first sequence can be put in a one-to-one correspondence with the second such that each term of the first sequence is less than or equal to its image in the second sequence. In particular* $\{(a_i, b_i)\}_{i=1}^{i=n}$ *is majorized by* $\{(a_i' b_i')\}_{i=1}^{i=n}$ *if* $a_i \leq a_i'$ *and* $b_i \leq b_i'$ *for all* $1 \leq i \leq n$.

1.3 Operations on Graphs

Next we cover the basic operations on graphs.

Definition 1.66 ([36]) *The union of two graphs G and H has*

$$G \cup H = \; < V(G) \cup V(H), E(G) \cup E(H) > .$$

Definition 1.67 ([36]) *The join of two graphs*

$$G \wedge H = \; < V(G) \cup V(H), E(G) \cup E(H) \cup E([V(H), V(G)]) > .$$

Definition 1.68 ([36]) *The direct product of two graphs*

$$G \times H = \; < \{v_{i,j} : v_i \in V(H) \wedge v_j \in V(G)\},$$

$$\{v_{i,j} v_{h,k} : v_i v_h \in E(H) \vee v_j v_k \in E(G)\} > .$$

Definition 1.69 ([36]) *The lexicographic product*

$$G \times_{lex} H = \; < \{v_{i,j} : v_i \in V(H) \wedge v_j \in V(G)\},$$

$$\{v_{i,j} v_{h,k} : v_j v_k \in E(G) \vee (v_i v_h \in E(H) \wedge j = k)\} > .$$

Definition 1.70 *The direct product of two digraphs*

$$\vec{G} \times \vec{H} = \; < \{v_{i,j} : v_i \in V(\vec{H}) \wedge v_j \in V(\vec{G})\},$$

$$\{v_{i,j} \vec{v}_{h,k} : v_i \vec{v}_h \in A(\vec{H}) \vee v_j \vec{v}_k \in A(\vec{G})\} > .$$

2. Irregular Labelings

Giving a brief overview of the history involved will motivate the subject matter of graph irregularity strength. We also introduce a core definition, $s(G)$, the irregularity strength of a graph G. Some upper bounds and existence theorems for the parameter $s(G)$ are given in section 2.5.

2.1 History

While graph theory seems to have originated in several different geographic areas at varying time-periods both because of and apart from earlier developments, the study of irregular graphs and digraphs can be traced directly to a seminal paper in the 1980s by Chartrand, Jacobson, Lehel, Oellerman, Ruiz, and Saba paper [9]. Notice that the first paper published on the subject matter, authored by Gyárfás, [27], was actually the second submitted paper on the topic of irregularity strength of graphs. While many authors had been aware of the fact that every simple graph has two vertices of the same degree, it was not until this time period that the concept of edge labels were used to distinguish the weighted degrees of a graph.

We proceed to define some terms such as an irregular labeling and the irregularity strength of a graph. The terms are analogous to the topics of an irregular labeling of a digraph and the irregularity strength of a digraph.

2.2 The Motivation behind Irregular Labelings

Notice that the degree sequence of a graph does not always give all the information possible about a graph. While some parameters of a graph are always given by the degree sequence (such as Δ and δ), these parameters are not sufficient to characterize the graph. Notice also that if S is the set of positive integer degrees which are elements of the degree sequence, there is only one example where the cardinality of $S = |V(G)|$; the only such example is the isolated vertex K_1. This graph has the degree sequence 0 and the degree set $\{0\}$. We have in this case that $|S_G| = |V(G)| = 1$. Except for this case there does not exist an injection from the vertex set of a simple graph into the integers. This follows from the following argument:

Given a simple graph G of order $n \geq 2$, there are n possible degrees for the vertices of the graph G, namely $0, 1, 2, ..., n-1$. However, if the degree function is an injection of the vertex set of G, which has cardinality n, it must also be a surjection. (It is easy to show that any injection from a finite set to a finite set of equal size is also a surjection). Thus, we get a contradiction because if a graph has order n and a degree of order $n-1$, that graph cannot also have a vertex of degree 0.

Theorem 2.1 ([10]) *No non-trivial simple graph has all vertices of distinct degree.*

It is clear by example that the previous theorem does not hold for multi-graphs whether or not we allow loops. We will show that for all simple graphs with the exception of simple graphs containing $2K_1$ or K_2 as components, there exists a map from the edge set to an multi-edge set $< E(G), f(e) >$, where $f : E(G) \longrightarrow Z^+$ such that the multi-graph which is the image of our graph under f has all vertices of distinct multi-degree. By an irregular graph we mean either K_1, or multi-graph whose vertices all have distinct degrees. Irregular labelings help demonstrate what an irregular multi-graph is with examples from graph theory.

Consider an edge labeling of a graph G $f : E(G) \to Z^+$. We define a vertex weighting induced by our edge labeling to be

$$g : V(G) \to Z^+$$

where

$$g(v) = \sum_{vx \in E(G)} f(vx).$$

Definition 2.2 ([9]) *If our vertex weighting is such that each vertex has a distinct weight we say our edge labeling is irregular.*

Irregular labelings are a well-researched topic. Changing the context slightly expands the topic in a new way which enhances the pre-existing research.

Now to show our previous claim, suppose that G has no trivial components of the form $2K_1$ or K_2. Then it is clear that any map f which is injective into the subset of Z^+ which contains only powers of 2 is an irregular labeling of G. The vertex sums are sums of binary powers and are all distinct unless two vertices are incident with all the same edges or two vertices are incident with no edges. Notice the map f can be *any* injection from $E(G)$ into $\{x : x = 2^k,\ k \in Z^+\}$. With this existence theorem clear we move on to consider irregular labelings of simple graphs G such that the largest value $k \in f(E(G))$ is relatively small compared to $2^{|E(G)|}$ where f is an irregular edge labeling.

Theorem 2.3 ([9]) *The irregularity strength of all connected graphs of order greater than or equal to 3 is defined.*

Proof: Let G be a connected graph of order ≥ 3. Let $E(G) = \{e_1, e_2, ..., e_q\}$. Assign the weight 2^{i-1} to the edge e_i for $1 \leq i \leq q$. Each vertex then is assigned a weight equal to the sum of distinct binary powers. Thus the weights are distinct because no two vertices are incident the same edges. ■

To see that the answer is not always of the correct order of magnitude, consider a class as broad as the connected graphs other than K_3. In these cases, one upper bound on $s(G)$ is $2^{|E(G)|}$, but there is a second upper bound which bounds the irregularity strength of this class of graphs above by $\leq n-1$, where n is the order of our connected graph.

Definition 2.4 ([9]) *The set of irregular labelings of G is denoted $I(G)$.*

We will in fact determine in some cases the smallest value of $k \in f(E(G))$ where f is an irregular labeling on the edge set of G across all $f \in I(G)$. We can now define the irregularity strength of a graph G.

Definition 2.5 ([9]) *The irregularity strength of G, denoted $s(G)$, is defined by*

$$s(G) = \min_{f \in I(G)} \max_{e \in E(G)} f(e).$$

The irregularity strength of a graph G gives the smallest arc multiplicity of an irregular multigraph M such that G underlies M. Now that we have shown that the irregularity strength of G is defined in a variety of examples, we can turn to the problem of determining the irregularity strength of a graph G. The determination of the irregularity strength of a graph G generally has two parts: first, an irregular labeling of the graph is exhibited, and second, we demonstrate

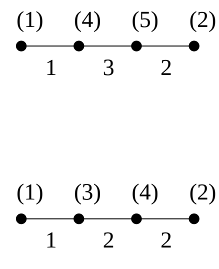


Figure 2.1: There is no unique irregular labeling of P_4.

that the largest label used in the given irregular labeling is in fact the minimum maximum labeling across all irregular labelings of G.

2.3 Lower Bounds

It is not clear in general how to determine $s(G)$. In many cases the following parameter $\lambda(G)$ is within an additive constant of $s(G)$ and in fact, in many of the cases, this constant $= 0$. One of the nice features of this lower bound, which is completely determined up to the degree sequence, is that proving $s(G) = \lambda(G)$ amounts to constructing an irregular labeling that achieves the lower bound.

Theorem 2.6 ([9]) *If G is connected with the order of $G \geq 2$ then let p_i represent the number of vertices of degree i in G. Then $s(G) \geq \frac{p_i+i-1}{i}$.*

Proof: Let ℓ be the number of labels required to label G. Then p_i, the number of vertices of degree i has $p_i \leq i\ell - i + 1$. So then $\ell \geq \frac{p_i-1}{i} + 1$. Thus $s(G) \geq \frac{p_i-1}{i} + 1 = \frac{p_i+i-1}{i}$. ■

The preceding theorem motivates the next theorem; we include both theorems to motivate a general lower bound in the study of graph irregularity strength.

Theorem 2.7 ([9]) *If G is of order ≥ 3 and p_i and p_j represent the vertices having degree i and j respectively $i \leq j$ in G, then*

$$s(G) \geq \left(\frac{(\sum_{k=i}^{j} p_k) + i - 1}{j} \right).$$

Proof: Let ℓ be the number of labels required to label G. Then $\sum_{k=i}^{j} p_k$, the number of vertices of degrees between i and j has $\sum_{k=i}^{j} p_k \leq j\ell - i + 1$. Thus $s(G) \geq \left(\frac{(\sum_{k=i}^{j} p_k)+i-1}{j} \right)$. ∎

Definition 2.8 *For a graph G with p_i vertices of degree i, let*

$$\lambda(G) = max_{0 \leq i,j \leq n} \left\{ \left(\frac{(\sum_{k=i}^{j} p_k) + i - 1}{j} \right) : i \leq j \right\}.$$

Notice that $\lambda(G)$ is a lower bound for $s(G)$.

These results form what was at one time a rough bound which later evolved into a function with a name and finally the standard lower bound. The thrust of the subject is just beginning to turn towards estimations of upper bounds and away from exploring the problems with the $\lambda(G)$ function.

2.4 Survey

There has been a great deal of collaboration in the subject of irregularity strength and it is rare that authors tackle the problem on their own. Unless there is a complete characterization of the values of $s(G)$ for all simple graphs G there will be a wealth of families of graphs whose irregularity strength can be partially or completely determined in the future. Because in many cases the irregular labelings of a graph are not unique, especially when the labeling allows labels greater than $s(G)$, finding an irregular labeling can be done in a relatively short amount of time.

The irregularity strength of some classes of graphs are known.

$$s(K_n) = 3 \quad n \geq 3.$$

$$s(P_n) = \begin{cases} \frac{n}{2} & \text{for } n \equiv \ 0 \bmod \ 4, \\ \\ \frac{n+1}{2} & \text{for } n \equiv 1, 3 \bmod \ 4, \\ \\ \frac{n+2}{2} & \text{for } n \equiv 2 \bmod \ 4; \end{cases}$$

$$s(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for } \ n \equiv 1 \bmod \ 4, \\ \\ \lceil \frac{n}{2} \rceil \ + \ 1 & \text{otherwise.} \end{cases}$$

There are a number of published papers on the topic of irregularity strength of graphs. The general strategy in the study of irregularity strength has been to pick a class of graphs, calculate λ and then produce a construction that achieves λ. There has been a movement toward calculating general bounds or finding a property which completely determines irregularity strength for a family of graphs. A table of results, Table 2.1, follows.

Most of these results are difficult to obtain. The constructions which show $s(G) = \lambda(G)$ are involved. But notice that one of the published results [32] (on tP_3) demonstrate that the lower bound of $\lambda(G)$ cannot be achieved using counting arguments that depend on the structure of the family of graphs being considered.

2.5 General Results and Bounds

Finding a general upper bound for the strength of all graphs G is difficult and depends in part on using spanning trees and the congruence method. Notice there is a simple example to which the general upper bound of $n-1$ for connected graphs of order n does not apply. This example (K_3) is unique and the irregularity strength achieves a value of $n = 3$.

Theorem 2.9 (Nierhoff [35], Aigner and Triesch [1]) *If the irregularity strength of a connected graph G is defined it obeys $s(G) \leq n-1$ unless $G = K_3$.*

Most of the upper bounds which have been determined for special classes of graphs are based on the minimum and maximum degree. The proofs rely on two techniques, either using the congruence method to find special spanning graphs and then determine a labeling of this spanning graph explicitly or through the use of the probabilistic method [19].

Theorem 2.10 (Frieze, Gould, Karonski, Pfender, [19]) *Let G be a graph with no isolated vertices or edges*

- If $\Delta \leq \lfloor (n/ln\ n)^{1/4} \rfloor,$ then $s(G) \leq 7n(\frac{1}{\delta} + \frac{1}{\Delta}),$
- If $\lfloor (n/ln\ n)^{1/4} \rfloor \leq \Delta \leq \lfloor n^{1/2} \rfloor,$ then $s(G) \leq 60n/\delta,$
- If $\Delta \geq \lfloor n^{1/2} \rfloor + 1,$ and $\delta \geq \lceil 6\ ln\ n \rceil,$ then $s(G) \leq 336(ln\ n)n/\delta.$

The following result is simpler to state and completes a program for bounding more sharply all graphs given Δ and δ.

Theorem 2.11 (FGKP [19]) *Let G be a graph with no isolated vertices or edges. If n is sufficiently large, then $s(G) \leq 14n/\delta^{1/2}$.*

Finally, we state a result of Ebert, Hemmeter, Lazebnik and Woldar [14]. As the number of available labels increases, the number of irregular labelings increases slower than a polynomial in the number of available labels. The polynomial has a leading exponent that is fixed two less than the cardinality of the edge set. The theorem is more formally stated in Theorem 2.13.

Definition 2.12 *Let $Irr(G, \zeta)$ be the number of irregular labelings of a graph when ζ labels are available.*

Theorem 2.13 (Ebert, Hemmeter, Lazebnik, and Woldar [14]) *$Irr(G, \zeta) - \zeta^q + c_1\zeta^{q-1} = O(\zeta^{q-2})$ as $\zeta \rightarrow \infty$ where $q = |E(G)|$.*

Authors	Graph	Strength	Year
Chartrand, Jacobson, Lehel, Oellerman, Ruiz, Saba [9]	K_n	3	1986
Gyárfás and CJLORS [27],[9]	$K_{m,m}$	3 m even 4 m odd	1988
Faudree, Jacobson Lehel and Shelp [16]	$K_{m(k)}$ $m \neq 2$	3	1989
Faudree, Jacobson Kinch and Lehel [15]	tK_p $p = 4$ $p \geq 5$	$\lceil \frac{4t+2}{3} \rceil$ $\lceil \frac{pt+p-1}{p-1} \rceil$	1991
Gyárfás [30]	$K_n - mK_2$	2 or 3 2 only if n = 4m or 4m+1, 4m-1	1989
Ebert, Hemmeter Lazebnik and Woldar [14]	$P_2 \times P_n$	$\lceil \lambda(G) + 1 \rceil$ n=1 mod 6 $\lceil \lambda(G) \rceil$ otherwise	1990
Ebert, Hemmeter Lazebnik and Woldar [14]	Wheel n= 4 or 5 $n \geq 6$	 $\lceil \lambda(G) + 1 \rceil$ $\lceil \lambda(G) \rceil$	1990
Ebert, Hemmeter Lazebnik and Woldar [14]	Q_k	$\lceil \lambda(G) \rceil$	1990
Togni [34]	$C_m \times C_n$	$\lceil (mn+3)/4 \rceil$	1997
Amar, Togni [2]	Tree with no vertices degree 2	number of pendant vertices	1998

Table 2.1: A summary of progress in the study of graph irregularity strength

3. Introduction to Digraph Irregularity Strength

In [30], Hackett extends the notion of irregularity strength to digraphs. We accomplish this extension in a different fashion, but some results overlap. Meanwhile in [12] a specific type of irregular digraphs is characterized. These irregular digraphs have irregularity strength 1 according to our definitions.

3.1 Irregular Labelings and the Main Definition

Consider an arc weighting of a digraph D $f : A(D) \to Z^+$. Define a vertex labeling induced by our arc weighting to be

$$g : V(D) \to (Z^+ \cup \{0\}) \times (Z^+ \cup \{0\}).$$

where

$$g(v) = (\sum_{vx \in A(D)} f(vx), \sum_{xv \in A(D)} f(xv)).$$

If the arc labeling is such that every vertex label is distinct, then the labeling is said to be irregular [18]. Let $I(D)$ denote the set of irregular labelings of a digraph D [18].

3.1.1 The parameter $\vec{s}$

We will define $\vec{s}(\vec{D})$, the irregularity strength of $\vec{D}$ in an analagous manner to that of graph irregularity strength and refer to an irregular labeling of $\vec{D}$ with s arc labels available as an irregular s-labeling. Furthermore, we may use the term *irregular s-weighting* of the digraph referring to the weighted vertex degree pairs if the context is perfectly clear and we want to emphasize some point about the vertex weights.

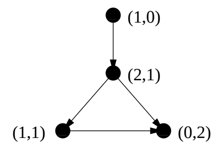


Figure 3.1: The paw has an irregular orientation.

Definition 3.1 ([18]) *The irregularity strength of a digraph D is defined as*

$$\vec{s}(D) = \min_{f \in I(D)} \max_{e \in A(D)} f(e).$$

Unlike graphs, there are digraphs D with $\vec{s}(D) = 1$, such as the transitive tournaments, a directed P_3, and the orientation of the paw pictured in Figure 3.1.

3.1.2 Graphs and Irregular Orientations

Notice that not every orientation of the paw is irregular. This is the case with many graphs. As we shall see here when we discuss the property of being an irregular digraph, there exist graphs having some, but not all orientations irregular.

3.2 Irregular Digraphs

In [12], Dziechcińska-Halamoda, Majcher, Michael and Skupień determine all the irregular digraphs on n vertices which have the minimum number of arcs possible. While there is no program for determining all the digraphs which are irregular given a specific order of the digraph in [12], there are a number of valuable ideas for transforming an irregular digraph with the minimum number of arcs to an irregular digraph with a greater number of arcs using arc swaps and arc additions. In particular, we explore the idea of cataloguing digraphs of a given irregularity strength in Section 3.2.3 with reference to Berge's Theorem.

3.2.1 The Seminal Idea for Arc Minimal Digraphs

Definition 3.2 *For a positive integer n, the nonnegative integers τ_n and m_n are defined such that $\tau_1 = 0$ and $m_1 = 1$ and*

$$n = 1 + 2 + ... + \tau_n + m_n, \quad 1 \leq m_n \leq \tau_n + 1$$

and so

$$\tau_n = \lfloor \sqrt{2n} - \frac{1}{2} \rfloor$$
$$m_n = n - \frac{1}{2}\tau_n(\tau_n + 1).$$

The digraphs D, which are irregular and have minimum arc sum can be characterized in terms of their degree sequence.

For a given positive integer n, denote by π_n^{min} the class of all sets π of ordered pairs of non-negative integers such that the following three conditions hold: (1) π is an n-element set; (2) the sum of the terms of π is (0,0); (3) the absolute values of the arguments of the terms of π have the minimum sum among all π satisying (1) and (2).

By $B_s[k]$, where $0 \leq k \leq s+1$, we denote a k-element subset of the set B_s.

Theorem 3.3 *Let n be a positive integer and let τ_n and m_n be given as above. For $n \geq 2$, $\pi \in \pi_n^{min}$ if and only if π has one of the following four forms:*

i. $\pi = \{(0,0)\}$ is the only member of π_n^{min} for n=1, $\pi_1^{min} = \{B_0\}$,

ii. $\pi = \bigcup_{i=0}^{\tau_n - 1} B_i \cup B_{\tau_n}[m_n]$,where the set $B_{\tau_n}[m_n]$ is balanced and m_n is even or τ_n is even,

iii. $\pi = \bigcup_{i=0}^{\tau_n - 1} B_i - \{(\alpha,\beta)\} \cup B_{\tau_n}[m_n + 1]$, where $(\alpha,\beta) \in B_{\tau_n - 1}$, the set $B_{\tau_n}[m_n + 1] \cup \{\alpha, \beta\}$ is balanced and both numbers τ_n and m_n are odd,

iv. $\pi = \bigcup_{i=0}^{\tau_n - 1} B_i \cup B_{\tau_n}[m_n - 1] \cup \{(\alpha,\beta)\}$ where $(\alpha,\beta) \in B_{\tau_n + 1}$, the set $B_{\tau_n}[m_n - 1] \cup \{(\alpha,\beta)\}$ is balanced and both numbers τ_n and m_n are odd.

Theorem 3.4 *Let n be a positive integer such that τ_n and m_n are both odd. Then a set π described in case (iii) of the previous theorem exists for any pair $(a,b) \in B_{\tau_n - 1}$ if $m_n < \tau_n$, otherwise $m_n = \tau_n$ and then π exists for $a = b = \frac{1}{2}(\tau_n - 1)$ only. Moreover, a set π described in case (iv) exists for any $(a,b) \in B_{\tau_n + 1}$ if $m_n \neq 1$, otherwise π exists for $a = b = \frac{1}{2}(\tau_n + 1)$ only.*

Theorem 3.5 *The minimum size of a digraph D of order n with $\vec{s}(D) = 1$ is equal to ϵ_n where*

$$2\epsilon_n = \begin{cases} \tau_n(n - \frac{1}{6}(\tau_n + 1)(\tau_n + 2)) + 1 & \text{if } \tau_n \text{ and } m_n \text{ are odd,} \\ \\ \tau_n(n - \frac{1}{6}(\tau_n + 1)(\tau_n + 2)) & \text{otherwise.} \end{cases}$$

The proof of Theorem 3.5 relies on the proof of Theorem 3.3. In [12], the authors give a construction and sketch the proof of Theorem 3.3. Some of the details are left to the reader. We have included a proof of Theorem 3.3 in Appendix B.

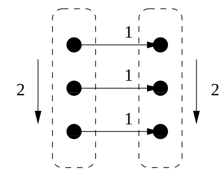


Figure 3.2: Realizing Arc Minimal Digraphs of Strength 2

3.2.2 Arc Minimal Digraphs with Arc Multiplicity 2

We use the same notation from Section 3.2.1. For this section, we use the notation $\vec{D}^2$ to describe the digraph whose adjacency matrix $= 2M$ if M is the adjacency matrix of $\vec{D}$; this is not to be confused with the square of a digraph, despite the identical notation. Furthermore, if there is a double arc $v_i\vec{v}_j$ write it as $2v_i\vec{v}_j$.

Example 1 We notice that the arc minimal digraphs with degree sequences of the form $\bigcup_{p=1}^{p=\tau_n} B_p = \{(a,b) : a+b = p,\ a,b \in N_0\}$ can always be realized in the following way:

If $p = 2k$ is even, then we can realize the sequence B_p as $T_k^2 \times \vec{K}_2$.

We can see this is arc minimal in the following way. Each vertex has at least one odd element in its degree pair since the two elements sum to an odd total. So there is at least one arc of multiplicity 1 incident each node in the realization. The fewest number of nodes thus occurs if every other arc has multiplicity 2. This is exactly the case above. Notice that there exists a realization of a digraphic sequence with t odd out-terms with just t arcs labeled with a 1. If a pair of vertices both have two arcs labeled 1 incident from each of them respectively, the intersection of their 1-labeled out-neighbors has cardinality either 0,1, or 2. In the case that the cardinality is 2, replace $v\vec{x}_1$, $v\vec{x}_2$, $u\vec{x}_1$, $u\vec{x}_2$ with a double arc from v to x_1 and a double arc from u to x_2. Similarly, given cardinality 1, we can replace $v\vec{x}_1$, $v\vec{x}_2$, $u\vec{x}_1$, $u\vec{x}_3$ with a double arc from v to x_1, an arc from u to x_2, and an arc from u to x_3. If every pair of vertices that are incident to an arc labeled 1 have disjoint out-neighborhoods, then there are exactly t arcs that are 1-labeled and t odd out-degrees.

If p is odd, the following digraph realizes B_p. Let $V(D) = \{v_1, ...v_{2k+1}\}$ where here $2k+1 = p$. Then let

$$\begin{aligned} A(D) = &\{2v_iv_j : 1 \leq i < j \leq k; j \neq i+1\} \\ &\cup \{v_iv_{i+1} : 1 \leq i \leq k-1\} \\ &\cup \{2v_iv_j : k+1 \leq i < j \leq 2k\} \\ &\cup \{2v_{2k+1}v_i : 2 \leq i \leq k-1\} \\ &\cup \{2v_iv_{i+k} : 1 \leq i \leq k\} \\ &\cup \{v_{2k+1}v_1, v_{2k+1}v_k\}. \end{aligned}$$

It is clear there is no realization of B_p with fewer arcs in this case. There are exactly $\frac{p+3}{2}$ arcs labeled 1.

If $B_p[m_n]$ has m_n non-zero, we cannot find a general construction for the case when $m_n \neq 0$.

We have just determined the minimal number of arcs in digraphs of strength 2 for cases when the number of vertices in the digraph has the form $\binom{p}{2}$ and constructed realizations of one instance of such digraphs in the case when p is odd and p is even. In the case where p is odd the minimum number of arcs is $\frac{\epsilon_p}{2} + \frac{p-1}{2} + 2$. In the case where p is even the minimum number of arcs is $\frac{\epsilon_p}{2} + \frac{p}{2}$.

It does not appear that we will be able to find a closed form (with proof) for the number of arcs in the arc minimal underlying digraph of an irregular super-digraph of order n, if we allow arc multiplicity m in the irregular super-digraph.

3.2.3 Berge's Theorem

Here we restate Berge's Theorem.

Theorem 1.60 *Let* $(r_1, s_1), (r_2, s_2), , , (r_n, s_n)$ *be pairs of integers with*

$$s_n \ \le s_{n-1} \ \le \ ... \ \le s_1.$$

The pairs $\{(r_k, s_k)\}_{k=1}^{k=n}$ *constitute the degree pairs of a digraph with maximum arc multiplicity* m *if and only if*

$$\sum_{j=1}^{j=k} r_j \le \sum_{i=1}^{i=n} \ \min\{s_i, mk\} \tag{3.1}$$

$$\sum_{i=1}^{i=n} s_i = \sum_{i=1}^{i=n} r_i. \tag{3.2}$$

Now consider all irregular sequences of vertex degree pairs of a given order and then use Berge's theorem to test and construct the graphic sequences. Once a single realization has been constructed we use capacitated arc-swaps to characterize the s-irregular digraphs fully. Despite the exhaustive nature of the program, note that if there exists a realization of an irregular sequence of degree pairs, there is an exact count on the number of irregular multi-digraphs which realize a given irregular di-sequence. There is thus an upper bound on the number of simple digraphs which realize an irregular di-sequence.

Theorem 3.6 *Given a realization of an irregular di-sequence the number of realizations of that di-sequence (with multiple arcs and loops allowed) is given* *by*

$$\frac{|A(D)|!}{\Pi_{v \in \vec{D}} deg^+ v! deg^- v!}.$$

Proof: This equality is proved by induction on the number of vertices and arcs. Clearly vertex-addition of v leaves the product fixed. Now add an arc a incident the new vertex v. By induction the number of arc-labeled realizations is $|A|!$ because we can leave the new arc fixed or swap the endpoint of a not incident with v with the corresponding endpoint of any of the $(|A|-1)$ arcs in a double arc swap for each of the $(|A|-1)!$ realizations. But then to reduce to the number of realizations up to arc-isomorphism divide by the order of the subgroup of the arc permutation group that fixes the number of arcs in each in-neighborhood and out-neighborhood. ■

3.3 Relationship with Graph Irregularity Strength

3.3.1 Nearly-Irregular Graphs

Definition 3.7 *The maximally near-irregular graph of order* n *is the graph with degree sequence* $1, 2, 3, ..., ((n-1)/2)^2, (n+1)/2, ..., (n-1)$ *when* n *is odd and* $1, 2, 3, ..., (n/2)^2, ((n+2)/2), ..., (n-1)$ *when* n *is even.*

Theorem 3.8 *The graph irregularity strength of the maximally near-irregular graph of order* n *is 2 for all* $3 \leq n$*.*

Proof: Suppose we have the maximally near-irregular graph G on n vertices. Then every vertex except 2 have different degrees. So the irregularity strength of G is greater than 1. But we can easily label the edges of G with two labels so that the corresponding multigraph N has a non-repeating degree sequence. In the case that n is even, label G with 2's and the change edge labels on the edge v_iv_{i+1} from 2 to 1 where $i \geq (n/2) + 1$ and i is even. In the case that n is odd label the edges with 1 and then change the edges v_iv_{i+1} from 1 to 2 where and $i \geq \lfloor n/2 \rfloor + 1$ and i is even. ■

Example 3.9 *For all such maximally near-irregular graphs, rank the vertices by degree, arbitrarily choosing which of the two vertices of the same degree comes first in the ranking. Then direct all arcs from the higher ranked vertex to the lower ranked vertex. We get an irregular orientation for the maximally near-irregular graph of each order in this way.*

Theorem 3.10 *Graph strength of underlying graph is always greater than or equal to digraph irregularity strength.*

Example 3.11 *Consider the Hackett graph $K_n - mK_2$ where $m = \sqrt{n}$ with no irregular orientation. Because the strength of the underlying graph is 2 [30], the strength of any orientation is ≤ 2. Therefore, this Hackett graph is fair (see Section 3.3.3).*

3.3.2 More Parameters

Definition 3.12 *The disparity of a digraph G,*

$$disp(G) = \max_{D_1 \cup D_2 \in O(G)} |\vec{s}(D_1) - \vec{s}(D_2)|$$

where O(G) is the set of orientations of a graph G.

It is not very surprising to find that the disparity of some classes of graphs go to infinity as the order of that class goes to infinity. However, in many cases the function that bounds the disparity from above for a class of graphs is less than linear as a function of order. A table of lower bounds of disparity for various graphs and classes of graphs follows (in Chapter 7) and determines the disparity for some graphs.

Definition 3.13 *We say an irregular labeling of a digraph with* mn *vertices is consecutive or sequential if the labels on the vertices form a rectangular lattice in the space* $R \times R$*.*

Definition 3.14 *We say an irregular labeling is perfectly irregular if it is consecutive and it is an irregular* $\vec{s}$*-labeling.*

Definition 3.15 *The value* s^H *is the minimum maximum value across all irregular labelings such that an irregular labeling is defined by an arc labeling which leaves the differences of in-label and out-label sums distinct.*

3.3.3 Fairness and Cost

Definition 3.16 *We define the irregularity cost of a digraph to be the minimum area of a square in the xy plane that includes the coordinates of every vertex weight of an irregularly labeled digraph.*

Definition 3.17 *A graph* G *is fair if* $disp(G) = 0$*.*

Definition 3.18 *A graph* G *is totally fair if* $\vec{s}(D) = 1$ *over all orientations* D *of* G*.*

Now, in fact no simple graph except K_1 and K_2 are such that every orientation is irregular. This fact amounts to the following Theorem.

Theorem 3.19 *No graphs are totally fair except K_1 and K_2.*

Proof: We will show every graph except K_2 and K_1 has an orientation with two vertices having the same outdegree, indegree pair.

Notice every graph has two vertices of the same degree with $|N(v) - [N(w)]| > 1$ unless $G \cong K_1$ or K_2.

So list the nodes of G left to right. If $v \equiv w$ exactly one neighbor of $v \in \{N(v) - N[w]\}$ is the unique vertex that follows v and w. If $v \not\equiv w$ list the nodes of G left to right , ..., w,v so that w and v are last. Now orient the arcs of G left to right.

That is, if z follows x in the queue orient the arc from x to z: $\vec{xz}$. ■

Notice that a necessary condition for a digraph to be irregular is that there are at most $k+1$ vertices of degree k in the underlying graph. Furthermore, it is easy to see that if a sequence with exactly $k+1$ vertices of degree k for all k has a realization as a union of cliques which can be oriented so that $s(D) = 1$ in the resulting digraph D. Now, we can see there are no irregular orientations of any bicliques except $K_{1,2}$ and $K_{1,1}$. In fact, the only irregular oriented trees are paths.

4. The Parameter $\vec{\lambda}$

The parameter $\vec{\lambda}$ is a useful parameter in establishing the irregularity strength of a digraph. Knowing $\vec{\lambda}$ can prevent attempts to irregularly label a digraph in a fashion that is clearly impossible (with too few labels available).

4.1 The Parameter $\vec{\lambda}$

Theorem 4.1 ([18]) *Let D be a digraph and let $U \subseteq V(D)$ be such that for all x in U, $i_1 \leq d^+(x) \leq i_2$ and $j_1 \leq d^-(x) \leq j_2$. Then*

$$\vec{s}(D) \geq \max_{U \subset V(D)} \{s : q_U(s) = 0\}$$

where $q_U(s) = (si_2 - i_1 + 1)(sj_2 - j_1 + 1) - |U|$.

Proof: Let D be a digraph with irregularity strength s. Then the degree of every vertex x in U must have $i_1 \leq d^+(x) \leq i_2$ and $j_1 \leq d^-(x) \leq j_2$. It follows that every vertex in U must have its weighted out-degree (in-degree) between i_1 and si_2 (j_1 and sj_2). We necessarily have that for all $U \subseteq V(D)$, $(si_2 - i_1 + 1)(sj_2 - j_1 + 1) \geq |U|$. Because this is the case for all such subsets $U \subseteq V(D)$, the theorem follows as stated above. ■

Definition 4.2 ([18]) *Define $\vec{\lambda}$ to be the largest zero of q_U for a given graph.*

The following corollary demonstrates the utility of this theorem.

Corollary 4.3 ([18]) *Let D be an arbitrary r-regular digraph of order $|D| = n$. Then $(\vec{s}r - r + 1)^2 \geq n$. Consequently, $\vec{s} \geq \lceil \frac{\sqrt{n}-1}{r} + 1 \rceil$.*

Corollary 4.4 ([18]) *Let D be an arbitrary digraph with k vertices of degree (1,1). Then $\vec{s} \geq \lceil \sqrt{k} \rceil$.*

Proof: Let s be the irregularity strength of D. Apply Theorem 4.1 with $U =$ the set of vertices of degree (1,1). We get $s^2 \geq k$ and so $\vec{s} \geq \lceil \sqrt{k} \rceil$. ∎

Corollary 4.5 ([18]) *Let D be an arbitrary union of directed cycles with size n; then $\vec{s}(D) \geq \lceil \sqrt{n} \rceil$. Let D be a union of t directed paths with orders k_1, k_2,...,k_t; then $\vec{s}(D) \geq \max\{t, \lceil \sqrt{\sum_{i=1}^{i=t}(k_i - 2)} \rceil\}$.*

Proof: For $\bigcup \vec{C}_{k_i}$, apply the previous corollary with $U = V(D)$. For $\bigcup \vec{P}_{k_i}$, consider three different applications of Theorem 4.1. Let U be the set of vertices of out-degree 0, in-degree 0, and degree (1,1), respectively. Since there are t vertices of out-degree 0, t vertices of in-degree 0, and $\sum_{i=1}^{i=t}(k_i - 2)$ vertices of degree (1,1), the corollary follows. ∎

The parameter $\vec{\lambda}$ is not accurate up to a constant in all cases. There is at least one family of disconnected graphs, $\{G_n\}_{i=1}^{i=n}$, such that there does not exist $c \in Z^+$ that gives $\vec{s}(G_n) < \vec{\lambda}(G_n) + c$ for all $n \in Z^+$. (See Chapter 8 for an example of such a family G_n.)

4.2 Minimizing and Maximizing $\vec{\lambda}$

The form of the polynomial that determines $\vec{\lambda}$ suggests that regular and near-regular orientations of graphs G will yield the smallest values of $\vec{\lambda}$. Given a polynomial of the form $ax^2 + bx + c = |U|$, the polynomial will have minimal values when it factors if the two factors are nearly equal to one another. That is if $xy = |U|$, minimizing $\max\{x, y\}$, is accomplished when $x = cy$ has $|c - 1| \leq \epsilon$ for the smallest ϵ possible.

Maximizing $\vec{\lambda}$ is usually accomplished by finding an orientation of a given graph with a large vertex set of sources or sinks, or a vertex set with large in-degree and small out-degree or vice versa. To see this, consider the equality $(\Delta_1\vec{\lambda} - \delta_1 + 1) \times (\Delta_2\vec{\lambda} - \delta_2 + 1) = |U|$. The largest root is maximized when one factor in $xy = |U|$ is much smaller than the other. Because the sum of the in-degrees equals the sum of the out-degrees, we get extremal examples for connected graphs like the out-directed star. All this said, given a path with several vertices of 0 in- or 0 out-degree, and many vertices of degree (1,1), it is possible that the vertex set that includes all the vertices of degree (1,1) produces $\vec{\lambda}$.

4.3 Discussion

There are situations where regular orientations of a graph G do not yield the smallest value of $\vec{\lambda}$ over all orientations of G. Consider, for instance a transitive tournament. While the transitive tournament is the only irregular orientation of the tournament and has irregularity strength equal to 1, a regular orientation of a tournament has $\vec{\lambda} = \vec{s} = 2$.

Example 4.6 *Consider the transitive tournament $\vec{T}_{2n+1}$. Consider also a regular tournament on $2n+1$ vertices $\vec{D}_{2n+1}$. Clearly $\vec{s}(\vec{T}_{2n+1}) = 1$ while $\vec{s}(\vec{D}_{2n+1}) = 2$. But notice $\lambda(k\vec{T}_{2n+1}) = k$ while $\lambda(k\vec{D}_{2n+1}) = \lceil \frac{\sqrt{kn}+k-1}{k} \rceil$ and we have $\lceil \frac{\sqrt{kn}+k-1}{k} \rceil \leq k$ as long as $n \leq k^3 - 2k^2 + k$.*

Also, given a graph that already has a near-irregular degree sequence, there may exist an orientation of strength 1 if we allow sources and sinks, but no such irregular orientation if we require every vertex has at least 1 in- and 1 out-degree. We consider irregular orientations of 2-irregular graphs in Chapter 9. Given two copies of an irregular graph, we can often show the irregularity strength is 2 by simply multiplying the arc multiplicity of one of the copies by 2 on every arc. In general, irregular labelings of digraphs is the study of spreading out clumps in the degree sequence so that this spreading process does not cause the clumps to then overlap. The parameter $\vec{\lambda}$ gives us a best-case scenario of how this might occur.

Example 4.7 *Given a regular digraph whose underlying graph is 2 colorable, we can often take two copies of the underlying graph orient them differently to form* $D_1 \cup D_2$ *and get* $\vec{s}(D_1 \cup D_2) = \max\{\vec{(}s(D_1), \vec{s}(D_2)\}$. *We give three examples:*

- $\vec{D_1} \cup \vec{D_2}$
  ($\vec{D_1}$ a di-regular $K_{2n,2n}$, $\vec{D_2}$ a $K_{2n,2n}$ where one partite set beats the other)

- $\vec{D_1} \cup \vec{D_2} = \vec{C}_{2n} \cup \vec{G_{2n}}$
  ($\vec{G_{2n}}$ is the anticycle on $2n$ vertices)

- $\vec{D_1} \cup \vec{D_2} = \vec{C}_4 \times \vec{C}_4 \cup \vec{G}$
  ($\vec{G} \equiv C_4 \times C_4$ with an orientation where 1 partite set beats the other).

Consider a final example of this degree-sequence clumping. Take a $\vec{C}_3$ and expand each of the vertices to sets of orders $k, 2k, 4k$ respectively. This gives an orientation of the complete 3-partite graph $K_{k,2k,4k}$. Here, $\lambda(\vec{D})$ is 2. This may not be immediately clear, but given two labels we can expand the three partite sets degree sets so that they do not overlap and in fact do not collide.

One of the shortcomings of the paramater $\vec{\lambda}$ is that we may have to check many degree sets to determine $\vec{\lambda}$. However, we can see that $\vec{\lambda}$ is always bounded below by $\frac{\epsilon_n}{|A(\vec{D})|}$ where n is the order of $\vec{D}$. Here ϵ_n is as defined in Theorem 3.5.

4.4 Disparity

A sufficiently large strength for one orientation of a graph is not enough to guarantee $disp(G) > 0$. The obvious counterexample is the matching nK_2. However, given an out-directed star forest $\vec{D}$ whose roots are of sufficiently large degree, and thus $\vec{s}(\vec{D})$ is large, has $disp(D) > 0$ where D is the graph underlying $\vec{D}$. Similar claims seem difficult to prove; it is difficult to prove bounds on $\vec{s}_{max}$, and difficult to prove bounds on $\vec{s}_{min}$. In, fact we have few upper bounds on the strength of a digraph unless there is a sufficiently large δ. We can state if $\delta(G) > \frac{n}{2}$, then $\vec{s}(G) < \frac{n}{2} + 1$, but little else.

Given a set of vertices of cardinality $|V(G)|$, of maximum degree k, we know that at most $\frac{|V|}{k+1}$ of these vertices are independent. Using this fact we can design a chromatic-type orientation that gives us that $\vec{s}(\vec{G}) > \frac{|V|}{(k+1)(k)}$. If furthermore, we have a 3-regular graph with a Hamiltonian cycle (see Chapter 7), we can use the congruence method to conclude that $disp(G) \geq |\frac{|V(G)|}{12} - \sqrt{|V|}|$. Furthermore, we have the following theorem:

Theorem 4.8 *If G is k-regular and has a Hamiltonian cycle, then* $disp(G) \geq |\frac{|V(G)|}{(k+1)k} - \sqrt{|V|}|$.

5. The Complete Graph and Tournaments

In this chapter we outline a constructive result on the topic of digraph irregularity strength. For now, we focus on establishing constructive results which completely determine the irregularity strength of some families of digraphs. The tournament is in a sense, the family of digraphs whose irregularity strength is the easiest to determine among all families of digraphs which are completely general. In fact the labeling is analogous to the labeling for the graph K_n. In all cases except when T_n is transitive, the tournament T_n has irregularity strength 2 and in all cases except when the tournament is transitive (or when $n \leq 3$), the same labeling algorithm applies. In the exceptional transitive case, the digraph irregularity strength of our digraph $\vec{T_n}$ has $\vec{s}(\vec{T_n}) = 1$. The transitive tournaments are the only digraphs whose underlying graph is regular and which have irregularity strength 1.

5.1 $s^H(T_n)$: A different definition for irregularity strength of digraphs

The next two definitions will be helpful in Chapters 5 and 6.

Definition 5.1 ([30]) *Define the deficiency of a vertex v to be* $\text{def}(v) = deg^+(v) - deg^-(v)$. *Then denote the difference in the weighted out-degree and the weighted in-degree relative to an arc labeling by* $\text{def}_f(v)$. *We will commonly write simply* $\text{def}(v)$ *relative to an arc labeling especially when we are using the* $\text{def}(v)$ *relative to an arc-labeling that will be altered later in a proof.*

Definition 5.2 ([30]) *We define an arc labeling f as before. Then a Hackett-irregular labeling of a digraph is an arc labeling such that the deficiency of every vertex is distinct. The Hackett-irregularity strength is denoted $s^H(D)$ and is the minimum maximum label used across all Hackett-irregular labelings of the digraph D.*

The number of arc-labels required to arc-label a given tournament so that the resulting multidigraph has every vertex of distinct deficiency is 2 [30]. Notice then that 2 is an upper bound for the irregularity strength under the definition we use.

5.2 Tournament Algorithm: $\vec{s}(T_n)$

In this section, we determine and prove the irregularity strength of every tournament T.

Theorem 5.3 ([18]) *The irregularity strength of a tournament T is 1 if T is transitive and 2 otherwise.*

Proof: Any transitive tournament is irregular and thus has irregularity strength 1. As such, we let T_n be a non-transitive tournament of order n and show that the irregularity strength of T_n is 2. The fact that $\vec{S}(T_n) \geq 2$ follows immediately from the fact that T_n is not irregular.

The only non-transitive tournament of order 3 is a directed cycle, which clearly has irregularity strength 2. Suppose therefore that $n \geq 4$. Since T_n is not transitive, there are distinct vertices z_1 and z_2 in $V(T_n)$ such that $deg(z_1) = deg(z_2)$. Let the remaining vertices of T_n, listed arbitrarily, be $z_3, \ldots, z_n$.

We now give an arc-labeling of T_n with the set $\{1, 2\}$. If i is even and $j \leq i$ let the arc between z_i and z_j be labeled with a 2. Otherwise, label the arc with a 1. Let $w(z_j)$ denote the sum of the in-weight and out-weight of z_j. Then

$$w(z_j) = \begin{cases} \frac{3n-j-2}{2} & j \text{ is odd and } n \text{ is odd,} \\ \\ \frac{3n+j-5}{2} & j \text{ is even and } n \text{ is odd,} \\ \\ \frac{3n-j-1}{2} & j \text{ is odd and } n \text{ is even,} \\ \\ \frac{3n+j-4}{2} & j \text{ is even and } n \text{ is even.} \end{cases}$$

Note that $w(z_i) \neq w(z_j)$ except in the case that $i = 1$ and $j = 2$. Suppose then that z_1 and z_2 do not have distinct vertex weights. Then $deg^+(z_1) = deg^+(z_2)$ and $deg^-(z_1) = deg^-(z_2)$. Represent the unweighted degree pairs of z_1 and z_2 by $(n-k, k)$ and $(n-k', k')$ respectively. Then the next equality follows $2(n-k-j)+j = 2(n-k'-j')+j'$ and the value on both sides of the equality is the weighted out-degree of z_1 and z_2 (where here j and j' are the number of vertices indexed by odd numbers which are dominated by z_1 and z_2 respectively). Next, the equality $2(k-h)+h = 2(k'-h')+h'$ equals the weighted in-degree of z_1 and z_2 follows. Here, h and h' are the number of vertices indexed by odd numbers which dominate z_1 and z_2 respectively. Considering these two equations together yields $4k = 4k'$ or rather $k = k'$; that is, as claimed, the vertices z_1 and z_2 agree in unweighted degree if they agree in weighted degree. Since $(deg^+(z_1), deg^-(z_1)) = (deg^+(z_2), deg^-(z_2))$, it is not difficult to show that

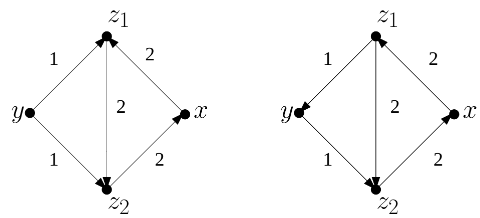


Figure 5.1: Two cases in the final step of the tournament algorithm

z_1 and z_2 must lie on a directed $\vec{C}_3$ in T_n. Let x be the other vertex on this directed $\vec{C}_3$ and select a fourth vertex y in T_n so that for one of z_1 and z_2, say z, one of x and y dominates or is dominated by z while z dominates or is dominated by the other. We guarantee the existence of such a vertex y because otherwise without loss of generality z_1 wins only once while z_2 loses only once. This last eventuality is prohibited since $n \geq 4$ and if $n \geq 4$, no two vertices can have the same degree under the above conditions. Now, perform the initial indexing so that one of x and y has edges labeled 2 going to z_1 and z_2 and one has edges labeled 1 adjacent with z_1 and z_2. Then there are only two possibilities for the orientations of these edges up to arc reversal, as given in the Figure 5.1.

If we switch the indices of $x = z_i$ and $y = z_j$ to $y = z_j$ and $w = z_i$ we do not change the total weight of any vertex with regard to the index of each vertex, but we do alter the in-weight and out-weight of both z_1 and z_2. Under the assumption that their quantities were equal under our prior ordering of $V(T_n)$, it is not difficult to see that they will be distinct after reordering. ■

The proof above can be generalized in various ways. We explore variations of the general method here in Chapter 5 and additionally in Chapter 6.

5.2.1 Variations on the Algorithm

It is immediately clear that the tournament algorithm extends to semi-complete digraphs. We have previously defined semi-complete digraphs; we have a single algorithm that determines a large portion of all digraphs. For the sparser, oriented graphs, we have a completely different story; there does not appear to be one simple construction that labels all oriented graphs.

Theorem 5.4 *The strength of a semi-complete digraph SC_n is one if its complement in the complete digraph $\vec{K}_n$ is irregular and 2 otherwise.*

Proof: Let SC_n be given. The proof given below is for $n \geq 4$. Notice $n = 2, 3$ can be done as exercises. Starting with the vertex of largest total degree and ending with the vertex of smallest total degree, index the vertices for the purpose of the labeling algorithm $2k, 2k-2, 2k-4, ..., k+2, k, k+1, k+3, ..., 2k-1$; then pick an underlying tournament $T_n \subset SC_n$ and carry out the tournament algorithm.

This gives two vertices of the same total weight and proceed to weight the extra arcs with a 1 or a 2. Either choice of 1 or 2 for the extra arcs only stratifies the vertex weights further unless there is a duplication from the tournament algorithm. That is, denote the total weight of z_j under this first round of labeling by $w(z_j)$ and obtain

$$w(z_j) = \begin{cases} \frac{3n-j-2}{2} + deg^{tot}_{SC_n - T_n}(z_j) & j \text{ is odd and } n \text{ is odd,} \\ \\ \frac{3n+j-5}{2} + deg^{tot}_{SC_n - T_n}(z_j) & j \text{ is even and } n \text{ is odd,} \\ \\ \frac{3n-j-1}{2} + deg^{tot}_{SC_n - T_n}(z_j) & j \text{ is odd and } n \text{ is even,} \\ \\ \frac{3n+j-4}{2} + deg^{tot}_{SC_n - T_n}(z_j) & j \text{ is even and } n \text{ is even.} \end{cases}$$

In fact, the indexing of the vertices assures that if

$$deg^{tot}_{SC_n - T_n}(z_j) \leq deg^{tot}_{SC_n - T_n}(z_i)$$

then $w'(z_j) \leq w'(z_i)$ where w' is the total weight of a vertex in T_n. That is, if v_j has a larger total weight than v_i in our labeled T_n, then v_i necessarily has a greater total weight than v_j after all the arcs in $A(SC_n) - A(T_n)$ are labeled with either a 1 or a 2.

Either the two vertices z_1 and z_2 have different total degree pairs or they are the only two vertices with the same total weight after we label the extra arcs. Suppose the vertices have the same total weight and the weighted degree pairs match. Then the vertices z_1 and z_2 lie on a directed triangle in T_n. As such, they lie on a directed triangle with another vertex, v. Furthermore, either we are in Case 1 and one of these two vertices, z_1, has an arc to and from another vertex, z_i, or we are in Case 2 and the two matching vertices z_1 and z_2 form an oriented split graph $\vec{G}$ with the tournament formed by z_1 and z_2 taken with the odd indexed vertices of SC_n joined to the vertices with even indices. (Here, invoke that $n \geq 4$.)

Case 1: Label the extra arcs with a 1 or 2 depending upon which choice gives either z_1 or z_2 a 2-loop with some z_i such that the 2 arcs in the 2-loop are labeled with a 1 and a 2 respectively. In this case, if the (out-weight, in-weight) pairs of z_1 and z_2 match after labeling the extra arcs a 1 and a 2, switch the arc labels on $\vec{z_1 z_i}$ and $\vec{z_i z_1}$ without loss of generality. (So that for instance if the label on $\vec{z_i z_1}$ is a 1 and $\vec{z_1 z_i}$ is a 2, then after the switch $\vec{z_i z_1}$ is labeled with a 2 and $\vec{z_1 z_i}$ is labeled with a 1.)

This maintains that there is only one tie in total weight and the 12 switch breaks the (out-weight, in-weight) tie between the total weight matched vertices z_1 and z_2.

Case 2: There are subcases depending on the structure of SC_n. The vertices z_1 and z_2 lie on a directed triangle with another vertex z_i. These two vertices z_1 and z_2 also lie on a triangle with another vertex (which is not directed), z_j. We will permute the indices of z_i and z_j as before in Theorem 5.2 and this will lead to an irregular labeling under the new permutation of indices.

Subcase 2.1: Neither z_i nor z_j is incident a 2-loop. Then if we permute z_i and z_j, this breaks the tie between z_1 and z_2 and leaves all other vertices with distinct total weights.

Subcase 2.2: Both z_i and z_j is incident a 2-loop that contains a 1-labeled and a 2-labeled arc. Permute the indices of z_i and z_j. If the new z_i is tied after the permutation, swap the 1 and the 2 label. This breaks the tie. Then, likewise, if the new z_j vertex is tied after the permutation, swap the 1 and the 2 label. This breaks the tie.

Subcase 2.3: The weight of z_i, $w(z_i) > w(z_j)$ and z_i is incident a 2-loop that contains a 1-labeled and a 2-labeled arc, but z_j is not incident a 2-loop. Permute the index i to j, but rather than permuting j to i, permute every even index less than i up by 2 and put z_j at index 4. Then the new z_j (previously indexed z_i) has a 2-loop that can be used to swap labels and break a tie; meanwhile, z_4 can only tie z_j. ■

5.2.2 Collections of Tournaments

In this section, we introduce a parameter called $\vec{\mu}$. We show that in all cases $\vec{s}(\bigcup_i T_i) \leq \vec{\mu}(\bigcup_i T_i)$.

Theorem 5.5 $\vec{s}(\bigcup_{i=1}^{i=k} T_i) \leq k+1$ *where all the* T_i *have the same order.*

Proof: Number the tournaments $T_1, T_2, ..., T_n$ and then label them according to the procedure in Theorem 5.1. Then, consider the sum of the in- and out-weights, each T_i has consecutive vertex labels with (at most) one repeat.

If n is even the sums of the weights are

$$\frac{3n-4}{2}, \frac{3n-2}{2}, \frac{3n-2}{2}, \frac{3n}{2}, \frac{3n+2}{2}, ..., 2n-3, 2n-2.$$

If n is odd the sums of the weights are

$$\frac{3n-5}{2}, \frac{3n-3}{2}, \frac{3n-3}{2}, \frac{3n-1}{2}, \frac{3n+1}{2}, ..., 2n-3.$$

In each case there are $n-1$ consecutive vertex labels. So operate on the labels of T_k by $k-1$ (adding $k-1$) to each label, obtaining a sequence of almost all distinct vertex weights where the duplications have distinct in- and out-weights by our previous discussion. So all vertex weights are distinct with $n+1$ labels, so that $\vec{s}(\vec{D}) \leq k+1$. ■

The following algorithm is also helpful for bounding the irregularity strength of collections of tournaments from above.

Definition 5.6 *Let* U_+ *be the set of vertices of a digraph* $\vec{D}$ *such that* $deg^+ v > 0$.

Definition 5.7 *Let* U_0 *be the set of vertices of a digraph* $\vec{D}$ *such that* $deg^+ v = 0$.

Definition 5.8 *Let* λ_+ *be the value we get as a maximum out-arc label in the labeling we get in the following way:*

Rank the vertices of U_+ *by out-degrees* $f : U_+ \to \{1, 2, ...|U_+|\}$ *is a bijection, and then queue the arcs once by* g *so that*

$$g : \{v_k \vec{w} : 1 \leq k \leq |U_+|\} \to \{1, 2, ...deg^+ v_1, deg^+ v_1 + 1, ..., |A(D)|\}$$

is a bijection (if the f*-building encounters ties in out-degree, break those ties arbitrarily).*

Now the label on $a_{(deg^+ v_j)+k}$ *where* $deg^+ v_j + 1 \leq k \leq deg^+ v_j + deg^+ v_{j+1}$ *is equal to* $\lfloor \frac{\vec{w}(v_j)+1}{deg^+ v_{j+1}} \rfloor$ *if* $k \leq \frac{\vec{w}(v_j)+1}{deg^+ v_{j+1}}$ *and* $\lceil \frac{\vec{w}(v_j)+1}{deg^+ v_{j+1}} \rceil$ *if* $k > \frac{\vec{w}(v_j+1)}{deg^+ v_{j+1}}$.

Definition 5.9 *We say* $\vec{\mu} = \lambda_+(U_+) + |U_0|$.

Discussion: The parameter $\vec{\mu}$ is an upper bound for $\vec{s}$ of a digraph $\vec{D}$ if the following algorithm terminates.

i. *Steps 1 through* $\lambda_+(U_+)$*.*

   Weight the out-arcs of the digraph as described. This amounts to a completed arc-labeling though we can see it may not be irregular.

ii. Consider the bigraph $[U_+, U_0]$.

iii. If the bigraph is acyclic the next step terminates in $\leq |U_0|$ steps, leaving us with an irregular arc labeling that is an irregular μ-labeling where $\mu \leq \vec{\mu}$ and clearly $\vec{\lambda} \leq \vec{s} \leq \mu$.

iv. *Steps* $\lambda_+(U_+)$ *through* $\lambda_+(U_+) + |U_0|$.

Rank the vertices in U_0 by $h : U_0 \to 1, 2, ...|U_0|$, sorting v_k below v_j if there exists a_r such that $v_k \sim a_r$ where for all t such that $a_t \sim v_j$, $r < t$

Then starting tie of greatest total weight in U_0 after Steps i-iii, then we pick the lowest ranked arc a_i incident the highest h-ranked vertex in the tie and add 1 to every label on every arc a_j $j \geq i$.

Supppose the algorithm cycles, that ties recur. There is no tie between $v, x \in U_+$ because if v is ranked higher by f than x no matter how many times we iterate step iv, steps i-iii guarantees v always has greater out-degree than x.

Suppose $v \in U_+$ and $x \in U_0$. This is a contradiction since x has out-degree 0.

Suppose $v, x \in U_0$. Then suppose h ranks v higher than x, we necessarily have that every time t in-arcs incident v increase by 1, and more than t arcs incident x increase by 1 due to the nature of h. Step iv breaks the tie between v and x, and they can never re-tie.

It remains to calculate $\lambda_+(U_+)$.

5.2.3 Multipartite Digraphs

Theorem 5.10 *The strength of* $K_{m(n)}$ *whose condensation is transitive is 2.*

Proof: Let the partite sets be labeled 1,...,m. Let the vertices be labeled by an ordered pair of two subscripts, one for the partite set containing it and one for a number between 1 and n. In this way order the vertices of each partite set 1 through n and get mn vertices each with a unique subscript (h, k) where h indicates a partite set and k indicates an element of the respective partite set. If $h < h'$ and $k \le k'$ the arc $v_{h,k}\vec{v}_{h',k'}$ is labeled with a 2, otherwise, label the arc with a 1.

We get

$$\vec{w}(v_{h,k}) = ((n-h)(k), (h-1)(n-k)+(n-h)(n-k)) = (k(n-h-1), (n-k)(n-1)).$$

Now suppose $\vec{w}(v_{h,k}) = \vec{w}(v_{h',k'})$. That is, suppose

$$(k(n-h-1), (n-k)(n-1)) = (k'(n-h'-1), (n-k')(n-1)).$$

Then $k = k'$ by consideration of the second arguments in the equality. And, using that $k = k'$ and the first arguments of the equality it follows $h = h'$. That, is the labeling is irregular. ■

There are a number of general constructions based on building irregular labelings from other irregular labelings.

Theorem 5.11 *Given an irregular s-labeling of a graph G, if we replace each vertex in G by an arbitrary (not necessarily identical) tournament of order n, then the strength of the resulting digraph is necessarily $\le 2s$. If all the tournaments are transitive, we can lower the bound to s.*

Proof: Suppose the tournaments are of equal size. Then, (*), if we weight the tournament according to our tournament algorithm, the tournament image of each point v has minimum total weight $2\vec{w}(v) + 1$ and maximum total weight $2\vec{w}(v)n+2(n-1)$, where $2\vec{w}(v)$ is twice the weighted degree of v in our irregularly labeled G. Then as long as the points in G are separated by 1, and we multiply the arcs by a factor of 2, the algorithm (*) amounts to an irregular labeling of the new digraph D. So, given an irregular s-labeling that separates points by 1 integer, multiply every edge label by 2. This gives us a greatest label of $2s$. If the tournaments are all transitive, we do not have to separate the points by 2 to begin with. ■

Theorem 5.12 *Given an irregular $\vec{s}$-labeling of a digraph, if we replace each vertex by an arbitrary (not necessarily identical) tournament of order n, then the strength of the resulting digraph is necessarily $\leq 2\vec{s}$. If all the tournaments are transitive, we can lower the bound to $\vec{s}$.*

Proof: Suppose we replace vertices of digraph D with tournaments of different size and replace the arcs with tournament joins. If vertex v associated with tournament T_v has different weighted degree in the irregular weighting of D than vertex x, then every vertex in T_x has different weighted degree from every vertex in T_v. Suppose the out-degrees agree on 2 of these vertices. Then the in-degrees do not: for $2nw^-(v) + 2(n-1) < 2n(w^-(v)+1)$. ■

Theorem 5.13 *If we replace every vertex of a directed cycle with a directed cycle; i.e., take the directed cross product of a cycle iteratively, then $\vec{D} \equiv \Pi_{i=1}^{i=k}\vec{C}_n$, and $\vec{s}(\vec{D}) \leq \lceil \sqrt{n}\, \rceil^k$*

Proof: Iteratively, multiply the weight of the arcs of the digraph being expanded by $\lceil \sqrt{n}\, \rceil$. Then weight each vertex of the cycles replacing the vertices with the weight of the root vertex plus the weight given by an irregular labeling of the cycle. Again, if the out-degrees of two points agree, the in-degrees do not: $\lceil \sqrt{n}\, \rceil w^-(v) + \lceil \sqrt{n}\, \rceil < (\lceil \sqrt{n}\, \rceil (w^-(v)+1)) + 1.$ ■

6. Small Irregularity Strength

6.1 $\vec{s} = 2$, and $disp = 1$

First we cite [30] and rewrite some theorems. Notice some of the results were actually proved in [28] and [16].

Theorem 6.1 ([30]) *Let G be a graph such that $s(G) = 2$ and $|V(G)| = n$. Then $|E(G)| \geq \lceil \frac{n^2-1}{8} \rceil$. For n=3 mod 4, $|E(G)| \geq \lceil \frac{n^2-1}{8} \rceil + 1$. Furthermore, there exist graphs H for which equality holds.*

Theorem 6.2 ([30]) *Let G be a graph such that $|V(G)| =$ and $s(G) = 2$. Then $|E(G)| \leq \binom{n}{2} - \frac{n-1}{4}$.*

Theorem 6.3 ([30]) *Let m, n and α be fixed positive integers with $2m \leq n \leq 4\alpha + 1$ and let M be a fixed m-element subset of the vertex set of K_n. If G is a graph obtained from K_n by deleting α edges of the complete subgraph induced by M, then $s(G) = 2$.*

Theorem 6.4 ([30]) *If G has $\binom{n}{2} - \lceil \frac{n-1}{4} \rceil$ edges and the complement of G is not a matching, then $s(G) = 2$.*

Theorem 6.5 ([30]) *If G is a graph obtained from K_n by deleting α edges, $\lceil \frac{n-1}{4} \rceil \leq \alpha \leq \frac{n}{2} - 1$, such that the removed edges induce a connected graph, then $s(G) = 2$.*

Theorem 6.6 ([30]) *If G=$K_n - K_m$ and $2m \leq n \leq 2m^2 - 2m + 1$, then $s(G) = 2$.*

Definition 6.7 *A split graph $SP_{p,q}$ has an independent set of size p joined to a clique of size q. We call a split graph $SP_{p,q}$ a Type 1 graph if $p = q^2 - q$.*

Definition 6.8 *A graph $K_n - K_{p,q}$ is called a cloister. A cloister with $p + q = \lfloor \sqrt{n} \rfloor$ is called a Type 2 graph.*

Theorem 6.9 *(i) Type 1 and Type 2 graphs have graph irregularity strength 2; (ii) Type 1 and Type 2 graphs have disparity 1.*

Proof: By previous work [30] Type 1 and Type 2 graphs have graph irregularity strength 2. Notice it follows they necessarily have digraph irregularity strength ≤ 2 for all orientations. If we take a transitive tournament of the same order as our Type 1 or Type 2 graph and remove either (1) a clique formed by the vertices $1, p, 2p, ..., p^2$ or (2) a biclique all of whose vertices in one partite set are ranked above all our vertices in the other partite set we get an irregular digraph whose underlying graph is our desired Type 1 graph or Type 2 graph. By the characterization of totally fair graphs, it follows that Type 1 and Type 2 graphs have disparity equal to 1. ■

6.2 If $n > 1$, $\vec{s}(\vec{K}_{n,n}) = 2$

The study of irregular labelings for digraphs actually began in [30], where it was required that two vertices have distinct deficiency for their weighted degrees to be distinct. We will use the definition of the deficiency of a vertex as well as the following lemmas in order to show $K_{n,n}$ is fair for all values of n. In the following discussion f is an arc-labeling, and $g(v) = ([h + \text{def}_f], w_f)$ where here $\text{def}_f(v)$ is $= \sum f(\vec{vx}) - \sum f(\vec{xv})$ and $w_f(v) = \sum f(\vec{vx}) + \sum f(\vec{xv})$.

Definition 6.10 *A* $(0,1)$*-labeling of a digraph* $\vec{D}$ *is a subdigraph* $D \subset \vec{D}$ *where the arcs labeled 1 are in* $A(D)$ *and the arcs labeled 0 are not in* $A(D)$.

Definition 6.11 *An irregular* $(0,1)$*-labeling of a digraph* $\vec{D}$ *is an irregular subdigraph* $D \subset \vec{D}$ *whose arcs are labeled 1 and all arcs of* $\vec{D} - D$ *are labeled 0.*

Definition 6.12 *An irregular* $(0,1)$*-labeling of a digraph that does not use vertex weight* $g(v) = (0,0)$ *is called a* $(0,0)$*-avoiding irregular* $(0,1)$*-labeling. Meanwhile, an arc labeling of a digraph that does not use vertex weight* $g(v) = (0,0)$ *is called a* $(0,0)$*-avoiding labeling.*

Theorem 6.13 *For* $n = 3$*, every orientation of* $K_{n,n}$ *has an* $(0,0)$*-avoiding irregular* $(0,1)$*-labeling.*

Proof: Consider the feedback graphs of size 1,2,3,4. There are two graphs of size 2; there are four graphs of size 3; and three graphs of size 4. In each case, the feedback graph gives rise to an irregular $(0,1)$-labeling. In some cases, the placement of the feedback digraph, i.e., which vertices are in which partite set, is important to consider. See Appendix A for proof by cases. These picture proofs label the digraph irregularly using the following function f for labeling. The function f labels the arc between v_{iA} and v_{jB} with a 1 if $i + j > 3$ and 0-label otherwise. The labeling is complete once the permutation of the vertex set is established. ■

Definition 6.14 *Let an orientation of* $K_{n,n}$*, called* $\vec{D}$ *be given. A function* $h : V(\vec{D}) \longrightarrow Z$ *is called a feasible initial condition in the* $(0,0)$*-avoiding case unless* $\vec{D}$ *is oriented so that team A beats team B,* $h|_A = d$*,* $h|_B = c$ *and* $c - d \in (2, 2n)$.

Theorem 6.15 *Every orientation of C_4 with every feasible initial condition has an $(0,0)$-avoiding irregular $(0,1)$-labeling with the following two exceptions:*

i. the digraph $\{(v_{1A}\vec{v}_{1B}), (v_{1B}\vec{v}_{2A}), (v_{2A}\vec{v}_{2B}), (v_{2B}\vec{v}_{1A})\}$ with the initial condition $h(v_{1A}) = h(v_{2A}) = h(v_{1B}) = h(v_{2B}) = c$.

ii. the digraph $\{(v_{1A}\vec{v}_{1B}), (v_{1B}\vec{v}_{2A}), (v_{1A}\vec{v}_{2B}), (v_{2B}\vec{v}_{2A})\}$ with the initial condition $h(v_{1A}) = c, h(v_{2A}) = c+6, h(v_{1B}) = c+4, h(v_{2B}) = c+2$.

In most applications of 6.15, we will use that the C_4 in question is a subdigraph of some superdigraph D and that there is a function h (which will be feasible by Lemma 6.20) imposed on the C_4 by the arcs from the C_4 to $D - C_4$.

Proof: The proof is by cases, see Appendix A for an explanation of how the proof works. ■

Lemma 6.16 *Let $h : V(\vec{K}_{3,3})$ be a feasible initial condition on an orientation of $K_{3,3}$, called $\vec{K}_{3,3}$. If the leave of a double star $\{xv_{1B}, xv_{2B}, xy, yv_{1A}, yv_{2A}\}$ where $h(x) \neq h(y)$, is one of the exceptional graphs (i.) or (ii.) from 6.15, then without loss of generality $h(x) \neq h(v_{1B})$, and the leave of $\{xv_{1B}, xv_{2B}, xy, v_{1B}v_{1A}, v_{1B}v_{2A}\}$ is not (i.) or (ii.).*

Proof: Case 1: If we have exceptional graph (i.) or (ii.) under forbidden conditions and we permute y and v_{1B} and get the same exceptional graph under forbidden conditions, then selecting v_{2B} (and permuting it with y) necessarily gives the other exceptional graph under non-forbidden feasible initial conditions except when we are in Case 2. We get

$$[h, h, h, h] \longrightarrow [h, h, h, h'] \neq [h_0, h_0+4, h_0+6, h_0+2]$$

or

$$[h_0, h_0+4, h_0+6, h_0+2] \longrightarrow [h_0, h_0+4, h_0+6, h'] \neq [h, h, h, h].$$

In the case that permuting y and v_{1B} does not return the same exceptional graph under forbidden conditions, it returns the same graph under allowable conditions, a graph different from the ones underlying (i.) and (ii.) or the other forbidden graph under allowable conditions:

$$[h_0, h_0+4, h_0+6, h_0+2] \longrightarrow [h_0, h', h_0+6, h_0+2] \neq [h, h, h, h],$$

or

$$[h, h, h, h] \longrightarrow [h, h', h, h] \neq [h_0, h_0+4, h_0+6, h_0+2].$$

Case 2: In the exceptional case that $h(x) = h(v_{1B})$ or $= h(v_{2B})$, and (i.) or (ii.) is the leave, then under one of the following permutations we get a non-forbidden leave as a subgraph:

$$[h_0, h_0+4, h_0+6, h_0+2] \longrightarrow [h_0, h_0+4, h_0+6, h_0+4] \neq [h, h, h, h]$$

$$\neq [h_0, h_0+4, h_0+6, h_0+2],$$

or under the permutation

$$[h_0, h_0+4, h_0+6, h_0+2] \longrightarrow [h_0, h_0+2, h_0+6, h_0+2] \neq [h, h, h, h]$$

$$\neq [h_0, h_0+4, h_0+6, h_0+2],$$

and we are done.

In the case that $h(x) = h(v_{1B}) = h(v_{2B})$, and (i.) is the leave, then we swap v_{1A} or v_{2A} for x. Since $h(v_{1A}) = h(v_{1B}) \neq h(y)$ this permutation gives a leave

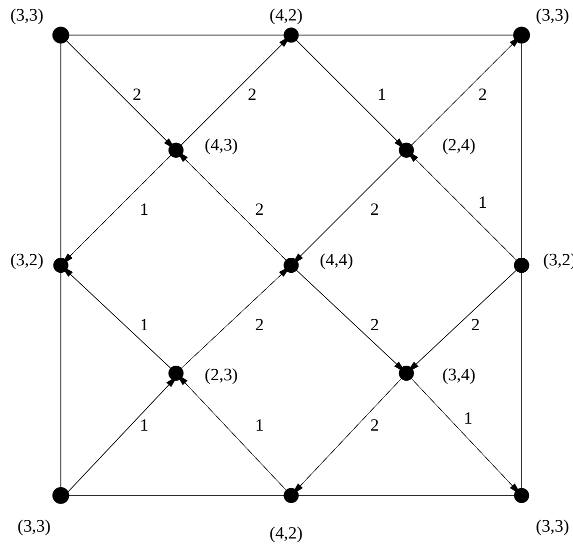


Figure 6.1: A $(0,0)$-avoiding irregular $(0,1)$-labeling of the Hamiltonian regular orientation of gives rise to this irregular 2-labeling of the same orientation of $K_{4,4}$ pictured here on the torus.

which does not have one of the forbidden orientation-initial feasible condition combinations. If the orientation changes to (ii.), the fact that

$$[h', h, h, h] \neq [h_0, h_0 + 4, h_0 + 6, h_0 + 2]$$

completes the proof. ∎

Theorem 6.17 *For any di-regular orientation of $G' \equiv \vec{K}_{n,n}$ with $n \geq 2$, there is a $(0,0)$-avoiding irregular $(0,1)$-labeling of G'.*

Proof: Let $A \cup B$ be the bipartition of $V(G'')$ in the discussion that follows. Pick a vertex $x \in A$ from G'. Label x as v_{1A}. On the ordering of the vertices we eventually select, label the arcs so that if

$$v_{iA}\vec{v}_{jB} \in A(K_{n,n}) \quad i \leq j$$

or if

$$v_{jB}\vec{v}_{iA} \in A(K_{n,n}) \quad j < i$$

then the arc is labeled with a 1; otherwise the arc is labeled with a 0. Continue selecting vertices of G'' in the following way and labeling the vertices v_{iB}, $v_{(i+1)A}$ for $1 \leq i \leq n-1$: pick v_{iB} and $v_{(i+1)B}$ so that they do not have the same number of 0-labeled victories. To see that this can always be done, consider the graph $G'' = G - \{x_{kA}, x_{jB} : 1 \leq k \leq i+1,\ 0 \leq j \leq i\}$. Suppose for every pair of vertices we select, one from each side of the partition, the two vertices have the same number of 0-labeled victories in G''. Then G'', whose underlying graph is $K_{m,m+1}$ for some m, has degree sequence $(a,b)^t$, $(a-1,b+1)^{m-t}$, $(a-1,b)^{t'}$, $(a,b-1)^{m+1-t'}$. Since we cannot select a vertex of degree (a,b) that beats a vertex of degree $(a,b-1)$ nor a vertex of degree $(a-1,b+1)$ that beats a vertex of degree $(a-1,b)$ by assumption, we must have that every vertex of degree (a,b) beats every vertex of degree $(a-1,b)$ beats every vertex of degree $(a-1,b+1)$ beats every vertex of degree $(a,b-1)$ beats every vertex of degree (a,b). But then we have a vertex of degree $(a-1,b+1)$ beats a vertex of degree $(a,b-1)$ contrary to assumption. So we get that $t=m$, and the equations

$$t'(a-1) + (m+1-t')a = mb$$

and

$$t'(b) + (m+1-t')(b-1) = ma$$

follow. From the first equation we get $a=b$. From the second equation we get that $t'+b = m+1$ or rather that $t' = a = b$. But notice that since $t=m$ and $m-t=0$ we get $a-1=0$ and $a=1$. The only way the selection process can stop is if there are three unordered vertices, they are the last three in the queue, and we get degrees $(0,1),(1,1),(1,0)$ in G''. In this case (*) order the vertices so that the vertex of degree $(1,0)$ is $v_{(n-1)B}$, the vertex of degree $(1,1)$ is v_{nA} and the vertex of degree $(0,1)$ is v_{nB}. Label the arc $v_{(n-1)B}\vec{v}_{nA}$ with a 1 and $v_{nA}\vec{v}_{nB}$ with a 0. Then $g(v_{nA}) = (0,n)$ and $g(v_{nB}) \neq g(v_{(n-1)B})$. However, the total weight of $v_{(n-1)B}$ and v_{nB} are identical to one another and distinct from all other vertices of G'. In general, no two vertices v, w can have the same weighted degree pair $g(v) = g(w)$ unless $g(v) = (i', \frac{n}{2} - i'') = g(w) = (j', \frac{n}{2} - j'')$ unless $i' + i'' = i = j = j' + j''$ where $v \in A$ and $w \in B$ and where i and j are the number of vertices v and w dominate respectively. That is, unless $v = v_{iA}$ and $w = v_{(i-1)B}$ for $i \leq n-1$ these two vertices cannot have the same weighted degree pair because of how we selected the vertices (in pairs) and because we handled the one exception where this selection process is not followed precisely (*). To see that there is a $(0,0)$-avoiding labeling, suppose $v_{(\frac{n}{2}+1)A}$ beats $v_{\frac{n}{2}B}$. Then simply reorder the vertices so that $kA \leftrightarrow (n-k+1)B$; i.e., flip the construction upside down (rotate by 180 degrees). ■

Corollary 6.18 *Every diregular orientation of $\vec{K}_{4,4}$ has a $(0,0)$-avoiding irregular $(0,1)$-labeling.*

Proof: The proof is by the above construction. (See the Figures 6.1 and 6.2.) ■

Definition 6.19 *An A beats B tournament is an orientation of $K_{n,n}$ where every vertex in partite set A dominates every vertex in partite set B.*

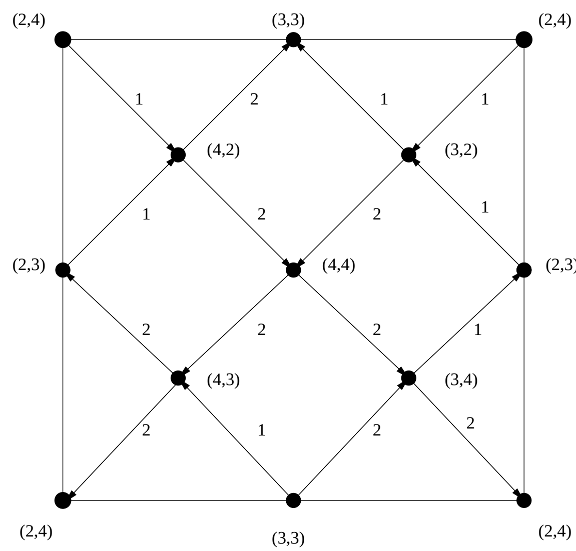


Figure 6.2: A $(0,0)$-avoiding irregular $(0,1)$-labeling of one of the regular orientations of $K_{4,4}$ gives rise to an irregular 2-labeling of the same orientation of $K_{4,4}$ pictured here on the torus.

Lemma 6.20 *Suppose $h : V(\vec{D}) \longrightarrow Z$ is feasible where $\vec{D}$ is an orientation of $K_{n,n}$ (with $n \geq 3$). Then define h'*

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).$$

where x and y are such that $[h + \text{def}](x) \neq [h + \text{def}](y)$. Then if h' is not feasible on $\vec{D} - \{x, y\}$ it follows that $\vec{D}$ has a $(0,0)$-avoiding irregular $(0,1)$-labeling.

Proof: Suppose $h, \vec{D}, x, y$ meet the hypotheses of the theorem and that h' (as defined above) is not feasible. Then there are four cases for the neighborhoods of x, y and we can show in each case that $\vec{D}$ has a $(0,0)$-avoiding irregular

$(0,1)$-labeling. Since h' is not feasible we have by definition of feasible, that $\vec{D} - \{x, y\}$ is an A beats B tournament such that $h|_B - h|_A \in (2, 2n)$. Suppose when we extend back to the digraph that $x \in A$ and $y \in B$. Index x as v_{1A} and the rest of the vertices of A as $v_{2A}, ..., v_{nA}$. Index y as v_{1B} and the rest of the vertices of B as $v_{2B}, ..., v_{nB}$. For the purposes of the proof, let f be the arc labeling that labels $v_{iA}v_{jB}$ with a 1 if $i + j \leq n + 1$ and labels $v_{iA}v_{jB}$ with the 0-label otherwise. Furthermore, under the current index and the labeling f, let k' be the index which has $[h + \text{def}](v_{k'A}) = [h + \text{def}](v_{k'B})$.

Case 1: Suppose B dominates x and y dominates A.

Case 1.1: Suppose $k' > \frac{1}{2}(n + 1)$. Suppose $k' + 2 \leq n$.

Case 1.1.1: Permuting $1A \leftrightarrow (k' + 1)A$ leads to (a) $[h + \text{def}](v_{1A}) = [h + \text{def}](v_{(k'+1)B})$, or (b) $[h + \text{def}](v_{1B}) = [h + \text{def}](v_{(k'+1)A})$.

Case 1.1.2: Permuting $1B \leftrightarrow (k' + 1)B$ leads to (c) $[h + \text{def}](v_{1A}) = [h + \text{def}](v_{(k'+1)B})$, or (d) $[h + \text{def}](v_{1B}) = [h + \text{def}](v_{(k'+1)A})$.

Notice there cannot be (a) and (c) : if (a), then $[h + \text{def}](v_{1A}) = h_1$ under the original ordering and $[h+\text{def}](v_{(k'+1)}) = h_2$ under the original ordering, then $h_1 + k = h_2$ and if (c), then $h_2 + k = h_1$, a contradiction. Meanwhile (b) and (d) cannot occur together by similar reasoning.

Case 1.1.3: Permuting $1 \leftrightarrow k' + 2$ leads to duplications: we have $[h + \text{def}](v_{1A}) = h_1$ under ordering in Case 1.1.1. And, $[h + \text{def}](v_{1B}) = h_2$ under the ordering in Case 1.1.2. So under the ordering in Case 1.1.3, $[h + \text{def}](v_{1A}) = h_1 + 1 + t$ and $[h + \text{def}](v_{1B}) = h_2 - 1 - t$. If $v_{1A}\vec{v}_{1B} \in A(\vec{D})$, then $t = -1$. Because $h_1 \neq h_2$, and because $[h + \text{def}](v_{(k'+2)A}) \neq [h + \text{def}](v_{(k'+2)B})$ under this new ordering, it follows this permutation of the order admits that f is a $(0,0)$-

avoiding irregular $(0,1)$-labeling of $\vec{D}$. (It is clear $[h+\text{def}](v_{jA}) \neq [h+\text{def}](v_{jB})$ for $j \neq k'+2$ because we permuted both v_{1A} and v_{1B} to a tier with index $> \frac{1}{2}(n+1)$.)

Case 1.1.4: Suppose $v_{1B}\vec{v}_{1A} \in A(\vec{D})$. Permute $1A \leftrightarrow (k'+2)A$ and $1B \leftrightarrow nB$. Then $[h+\text{def}](v_{1A}) = h_1 \neq [h+\text{def}](v_{(k'+2)B})$ under this ordering because $h_1 = [h+\text{def}](v_{(k'+1)B})$ under this ordering. It is also clear that $[h+\text{def}](v_{1B}) \neq [h+\text{def}](v_{nA})$ under this ordering by similar reasoning as long as $n > 4$. If $n = 4$ on the other hand, then $k' = 2$ and $k > 2$, a contradiction. This concludes Case 1.1.

Case 1.2: Suppose $k' > \frac{1}{2}(n+1)$. Suppose $k'+2 > n$. Permute $1 \leftrightarrow k'+1$. Then $[h+\text{def}](v_{1A}) = h_1 + t$ and $[h+\text{def}](v_{1B}) = h_2 - t$. Because $h_1 = h_2$, this concludes the Case 1.2.

Case 1.3: Suppose $n = k' > \frac{1}{2}(n+1)$.

Case 1.3.1: Permuting $1A \leftrightarrow k'A$ leads to (a) $[h+\text{def}](v_{1A}) = [h+\text{def}](v_{k'B})$, or (b) $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{k'A})$.

Case 1.3.2: Permuting $1B \leftrightarrow k'B$ leads to (c) $[h+\text{def}](v_{1A}) = [h+\text{def}](v_{k'B})$, or (d) $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{k'A})$.

Case 1.3.3: Permuting $1 \leftrightarrow k'$ leads to a duplication. Let $[h+\text{def}](v_{1A}) = h_1$ under ordering in Case 1.1.1. And, $[h+\text{def}](v_{1B}) = h_2$ under the ordering in Case 1.1.2. So under the ordering in Case 1.1.3, $[h+\text{def}](v_{1A}) = h_1 + 1 + t$ and $[h+\text{def}](v_{1B}) = h_2 - 1 - t$. If $v_{1A}\vec{v}_{1B} \in A(\vec{D})$, then $t = -1$ and because $h_1 = h_2$, and because $[h+\text{def}](v_{k'A}) \neq [h+\text{def}](v_{k'B})$ under this ordering, it follows this permutation of the order admits that f is a $(0,0)$-avoiding irregular $(0,1)$-labeling of $\vec{D}$. (It is clear $[h+\text{def}](v_{(j)A}) \neq [h+\text{def}](v_{(j)B})$ for $j \neq k'+2$

because we permuted both v_{1A} and v_{1B} to a tier with index $> \frac{1}{2}(n+1)$.)

Case 1.3.4: Suppose $v_{1B}\vec{v}_{1A} \in A(\vec{D})$ (and so $t = 1$). Permute $1A \leftrightarrow k'A$, $1B \leftrightarrow (k'-1)B$. As long as $k' = n > 2$, it follows that this will not lead to a duplication because $v_{1A}\vec{v}_{1B}$ is labeled with a 0-label under this ordering and the condition $3 \leq k'$. So then $[h+\text{def}](v_{1A}) = h_1 + 1 \neq h_1 = [h+\text{def}](v_{k'B})$. Furthermore, $[h+\text{def}](v_{1B}) = h_2 \neq h_2 + 1 = [h+\text{def}](v_{(k'-1)A})$. Finally, the 2 vertices permuted to the top tier do not fix $[h+\text{def}]$. This concludes case 1.3.

Case 1.4: Suppose $k' \leq \frac{1}{2}(n+1)$.

Case 1.4.1: Permuting $1A \leftrightarrow k'A$ leads to (a) $[h+\text{def}](v_{1A}) = [h+\text{def}](v_{k'B})$, or (b) $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{k'A})$.

Case 1.4.2: Permuting $1B \leftrightarrow k'B$ leads to (c) $[h+\text{def}](v_{1A}) = [h+\text{def}](v_{k'B})$, or (d) $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{k'A})$.

Notice there cannot be (a) and (c) : if (a), then $[h+\text{def}](v_{1A}) = h_1$ under the original ordering and $[h+\text{def}](v_{k'A}) = h_2$ under the original ordering, then $h_1 + k = h_2$ and if (c), then $h_2 + k = h_1$, a contradiction. Meanwhile (b) and (d) cannot occur together by similar reasoning.

Case 1.4.3: If we permute $1 \leftrightarrow k'$, we get a duplicated vertex pair. Suppose then $[h+\text{def}](v_{1A}) = h_1 + t = [h+\text{def}](v_{1B}) = h_2 - t$, under this ordering; ie. define t in this fashion.

Case 1.4.4: Permuting $1A \leftrightarrow k'A$, $1B \leftrightarrow nB$ leads to a duplication. Then $[h+\text{def}](v_{1A}) = h_1 + t \neq [h+\text{def}](v_{k'B})$ where t equals +1 or -1 (depending on the arc direction between v_{1A} and v_{1B}) under this ordering because $h_1 = [h+\text{def}](v_{k'B})$ under the ordering 1.2.1. If $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{nA}) = h_n$, then $h_2 - k = h_n$ for non-zero k unless $k' = n$, contradiction.

Case 1.4.5: Permuting $1A \leftrightarrow nA$, $1B \leftrightarrow k'B$ permits a duplication. Then $[h{+}\text{def}](v_{1B}) = h_1 - t \neq [h{+}\text{def}](v_{k'A})$ where t equals +1 or -1 (depending on the arc direction between v_{1A} and v_{1B}) under this ordering. Now $h_2 = [h{+}\text{def}](v_{k'A})$ under the ordering 1.4.4. If $[h{+}\text{def}](v_{1B}) = [h{+}\text{def}](v_{nA}) = h_n$, then $h_1 + k = h_n$ for non-zero k unless $k' = n$, contradiction. But $h_1 = h_2 - 2t$. So $h_1 + 2k = h_2$ so that $t = k = 1$. But then $k' = 2$, $n = 3$.

Case 1.4.6: $k' = 2$, $n = 3$ and Cases 1.2.1-1.2.5 fail to resolve duplications. Permute $1 \leftrightarrow n$. Then $h_1 + 2 \neq h_2 - 2$, so that $[h + \text{def}](v_{1A}) \neq [h + \text{def}](v_{1B})$ under this ordering. Furthermore $[h + \text{def}](v_{k'A}) \neq [h + \text{def}](v_{k'B})$. This can be seen in 2 ways: (1) we permuted $1 \leftrightarrow n$, (2) $h_1 - 1 \neq h_2 + 1 = h_1 + 2 + 1 = h_1 + 3$. This concludes Case 1.4, and therefore concludes Case 1.

Case 2: In this case suppose x is dominated by B, but y dominates some of A and is dominated by some of A.

Permute v_{1B} and v_{nA} maintaining that $v_{(k'+1)A}$ beats v_{1B} and v_{1B} beats $v_{k'A}$.

Case 3: In this case suppose x is dominated by some of B and dominates some of B. Meanwhile, y is dominated by some of A and dominates some of A.

Permute v_{1A} and v_{nB} maintaining that $v_{(k'+1)B}$ beats v_{1A} and v_{1A} beats $v_{k'B}$.

Case 4: In this case x dominates A and y is dominated by B.

Case 4.1: Suppose x beats y. Permute v_{1A} and v_{1B} with $v_{k'A}$ and $v_{k'B}$.

Case 4.2: Suppose y beats x.

Case 4.2.1: Suppose $k' > \frac{1}{2}(n + 1)$.

Case 4.2.1.1: Permuting $1A \leftrightarrow (\max\{n, k'{+}1\})A$ leads to (a) $[h{+}\text{def}](v_{1A}) = [h + \text{def}](v_{(k'+1)B})$, or (b) $[h + \text{def}](v_{1B}) = [h + \text{def}](v_{(k'+1)A})$.

Case 4.2.1.2: Permuting $1B \leftrightarrow (\max\{n, k'+1\})B$ leads to (c) $[h+\text{def}](v_{1A}) = [h+\text{def}](v_{(k'+1)B})$, or (d) $[h+\text{def}](v_{1B}) = [h+\text{def}](v_{(k'+1)A})$.

Notice there cannot be (a) and (c) : if (a), then $[h+\text{def}](v_{1A}) = h_1$ under the original ordering and $[h+\text{def}](v_{(k'+1)}) = h_2$ under the original ordering, then $h_1 + k = h_2$ and if (c), then $h_2 + k = h_1$, a contradiction. Meanwhile (b) and (d) cannot occur together by similar reasoning.

Case 4.2.1.3: Permute $1 \leftrightarrow \max\{n, k'+2\}$. Then $[h+\text{def}](v_{1A}) = h_1$ under ordering in Case 4.2.1.1. And, $[h+\text{def}](v_{1B}) = h_2$ under the ordering in Case 4.2.1.2. So under the ordering in Case 4.2.1.3, $[h+\text{def}](v_{1A}) = h_1 + 1 + t$ and $[h+\text{def}](v_{1B}) = h_2 - 1 - t$. If $v_{1A}\vec{v}_{1B} \in A(\vec{D})$, then $t = -1$. Because $h_1 = h_2$, and $[h+\text{def}](v_{(k'+2)A}) \neq [h+\text{def}](v_{(k'+2)B})$ under this ordering, it follows this permutation of the order admits that f is a $(0,0)$-avoiding irregular $(0,1)$-labeling of $\vec{D}$. (It is clear $[h+\text{def}](v_{jA}) \neq [h+\text{def}](v_{jB})$ for $j \neq k'+2$ because we permuted both v_{1A} and v_{1B} to a tier with index $> \frac{1}{2}(n+1)$.) This holds as long as $k'+2 \leq n$.

Case 4.2.1.4: Suppose $v_{1B}\vec{v}_{1A} \in A(\vec{D})$ (and so $t = -1$). Permute $1A \leftrightarrow k'A$, $1B \leftrightarrow (k'-1)B$. As long as $k' = n > 2$, it follows that this will not lead to a duplication because $v_{1A}\vec{v}_{1B}$ is labeled with a 0-label under this ordering and the condition $3 \leq k'$. So then $[h+\text{def}](v_{1A}) = h_1 + 1 \neq h_1 = [h+\text{def}](v_{k'B})$. Furthermore, $[h+\text{def}](v_{1B}) = h_2 \neq h_2 + 1 = [h+\text{def}](v_{(k'-1)A})$. Finally, the two vertices permuted to the top tier do fix $[h+\text{def}]$.

Case 4.2.1.5: Suppose $k'+2 > n$. Permute $1 \leftrightarrow \max\{n, k'+1\}$. Suppose $k'+1 \leq n$. It still follows that $h_1 - 1 \neq h_2 + 1$, so that $[h+\text{def}](v_{1A}) \neq [h+\text{def}](v_{1B})$. Furthermore, it is clear that no vertices from any other tiers have

$g(v_{jA}) = g(v_{jB})$.

Case 4.2.1.6: If $k' + 1 > n$, then permute $1A \leftrightarrow k'$.

Case 4.2.1.6.1: Permuting $1A \leftrightarrow k'A$ leads to (a) $[h + \text{def}](v_{1A}) = [h + \text{def}](v_{k'B})$, or ($b$) $[h + \text{def}](v_{1B}) = [h + \text{def}](v_{k'A})$.

Case 4.2.1.6.2: Permuting $1B \leftrightarrow k'B$ leads to (c) $[h + \text{def}](v_{1A}) = [h + \text{def}](v_{k'B})$, or ($d$) $[h + \text{def}](v_{1B}) = [h + \text{def}](v_{k'A})$.

Notice there cannot be (a) and (c) : If (a), then $[h + \text{def}](v_{1A}) = h_1$ under the original ordering and $[h + \text{def}](v_{k'A}) = h_2$ under the original ordering, then $h_1 + k = h_2$ and if (c), then $h_2 + k = h_1$, a contradiction. Meanwhile (b) and (d) cannot occur together by similar reasoning.

Case 4.2.1.6.3: Permute $1 \leftrightarrow k'$: We have $[h + \text{def}](v_{1A}) = h_1$ under ordering in Case 4.2.1.6.1. And, $[h + \text{def}](v_{1B}) = h_2$ under the ordering in Case 4.2.1.6.2. So under the ordering in Case 4.2.1.6.3, $[h + \text{def}](v_{1A}) = h_1 + t$ and $[h + \text{def}](v_{1B}) = h_2 - t$. For, in any case, either $v_{1A}\vec{v}_{1B}$ or $v_{1B}\vec{v}_{1A} \in A(\vec{D})$, so $t > 0$ and because $h_1 = h_2$, and because $[h + \text{def}](v_{k'A}) \neq [h + \text{def}](v_{k'B})$ under this ordering, it follows this permutation of the order admits that f is a $(0, 0)$-avoiding irregular $(0, 1)$-labeling of $\vec{D}$. (It is clear $[h + \text{def}](v_{jA}) \neq [h + \text{def}](v_{jB})$ for $j \neq 1$ under this order because k' was the tier with duplications.)

Definition 6.21 *An* (n)*-avoiding* $(0, 1)$*-labeling of* $\vec{D}$ *where* $\vec{D}$ *is an orientation of* $K_{n,n}$ *is a* $(0, 0)$*-avoiding irregular* $(0, 1)$*-labeling on* $\vec{D} - \{x, y\}$ *where* x *and* y *come from opposite partite sets.*

Lemma 6.22 *Suppose* $h : V(\vec{D}) \longrightarrow Z$ *is feasible where* $\vec{D}$ *is an orientation of* $K_{n,n}$ *(with* $n \geq 3$*). Then if* h *is not feasible on* $\vec{D} - \{x, y\}$ *it follows that*

$\vec{D} - \{x, y\}$ *is feasible for some other choice of* x *and* y *such that* $h(x) \neq h(y)$.

Proof: Let $A \cup B$ be the bipartition of the vertex set of $\vec{D}$. If $\vec{D} - \{x, y\}$ is not feasible, then $h|_A(\vec{D}) \neq h|_B(\vec{D})$. Pick any vertices from $A - \{x\}$ and $B - \{y\}$ respectively (unless one of the vertices is incident with a unique feedback arc in which case, do not pick the other endpoint of that arc). The remaining graph is feasible. For, without loss of generality, either $h(x) \neq h|_A$ or x is incident a feedback arc. ■

Theorem 6.23 *Every orientation of* $K_{4,4}$ *has a* $(0, 0)$*-avoiding irregular* $(0, 1)$*-labeling.*

Proof: The orientation $\vec{K}_{4,4}$ is not regular, so assign a double star xv_4,xv_5,xv_6,xy,yv_1,yv_2,yv_3 such that $\mathrm{def}(x) \neq \mathrm{def}(y)$ and label all its arcs with 1. Let $K_{3,3}$ be the digraph induced by $V(\vec{K}_{4,4}) - \{x, y\}$. Either there is a pair of vertices v and w such that $[h' + \mathrm{def}_{G'-K_{3,3}}](v) \neq [h' + \mathrm{def}_{G'-K_{3,3}}](w)$ where

$$\begin{aligned} h'(v) &= h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}). \end{aligned}$$

or the labeling $f' = 1 - f$ on the arcs $K_{3,3}$ completes the labeling; that is, if $f^* = 1$ on $G'' \equiv G' - K_{3,3}$ and $f^* = f'$ on $K_{3,3}$, then f^* is a $(0, 0)$-avoiding irregular $(0, 1)$-labeling. (Here, h' is feasible or we can label the digraph completely by Lemma 6.20.) For consider, the vertex weights of x and y are n and larger than any other vertex in the labeled digraph. But $[h + \mathrm{def}](x) \neq [h + \mathrm{def}](y)$.

(From this comment it should be clear the labeling is $(0,0)$-avoiding.) Then f^* is distinct on $K_{3,3}$ by definition of h' and f'. The function f is given by Theorem 6.13. In the case $[h' + \text{def}_{G''}]v \neq [h' + \text{def}_{G''}]w$, it is clear that there is a way to complete the arc-labeling so that it will be irregular, by induction from Theorem 6.15, and Lemmas 6.16, and 6.22. We have $h'(v) \neq h'(w)$ for some choice of v and w that leaves us with a C_4 under conditions for which there exists a $(0,0)$-avoiding irregular $(0,1)$-labeling, call it f. Let $f^* = f$ on $G''' = G'' - \{v, w\}$. Then $f^* = 0$ on $G'' - G'''$ and 1 on $G' - G''$. So then $[h + \text{def}_{f^*}](v) \neq [h + \text{def}_{f^*}](w)$ and clearly f^* is a $(0,0)$-avoiding irregular $(0,1)$-labeling. ■

Lemma 6.24 *Let $n \geq 3$ and let an orientation of $K_{n,n}$, $G' \equiv \vec{K}_{n,n}$, with a feasible initial condition h such that $[h + \text{def}](x) \neq [h + \text{def}](y)$ for x and y in $V(\vec{K}_{n,n})$ be given. Then there are two possibilities. The first possibility (i) is the following. The orientation $\vec{K}_{n,n}$ is a team tournament where team A beats team B and $[h+\text{def}](x)$ is constant for x in partite set A under the labeling $f_1(a_i) = 1$ for all $a_i \in A(\vec{K}_{n,n})$. Also, $[h + \text{def}](y)$ is fixed for all y in partite set B under the same arc labeling and $h(x) = h(y) - 2(n-1)$. The other possibility (ii) is we can find some pair of vertices which we call x and y such that $h(x) \neq h(y)$ and $[h + \text{def}_{G' - \{x,y\}}](v)$ is not fixed over the remainder of the vertex set when we cut vertices x and y, i.e., not fixed over $V(\vec{K}_{n,n}) - \{x, y\}$.*

Proof: Let $G' \equiv \vec{K}_{n,n}$ and h meet the conditions of the theorem. Let $A \cup B$ be the bipartition of $V(\vec{K}_{n,n})$. We want to show if case (ii) does not hold, case (i) holds. Suppose for all choices of x, y where $h(x) \neq h(y)$, we get a fixed value for $[h + \text{def}_{G' - \{x,y\}}](v)$ for $v \in V(\vec{K}_{n,n})$. If there is a fixed h value on $V|_A$, every distinct h value on $V|_B$ is associated with a unique neighborhood. To prove this last assertion observe that it cannot be the case that $h(x) = h(z)$ and $(N^+(z), N^-(z)) \not\equiv (N^+(x), N^-(x))$ for $z \in V(G') - \{x, y\}$, for if this is the case, (ii) results. Likewise for y. Furthermore, if there are three distinct h values as h ranges over either side of the partition, case (ii) results. The only possibility is that there are two or fewer h values on either side of the partition. Suppose there is only one h value $= h_0$ on $V|_A$. There are fewer than three h values on $V|_B$. If there are two distinct h values on $V|_B$, neither can match the h value on $V|_A$. Every vertex in partite set A has the same neighborhood, so that every vertex in partite set B is a source (if its h value is h_1), or a sink (if its h value is h_2). Given that $h_0 \neq h_1$ and $h_0 \neq h_2$, case (ii) results. Given that $h_0 = h_1$, it is clear that if we cut x and y where $h(x) = h_0$ and $h(y) = h_2$, then because $n \geq 3$ there are at least two vertices in partite set A with the same value $[h + \text{def}_{G'-\{x,y\}}] = h_0 + \text{def}_0$, but that $[h + \text{def}_{G'-\{x,y\}}]$ of an h_1 vertex, $= h_1 + \text{def}_1$, cannot have $[h_1 + \text{def}_1] = [h_0 + \text{def}_0]$ unless all the vertices in set A and all the h_1 vertices in set B are without loss of generality sources in $G' - \{x, y\}$. This contradicts the fact that the vertices in partite sets A and B are incident.

If there are two h values on $V|_A$, the only remaining possibility is that there be two h values on $V|_B$. Then we have h_{1A}, h_{2A} vertices and neighborhoods and h_{1B}, h_{2B} vertices and neighborhoods. We necessarily have $h_{iA} = h_{iB}$ for $i = 1, 2$. Furthermore, without loss of generality $[h + \text{def}](v_{2A}) + 1 = [h + \text{def}](v_{1B}) - 1$, and $[h + \text{def}](v_{1A}) - 1 = [h + \text{def}](v_{2B} + 1)$. But then $\text{def}(v_{2A}) - 2 = \text{def}(v_{1B})$ and $\text{def}(v_{1A}) - 2 = \text{def}(v_{2B})$. Therefore we have sets A_1,A_2,B_1,B_2 where A_1 beats B_2

beats A_2 beats B_1 beats A_1. The sizes of the sets are

$$|B_2| - |B_1| - 2 = |A_1| - |A_2|.$$

However, then $2|B_2| - 2 = 2|A_1|$ and $2|B_1| + 2 = 2|A_2|$. Therefore, $|A_1| = |A_2| = |B_2| - 1 = |B_1| + 1$. But then there are 2 def values on $V|_A$ and we get h_1+def$_0$+2 and h_2+def$_0$−2 for values of $[h+\text{def}]$ on $V|_A$ and h_1+def$_0$ and h_2+def$_0$ on $V|_B$. If we assign an h_1 vertex from A then and an h_2 vertex from B to x and y, we get that we have a vertex from A with $[h+\text{def}_{G'-\{x,y\}}](v_{2A}) = h_2+\text{def}_0$ and a vertex from B with $[h+\text{def}_{G'-\{x,y\}}](v_{1B}) = h_1+\text{def}_0-2$ when we cut x and y. So then $h_1 - 2 = h_2$. Then assign an h_2 vertex from A and an h_1 vertex from B to x and y. We get that we have a vertex from A with $[h + \text{def}_{G'-\{x,y\}}](v_{1A}) = h_1 + \text{def}_0 + 4$ and a vertex from B with $[h + \text{def}_{G'-\{x,y\}}](v_{2B}) = h_2 + \text{def}_0 - 2$ when we cut x and y. So then $h_2 - 2 = h_1 - 2 - 2 = h_1 - 4 = h_1 + 4$. This final contradiction completes the proof.

So there is one h value on $V|_A$ and $V|_B$ and one associated neighborhood and case (i) results. To see that $[h + \text{def}](x) = [h + \text{def}](y) - 2(n - 1)$ also results, notice that if $\text{def}(x) = n$, $\text{def}(y) = -n$, (because team A beats team B) and h is fixed on A and h is fixed on B. When we cut x and y, we get $\text{def}(v_{1A}) + h(v_{1A}) = \text{def}(v_{2B}) + h(v_{2B})$ and so $n - 1 + h(v_{1A}) = -n + 1 + h(v_{1B})$ and so $h(v_{1A}) - h(v_{1B}) = -2(n - 1)$. ■

Theorem 6.25 *For all $n \geq 3$ and every feasible initial condition $h : \vec{K}_{n,n} \longrightarrow Z$ other than $[h + \text{def}](v) = c$ there is an $(0,0)$-avoiding irregular $(0,1)$-labeling.*

Proof: Let the bipartition of $V(\vec{K}_{n,n})$ be $A \cup B \equiv V(\vec{K}_{n,n})$. Suppose $n \leq 4$. Consider digraph $G' = \vec{K}_{n,n}$ with the initial condition h the zero function. Then

6.13 and 6.23 verify the theorem under these conditions, to get an irregular labeling f^*.

Let

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).$$

If $[h' + def_{G'-\{x,y\}}]$ is fixed over $G'' \equiv G' - \{x, y\}$, then apply Theorem 6.15 and Lemma 6.16 in the case $n = 3$, apply induction in the case $n = 4$ and then in both cases utilize $f' = 1 - f$ where f' is an irregular labeling of G'' under h' to complete the irregular $(0, 1)$-labeling of G'. (Here h' is feasible by Lemma 6.20.) To see that in this case, $f^* = f'$ on G'' and $f^* = 1$ on $G' - G''$ is irregular and $(0, 0)$-avoiding, notice that $[h + \text{def}]$ is distinct on $V(G'')$ under f^* by definition of h' and f' and that x and y have the greatest total weight and furthermore $[h + \text{def}]x \neq [h + \text{def}]y$ under f^*. Finally, f^* is $(0, 0)$-avoiding because the total weight of both x and y is n. On the other hand, suppose $[h' + \text{def}_{G''}]$ is not fixed. Assign a G''-spanning double star of 0-labels to arcs incident with x_1 and y_1 where x_1 and y_1 are as in Lemma 6.24, Case (ii). (The function h' remains feasible on this restriction by Lemma 6.22.) Call this assignment of 0-labels f''. Let $G''' = G'' - \{x_1, y_1\}$ under $h'' = h'$. The labeling can be completed by producing (f''', G''', h'') which exists by induction. Again form f^* where

$$f^*(a) = 1 \quad \text{if } a \in A(G' - G'')$$

$$f^*(a) = 0 \quad \text{if } a \in A(G'' - G''')$$

$$f^*(a) = f'''(a) \quad \text{if } a \in A(G''')$$

To see that f^* is irregular and $(0,0)$ avoiding notice that f^* produces $g(x) \neq g(y)$ and that f^* produces $g(x_1) \neq g(y_1)$ because $[h + \text{def}]x \neq [h + \text{def}]y$ and $h'(x_1) \neq h'(y_1)$. Finally observe that f^* is distinct on G''' and that the total weights of the vertices in G''' are properly between n (the weight of x) and 1 (the weight of x_1). Since the weight of x_1 and y_1 are both greater than zero it is clear the labeling is $(0,0)$-avoiding.

If x and y do not exist as in case (ii), then the digraph G'' is in Case (i) of Lemma 6.24. Queue the vertices of G'' and each partite set $x_1 = v_{1A}, v_{2A}, ..., v_{(n-1)A}$; $y_1 = v_{1B}, v_{2B}, ..., v_{(n-1)B}$ respectively. Label the arc between v_{iA} and v_{jB} with a 1 if $i + j > n - 1$ and a zero otherwise. It follows that $[h + \text{def}](v_{iA}) - [h + \text{def}(v_{iB})] = 2i - 2(n-2) < 0$ for all tiers of the construction. Furthermore, different tiers have different total weight. In this case, f^* induces $g(v)$ which is injective over $V(G'')$; also, the total weight of x and y is n which assures the labeling is $(0,0)$-avoiding and additionally demonstrates that because this is greater than the total weight of any of the vertices in G'' and because $[h + \text{def}]x \neq [h + \text{def}]y$ we have the $(0,1)$-labeling f^* is irregular. Now we consider the cases when $n \geq 5$.

Case 1: Now suppose $n \geq 5$, n odd. In this case, we can assign a G'-spanning double star of 1-labels centered at $\{x, y\}$ that have $[h + \text{def}](x) \neq [h + \text{def}](y)$.

Subcase 1.1: If the set $V(G') - \{x, y\}$ fixes $[h + def]$ use the labeling of $f' = 1 - f$ of $G'' = G' - \{x, y\}$ under h' where

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$

$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n})$$

where f is available by induction. (The function h' is either feasible or the labeling can be completed immediately by Lemma 6.20.) Define

$$f^*(a) = 1 \quad \text{if } a \in A(G' - G'')$$

$$f^*(a) = f'(a) \quad \text{if } a \in A(G'' - G''').$$

To see that f^* is irregular notice, $[h + \text{def}](x) \neq [h + \text{def}](y)$ and that f^* is irregular on G'' by definition of h' and f'. Furthermore, f is $(0, 0)$-avoiding and so the total weight of x and y is n which is greater than any of the vertices in G'' and large enough to show that f^* is $(0, 0)$-avoiding.

Subcase 1.2: If the function $[h' + \text{def}_{G''}]$ is not fixed use Lemma 6.24 to obtain a pair of vertices x_1 and y_1 both $\in V(G'')$ which when labeled in a G''-spanning double star of 0-labels, leaves $[h' + \text{def}_{G''-\{x_1,y_1\}}]$ non-fixed over $G''' \equiv G'' - \{x_1, y_1\}$ and $h'(x) \neq h'(y)$. (The function h' remains feasible by Lemma 6.22.) Use the inductive hypothesis to compute a labeling f''' on G'''

under $h'' = h'$ Define f^*

$$f^*(a) = 1 \quad \text{if } a \in A(G' - G'')$$

$$f^*(a) = 0 \quad \text{if } a \in A(G'' - G''')$$

$$f^*(a) = f'''(a) \quad \text{if } a \in A(G''').$$

Case 2: Now suppose $n \geq 6$ is even. This case is identical to Case 1.

■

Definition 6.26 *An (n)-avoiding irregular 2-labeling of a digraph $\vec{D}$ is an irregular 2-labeling such that no vertex has total weighted degree $= n$ under the irregular 2-labeling. An $(2n)$-avoiding irregular 2-labeling of a digraph $\vec{D}$ is an irregular 2-labeling such that no vertex has total weighted degree $= 2n$ under the irregular 2-labeling.*

Definition 6.27 *Let an orientation of $K_{n,n}$, called $\vec{D}$ be given. A function $h : V(\vec{D}) \longrightarrow Z$ is called a feasible initial condition in the (n)-avoiding case unless unless $\vec{D}$ is oriented so that team A beats team B, $h|_A = d$, $h|_B = c$ and $c - d \in (2n, 4n-2)$, and in the $(2n)$-avoiding case unless $\vec{D}$ is oriented so that team A beats team B, $h|_A = d$, $h|_B = c$ and $c - d \in (2n+2, 4n)$.*

Lemma 6.28 *Suppose $h : V(\vec{D}) \longrightarrow Z$ is feasible where $\vec{D}$ is an orientation of*

$K_{n,n}$ *(with* $n \leq 3$*). Then define* h'

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).$$

where x *and* y *are such that* $[h+\text{def}](x) \neq [h+\text{def}](y)$*. Then if* h' *is not feasible on* $\vec{D} - \{x, y\}$ *it follows that for some other choice of* x *and* y*,* h' *is feasible on* $\vec{D} - \{x, y\}$*.*

Proof: Let $A \cup B$ be the bipartition of $V(\vec{D})$. Without loss of generality, pick an element z of B such that it is not the only endpoint of a feedback arc incident x. (The situation could be reversed selecting a vertex that is not the only endpoint of a feedback arc incident y.) Pick an element $w \neq x$ from A. Then h' is feasible on $\vec{D} - \{w, z\}$. For either $h'(x) \neq h'|_A$ or x is incident with a feedback arc. ■

Lemma 6.29 *Suppose* $h : V(\vec{D}) \longrightarrow Z$ *is feasible where* $\vec{D}$ *is an orientation of* $K_{n,n}$*. Then define* h' *by*

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).$$

where x *and* y *are such that* $[h+\text{def}](x) \neq [h+\text{def}](y)$*. Then if* h' *is not feasible on* $\vec{D} - \{x, y\}$ *it follows that* $\vec{D}$ *has an* (n)*-avoiding irregular 2-labeling.*

Proof: The proof for Lemma 6.20 applies. ■

Theorem 6.30 *For $n = 3$, every orientation of $K_{n,n}$ has an (n)-avoiding and a $(2n)$-avoiding irregular 2-labeling.*

Proof: Consider the feedback graphs of size 1,2,3,4. There are two graphs of size 2; there are four graphs of size 3; and three graphs of size 4. (In some cases 2 placements of the feedback graphs must be considered.) In each case, the vertices of the orientation can be permuted so that if $i + j > k$ then the arc between v_{iA} and v_{jB} is labeled with a 2 and otherwise the arc is labeled with a 1 is an irregular labeling. The value of k is 3 in the (n)-avoiding case and 4 in the $(2n)$-avoiding case. See Appendix A for the completed proof. ■

Theorem 6.31 *Every orientation of C_4 with every feasible initial condition has an (n)-avoiding and a $(2n)$-avoiding irregular 2-labeling except in the two cases forbidden by Theorem 6.15.*

Proof: The proof is by cases, Appendix A. ■

Lemma 6.32 *Let $h : V(\vec{K}_{3,3})$ be a feasible initial condition on an orientation of $K_{3,3}$, called G'. If the leave of a double star $\{xv_{1B}, xv_{2B}, xy, yv_{1A}, yv_{2A}\}$ where $[\text{def} + h](x) \neq [\text{def} + h](y)$, is one of the exceptional graphs (i.) or (ii.) from Theorem 6.15, then without loss of generality $[\text{def} + h](x) \neq [\text{def} + h](v_{1B})$, and the leave of $\vec{K}_{3,3} - \{xv_{1B}, xv_{2B}, xy, v_{1B}v_{1A}, v_{1B}v_{2A}\}$ is not (i.) or (ii.) under the initial condition h' with h' defined by*

$$h'(v) = h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n})$$
$$h'(v) = h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).$$

Proof:

Case 1: If we have an exceptional graph (i.) or (ii.) under forbidden conditions, and we select v_{1B}, and get the same exceptional graph under forbidden conditions, then selecting v_{2B} necessarily gives the other exceptional graph under non-forbidden feasible initial conditions except when we are in Case 2. We get

$$[h', h', h', h'] \longrightarrow [h', h^*, h', h'] \neq [h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2],$$

or

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0, h^*, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h'].$$

If permuting v_{1B} and y does not return the same exceptional graph under forbidden conditions, it returns a graph other than the ones underlying (i.), or (ii.), or the same exceptional graph under allowable conditions, or the other graph from the set under allowable conditions:

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0, h^*, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h']$$

$$[h', h', h', h'] \longrightarrow [h', h^*, h', h'] \neq [h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2].$$

Case 2: In the exceptional case that $[\text{def} + h](x) = [\text{def} + h](v_{1B})$ or $= [\text{def} + h](v_{2B})$, and (ii.) is the leave, then we swap v_{1A} or v_{2A} for x, whichever vertex does not have the same $[h+\text{def}]$ value as y; they are not the same. Under this permutation, there is either

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0 + 4, h'_0 + 4, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h'],$$

or

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h'],$$

or

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0 + 2, h'_0 + 4, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h'],$$

or

$$[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \longrightarrow [h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2] \neq [h', h', h', h'],$$

and of course the charge matrices under permutation do not match $[h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2]$.

In the case that $h'(x) = h'(v_{1B}) = h'(v_{2B})$, and (i.) is the leave, then we swap v_{1A} or v_{2A} for x. Since $[\text{def} + h'](v_{1A}) = [\text{def} + h'](v_{1B}) \neq [\text{def} + h'](y)$ this permutation gives a leave which does not have one of the forbidden orientation-initial feasible condition combinations. If the orientation changes to (i.), the fact that

$$[h^*, h', h', h'] \neq [h'_0, h'_0 + 4, h'_0 + 6, h'_0 + 2]$$

completes the proof. ■

Theorem 6.33 *For $n = 3$ every feasible initial condition $h : \vec{K}_{n,n} \longrightarrow Z$ gives rise to a $(2n)$- and (n)-avoiding irregular 2-labeling.*

Proof: Let an orientation of $K_{3,3}$, called $G' \equiv \vec{K}_{3,3}$, be given.

Suppose $[h + \text{def}]$ is fixed by the vertex set of G'. Obtain a $(0, 0)$-avoiding $(0, 1)$ labeling f from Theorem 6.25 and let $f' = 1 + f$; then f' is a $(2n)$-avoiding irregular 2-labeling of G'. Suppose the labeling were not irregular. Then there exists x and y in G' such that under h' we have $g_f(x) = g_f(y)$, contrary to construction.

Let x and y be such that $[h + \text{def}](x) \neq [h + \text{def}](y)$. Label a G'-spanning double star centered at x and y with 1-labels. Then $G' - \{x, y\}$ is an orientation of C_4. For the restriction of h' to $V(G' - \{x, y\})$, define

$$
\begin{aligned}
h'(v) &= h(v) + 1 \quad \text{if } \vec{vx} \in A(\vec{K}_{n,n}) \\
h'(v) &= h(v) - 1 \quad \text{if } \vec{xv} \in A(\vec{K}_{n,n}) \\
h'(v) &= h(v) + 1 \quad \text{if } \vec{vy} \in A(\vec{K}_{n,n}) \\
h'(v) &= h(v) - 1 \quad \text{if } \vec{yv} \in A(\vec{K}_{n,n}).
\end{aligned}
$$

(The function h' is feasible by Lemmas 6.28 and 6.29.) Notice, Lemma 6.32 allows that x and y can be selected so that Theorem 6.31 applies, and there exists an (n)-avoiding irregular 2-labeling of the oriented C_4, call it f'. Define $f = f'$ on $G' - \{x, y\}$ and $f = 1$ on $G' - (G' - \{x, y\})$. Then f is a $(2n)$-avoiding irregular 2-labeling of G'. To see this, notice there are vertices in both partite sets incident with only 1-labels: x and y. Consequently, the vertices x and y are the only two vertices of total weight $= 3$, and they have $g_f(x) \neq g_f(y)$. Finally, the vertices in $G' - \{x, y\}$ have g_f distinct by the definition of h' and f.

In the (n)-avoiding case, suppose $[h + 2\text{def}]$ is fixed by the vertex set of G'. Obtain a $(0,0)$-avoiding $(0,1)$ labeling f from Theorem 6.25, and let $f' = 2 - f$; then f' is an (n)-avoiding irregular 2-labeling of G'. Suppose the labeling were not irregular. Then $g_f(x) = g_f(y)$, under $h' = 0$, contrary to construction.

On the other hand, let x and y be such that $[h + 2\text{def}](x) \neq [h + 2\text{def}](y)$. Label a G'-spanning double star centered at x and y with 2-labels. Then $G' - \{x, y\}$ is an orientation of C_4. For the restriction of h' to $V(G' - \{x, y\})$, define h' which is feasible by Lemma 6.29:

$$\begin{aligned} h'(v) &= h(v) + 1 \quad && \text{if } \vec{vx} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) - 1 \quad && \text{if } \vec{xv} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) + 1 \quad && \text{if } \vec{vy} \in A(\vec{K}_{n,n}) \\ h'(v) &= h(v) - 1 \quad && \text{if } \vec{yv} \in A(\vec{K}_{n,n}). \end{aligned}$$

Notice, Lemma 6.32 allows that x and y can be selected so that Theorem 6.31 applies and there exists an $(2n)$-avoiding irregular 2-labeling of the oriented C_4, call it f'. Define $f = f'$ on $G' - \{x, y\}$ and $= 2$ on $G' - (G' - \{x, y\})$. Then f is an (n)-avoiding irregular 2-labeling of G'. To see this, notice there are vertices in both partite sets incident with only 2-labels: x and y. The vertices x and y are the only two vertices of total weight $= 6$ and they have $g_f(x) \neq g_f(y)$. Finally, the vertices in $G' - \{x, y\}$ have g_f distinct by the definition of h' and f.

■

Theorem 6.34 *Every diregular orientation of $\vec{K}_{4,4}$ has an (n)-avoiding and a*

$(2n)$-avoiding irregular 2-labeling. Moreover, the (n)-avoiding labelings can be formed from the $(2n)$-avoiding labelings and vice versa because the digraph is regular.

Proof: The proof is by two part construction. There are two cases to consider, one where the feedback graph is a Hamiltonian cycle and one where the feedback graph is a spanning pair of disjoint C_4s. To see that the theorem holds, notice the two figures Figure 6.1 and 6.2 show that there is an (n)-avoiding irregular 2-labeling in each case. In each case, let f be the function which describes the 2-labeling. Let $f'(a) = 3 - f(a)$ for all arcs a in the digraph D, and obtain a 2-labeling in each case which is $(2n)$-avoiding. It is easy to observe that the labeling is irregular and the fact that the labeling is $(2n)$-avoiding in each case follows from the fact that there is no vertex of total weight (n) under f'; if $g_{f'}(v) = (n, n)$ then the arcs incident v are labeled with 2 under f' which by definition of f' implies all the arcs incident with v are labeled with a 1 under f. So then f is not (n)-avoiding contrary to construction. ■

Theorem 6.35 *Every orientation of $K_{4,4}$ has an (n)-avoiding irregular 2-labeling.*

Proof: Let an orientation of $K_{4,4}$ called G' be given. The orientation is not regular so that there exists a pair of vertices x and y such that $2\text{def}(x) \neq 2\text{def}(y)$. Label a G'-spanning double star centered at x and y with 2-labels. Let h' be defined by $h'(v) = h+2$ if v dominates x or y and $h'(v) = h-2$ if v is dominated by x or y. If $[h+\text{def}]$ on is fixed by $V(G'-\{x, y\})$ then obtain the (n)-avoiding 2-labeling f which irregularly labels of $G'-\{x, y\}$ under h'. Let $f' = 3-f$ and label

$G' - \{x, y\}$ by f'. Let f^* be defined by $f^*(a) = 2$ for $a \in A(G' - (G' - \{x, y\}))$ and $f^*(a) = f'(a)$ otherwise. Then f^* is an (n)-avoiding irregular 2-labeling of G'. We have $g_{f^*}(x) \neq g_{f^*}(y)$. By definition h' and by definition f' it follows that g_{f^*} is distinct on $V(G' - \{x, y\})$. The total weight of x and y is equals 8 and the total weight of any other vertex is less than 8 since f was (n)-avoiding and we considered $f' = 3 - f$ (which is thus necessarily $(2n)$-avoiding). So the function f^* is (n)-avoiding, irregular, and uses two labels. If $[h + \text{def}]$ is not fixed by the vertices in $G' - \{x, y\}$ it follows that by Theorem 6.33 there exists f, an irregular $(2n)$-avoiding labeling that uses two labels under h'. So let $f' = 3 - f$ and let $f^*(a) = 2$ for $a \in A(G' - (G' - \{x, y\}))$ and $f^*(a) = f'(a)$ if $a \in A(G' - \{x, y\})$. This labeling f^* is (n)-avoiding irregular and uses two labels. In each case h' is and remains feasible by Lemma 6.28 and 6.29 or else we can complete the labeling in any case by Lemma 6.30 and 6.29. ■

Theorem 6.36 *Every orientation of $K_{4,4}$ has a $(2n)$-avoiding irregular 2-labeling.*

Proof: Assign a double star xv_4,xv_5,xv_6,xy,yv_1,yv_2,yv_3 and such that $\text{def}(x) \neq \text{def}(y)$ and label all its arcs with 1. If the resulting $K_{3,3}$ is such that $2\text{def}(v)$ is fixed, use the labeling $f' = 3 - f$ where f is given by Theorem 6.30 and is $(2n)$-avoiding to complete the arc labeling; the completed arc labeling is $(2n)$-avoiding and uses two labels; to see that it is irregular check the weighted degree pairs on $V(\vec{K}_{3,3})$. For two degree pairs to match, we would have to have $g_f(v) = g_f(w)$ under the original labeling and this is contrary to the fact that f is irregular on $\vec{K}_{3,3}$. In the remaining case, use Theorem 6.33 directly, this time using an f given by the theorem which is (n)-avoiding. (Here we have a feasible initial condition induced by the arcs of x and y.) Similar to the previous theorem, we define h' in the standard way and notice it is and remains feasible by previous lemmas. ■

Theorem 6.37 *For all $n \geq 3$ and every feasible initial condition $h : \vec{K}_{n,n} \longrightarrow Z$ where $[h + \text{def}]$ is not fixed there is an (n)-avoidable irregular 2-labeling. If $[h + 2\text{def}]$ is not fixed, there is a $(2n)$-avoidable irregular 2-labeling.*

Proof: Let a digraph $G' = \vec{K}_{n,n}$ with the initial condition $h = 0$ and $n \leq 4$ be given. Then by Theorems 6.30, 6.35, and 6.36 the theorem is verified. Now consider the case where h is not trivial. For $n = 3$, Theorem 6.33, establishes the result. For $n = 4$, in the $(2n)$-avoiding case, pick x and y such that $[h+\text{def}]$ is not identical on x and y and label a double star of 1-labels that spans the digraph and is centered at these two vertices. Then if the remaining digraph, $G' - \{x, y\}$, has $[h+2\text{def}]$ non-fixed, apply Theorem 6.33 in the (n)-avoiding case. (Augment h to h' as before.) The labeling obtained by pasting together the labeling from Theorem 6.33 and the labeling on the rest of the digraph is irregular: the vertices $v \in V(\vec{K}_{3,3})$ all have $g(v)$ distinct and x and y have the smallest total weight of any vertices in the digraph less than the total weight of the vertices in the $K_{3,3}$. If the remaining digraph has $[h+2\text{def}]$ fixed, apply Theorem 6.33 to obtain f, an $(2n)$-avoiding irregular 2-labeling and use $f' = 3 - f$ to label the arcs of the $K_{3,3}$. Again, the vertices of $\vec{K}_{3,3}$ all have $g(v)$ distinct and x and y have the lowest possible total weight less than the weight of any of the other vertices. By choice of x and y, $g(x) \neq g(y)$. Let $n \geq 5$ be odd. Then there exists a pair of vertices x and y such that labeling a double star incident with these two vertices x and y alternatively with 1 labels or 2 labels leaves $[h + \text{def}]$ or $[h + 2\text{def}]$ of x and y distinct. In the case that $[h + 2\text{def}]$ or $[h + \text{def}]$ is not fixed (on $V - \{x, y\}$); induction on the remaining vertices $V - \{x, y\}$ gives (n)- and $(2n)$-avoiding irregular 2 labelings and completes the proof. In the case that $[h + 2\text{def}]$ is fixed use $3 - f$ where f is given by induction and (n)-avoiding, or in the case where $[h + \text{def}]$ is fixed use $3 - f$ where f is given by induction and $(2n)$-avoiding.

In the case where $n \geq 6$ and even, $\vec{K}_{n,n}$ is not regular, so pick x and y such that either $[h + \text{def}]$ or $[h + 2\text{def}]$ are distinct depending on whether we want an (n)- or a $(2n)$-avoiding irregular labeling and label a double star with 1 labels or 2 labels. Then in the case that $[h + \text{def}]$ or $[h + 2\text{def}]$ is fixed on the remaining vertices use $3 - f$ where f is given in either the $(2n)$- or (n)-avoiding case. In the case that these values are not fixed use induction. ■

This last theorem, taken together with Theorem 6.17 shows that $\vec{s}(\vec{K}_{n,n}) = 2$ where $\vec{K}_{n,n}$ is any orientation of $K_{n,n}$ and $n > 1$.

7. On Path Irregularity Strength and Disparity

Some of the following labeling algorithms depend on a lemma called the Rooted Circuit Packing Lemma (RCPL). For the purpose of this chapter, let $\vec{K}_n$ denote the complete symmetric digraph with loops on n vertices.

7.1 Paths, Cycles, and the Rooted Circuit Packing Lemma

Consider the problem of labeling a directed cycle and directed path irregularly. Because every vertex of $\vec{C}_k$ has degree (1,1), an irregular labeling of $\vec{C}_k$ corresponds to a cyclic integer sequence of length k where no consecutive ordered pair occurs more than once. These sequences are similar to deBruijn sequences [8]. Finding a k-element deBruijn-like sequence over an alphabet of n symbols is equivalent to finding a circuit of length k in $\vec{K}_n$. As an example, the following figure gives the irregular labeling of $\vec{C}_8$ associated with the deBruijn-like sequence 11312332 as well as the associated circuit in $\vec{K}_3$ [18]. It is shown

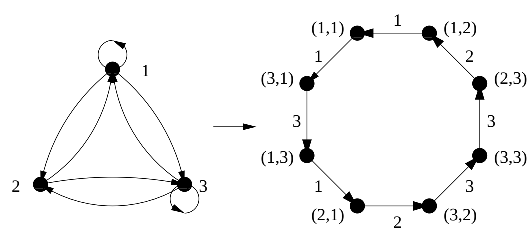


Figure 7.1: The closed trail in $\vec{K}_3$ and irregular labeling of $\vec{C}_8$ associated with the deBruijn-like sequence 11312332.

in Lemma 7.5 that $\vec{K}_n$ contains a circuit of length k for any $k \leq n^2$. This fact, along with Corollary 4.4, makes it easier to calculate the irregularity strength of $\vec{C}_k$.

Theorem 7.1 ([18]) *For any directed cycle $\vec{C}_n$ of order n, $\vec{s}(\vec{C}_n) = \lceil \sqrt{n} \rceil$.*

In a similar manner, circuits in $\vec{K}_n$ generate irregular labelings of $\vec{P}_k$ ($k \leq n^2$). The directed path $\vec{P}_k$ has $k-2$ vertices of degree $(1,1)$, each of which would correspond to a distinct arc in $\vec{K}_n$ under an irregular labeling.

Theorem 7.2 ([18]) *For any directed path $\vec{P}_n$ of order $n \geq 3$, $\vec{s}(\vec{P}_n) = \lceil \sqrt{n-2} \rceil$.*

For completeness, note $\vec{P}_2$ has irregularity strength 1.

7.1.1 Unions of Directed Cycles

The problems of irregularly labeling a disjoint union of directed paths and cycles are slightly different. An analysis identical to that given above shows a set of arc-disjoint circuits in $\vec{K}_n$ of lengths n_1,...,n_k correspond to an irregular labeling of $\bigcup_{i=1}^{i=k} C_{n_i}$. We apply the following theorem of Balister [4] about packing arc-disjoint circuits into a complete symmetric digraph without loops to get a general result about the irregularity strength of a union of cycles.

Theorem 7.3 ([4]) *If $\sum_{i=1}^{i=k} n_i \leq n(n-1)$ and $n_i \geq 2$ for $i = 1, ..., k$ then the complete symmetric digraph of order n* without *loops contains k arc-disjoint circuits of lengths $n_1, ..., n_k$ except in the case when $n = 6$, $k = 10$, and all $n_i = 3$.*

Using the circuits from Theorem 7.3 and the fact that $\vec{K}_6$ decomposes into 12 circuits of length 3, we get the following corollary.

Corollary 7.4 ([18]) *Let $D = \bigcup C_{n_i}$, with $n_i \geq 2$ and $\sum n_i = m$. If $(n-1)^2 < m \leq n(n-1)$ then $\vec{s}(D) = n$.*

7.1.2 Unions of Directed Paths

Recall that an irregular labeling of a union of directed paths requires that each out-leaf (in-leaf) has a distinct label. For this reason, Theorem 7.3 does not apply in this case. The following rooted circuit packing lemma helps to construct an irregular labeling of minimum strength for a union of directed paths. Let $V(\vec{K}_n) = \{v_1, v_2, ..., v_n\}$ and say that a circuit C is rooted at v_i if $v_i \in V(C)$. Additionally, a circuit C in $\vec{K}_n$ is symmetric if $\vec{uv} \in A(C)$ implies $\vec{vu} \in A(C)$.

Lemma 7.5 (The Rooted Circuit Packing Lemma [18]) *Let $n \geq 3$ be an integer, and let $n_1, \ldots, n_k$ be nonnegative integers such that $k \leq n$ and $\sum_{i=1}^{i=k} n_i \leq n^2$. Then there exist arc-disjoint circuits $C_1, \ldots, C_k$ in $\vec{K}_n$ such that each C_i has length n_i and is rooted at v_i, except in the case where $n = 3,\ k = 3,\ n_1 = n_2 = 2$ and $n_3 = 5$.*

Proof: The proof is by double induction on n and k. Let $k \leq n$, and $\{n_i\}_{i=1}^{k}$ be non-decreasing such that $\sum_{i=1}^{i=k} n_i \leq n^2$. We prove the slightly stronger statement that for $n \geq 3$ there exists, with the stated exception, a set of arc-disjoint symmetric rooted circuits $\{C_i\}_{i=1}^{i=k}$ in $\vec{K}_n$ having the stated properties. For the basis step, we need to verify all cases where $n = 3$, and the base case $k = 1$ for arbitrary n. It is easy for the reader to check that every choice of $\{n_i\}_{i=1}^{i=3}$ for $n = 3$ except for $n_1 = 2, n_2 = 2, n_3 = 5$ will result in an appropriate packing. To show that the exceptional case is not feasible, we simply note that the only way to remove two 2-cycles from $\vec{K}_3$ leaves a disconnected digraph of size 5.

To show that the case $k = 1$ holds for all n, let $v_i\vec{v}_j$ denote the 2-cycle $v_iv_jv_i$ where $i < j$. Order the 2-cycles so that $v_i\vec{v}_j$ is ordered before $v_k\vec{v}_h$ if $i < k$, or $i = k$ and $j < h$ (in other words, via the lexicographic ordering). If $n_1 \leq n(n-1)$ is odd, let $E(C_1)$ be the loop at v_1 and the first $\frac{n_1-1}{2}$ 2-cycles. If $n_1 \leq n(n-1)$ is even, let $E(C_1)$ be the first $\frac{n_1}{2}$ 2-cycles. If $n_1 > n(n-1)$, add any $n_1 - n(n-1)$ remaining loops of $\vec{K}_n$ to the circuit composed of all of the $\binom{n}{2}$ 2-cycles of $\vec{K}_n$.

For the remainder of the proof, retain the notation $v_i\vec{v}_j$ to represent a 2-cycle, but it is no longer necessary to require $i < j$. Moving forward, let $n \geq 4$, $k \leq n$, and suppose the theorem holds for all smaller values of n and k. Consider two cases:

Case 1: $n_k \geq 2n - 1$.

In the case $n = 4,\ n_1 = 2,\ n_2 = 2,\ n_3 = 5,\ n_4 = 7$ refer to the following figure. For the remainder of the cases we are free to use the inductive hypothesis.

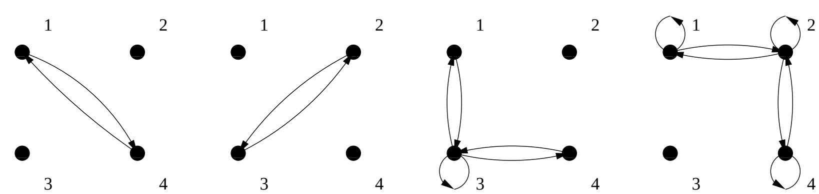


Figure 7.2: A symmetric rooted circuit decomposition of $\vec{K}_4$ with $n_1 = 2$, $n_2 = 2$, $n_3 = 5$, $n_4 = 7$.

Since $\sum_{i=1}^{i=k-1} n_i \leq (n-1)^2$, there are $k-1$ edge-disjoint symmetric rooted circuits $\{C_i\}_{i=1}^{i=k-1}$ in $\vec{K}_n - v_k$. Note that the removal of these $k-1$ symmetric circuits from $\vec{K}_n$ leaves only loops and 2-cycles, in particular the loop at v_k and all 2-cycles of the form $v_i\vec{v}_k$. We construct C_k using all 2-cycles of the form $v_i\vec{v}_k$ and enough additional 2-cycles and loops to assure that C_k has a total of n_k

arcs. Then $\{C_i\}_{i=1}^{i=k}$ is the required set of symmetric rooted circuits.

Case 2: $n_k < 2n - 1$.

Let $\sigma = 2n - 1 - n_k$.

Subcase 2.1: $(\sigma = 1)$

If $n_i \leq 2$ for all $1 \leq i \leq k-1$ then as $n \geq 4$ it is not difficult to pack a set of symmetric rooted circuits $\{C_i\}_{i=1}^{i=k}$ in $\vec{K}_n$. Consequently, assume $n_{k-1} \geq 3$ and we let $n'_{k-1} = n_{k-1} - 3$. Then $\sum_{i=1}^{i=k-2} n_i + n'_{k-1} \leq (n-1)^2 - 2$ for $\sum_{i=1}^{i=k-1} n_i \leq (n-1)^2 + 1$. By induction, $\vec{K}_n - v_k$ contains arc-disjoint symmetric rooted circuits $\{C_i\}_{i=1}^{i=k-2} \cup \left\{C'_{k-1}\right\}$ where C_i has length n_i for $1 \leq i \leq k-2$ and C'_{k-1} has length n'_{k-1}. Additionally, as the lengths of these circuits sum to at most $(n-1)^2-2$, removing these $k-1$ circuits from $\vec{K}_n - v_k$ leaves at least one 2-cycle or two loops.

Assume that the 2-cycle $\vec{v_i v_j}$ is in $\vec{K}_n - v_k$ and does not lie in $C_1, \ldots, C_{k-2}$ or C'_{k-1}. Augment C'_{k-1} to create a symmetric circuit of length n_{k-1} rooted at v_{k-1} by adding the 2-cycle $\vec{v_{k-1} v_k}$ and the loop at v_k to C'_{k-1}. Construct the circuit C_k by taking the $n-2$ 2-cycles adjacent to v_k in $\vec{K}_n$ that do not lie in C_{k-1}, along with the 2-cycle $\vec{v_i v_j}$. Then C_k is symmetric and rooted at v_k with $n_k = |E(C_k)|$. Therefore $\{C_i\}_{i=1}^{i=k}$ is a set of arc-disjoint symmetric rooted circuits of the desired lengths.

Now suppose that the loops at v_i and v_j, where neither i nor j is equal to k remain when we remove $C_1, \ldots, C_{k-2}, C'_{k-1}$ from $\vec{K}_n - v_k$. If v_{k-1} is not incident with either of these loops, construct C_k from these loops and all of the 2-cycles containing v_k except $\vec{v_{k-1} v_k}$ and augment C'_{k-1} with the loop at v_k and the 2-cycle $\vec{v_{k-1} v_k}$. If, without loss of generality $i = k-1$ is incident with one of these loops, then we augment C'_{k-1} by adding $v_{k-1}v_kv_{k-1}$ and the loop at v_{k-1} to build C_{k-1}. Construct C_k by adding the loops at v_k and v_j and of the 2-cycles incident with v_k except $\overrightarrow{v_{k-1}v_k}$. Then $\{C_i\}_{i=1}^{i=k}$ is a set of symmetric rooted circuits that have the desired lengths.

Subcase 2.2: $(\sigma > 1)$

If $\sum_{i=1}^{i=k-1} n_i \leq (n-1)^2$, the goal is to remove v_k from $\vec{K}_n$ and invoke the inductive hypothesis for this copy of $\vec{K}_n - v_k$. Since $\sigma > 1$, apply the inductive hypothesis except when $n = 4$, $n_1 = 2$, $n_2 = 2$, $n_3 = 5$, and $n_4 = 5$ or 6. These cases follow from the decomposition given above for $n = 4$, $n_1 = 2$, $n_2 = 2$, $n_3 = 5$, $n_4 = 7$.

In any other case, apply the induction hypothesis to obtain symmetric circuits $C_1, \ldots, C_{k-1}$ in $\vec{K}_n - v_k$ with each C_i rooted at v_i. Since n_k is at most $2n-3$, construct C_k using any $\lfloor \frac{n_k}{2} \rfloor$ 2-cycles containing v_k and, if n_k is odd, the loop at v_k. Then, each of the circuits $C_1, \ldots, C_k$ is symmetric and rooted at the vertices $n_1, \ldots, n_k$, respectively.

Otherwise, $\sum_{i=1}^{i=k-1} n_i \geq (n-1)^2 + 1$. Since the n_i are non-decreasing, both n_{k-1} and n_k are at least n and therefore σ is at most $n-1$. Let $n'_{k-1} = n_{k-1} - \sigma \geq 1$. Invoke the inductive hypothesis and get a collection of symmetric rooted circuits $\{C_i\}_{i=1}^{i=k-2} \cup C'_{k-1}$ with C'_{k-1} having length n'_{k-1}. Note that only one of n_k and σ can be odd (since their sum is odd). If σ is odd, we create the symmetric circuit C_{k-1} of length n_{k-1} by adding the loop at v_k, the 2-cycle $\overrightarrow{v_{k-1}v_k}$ and $\frac{\sigma-3}{2}$ 2-cycles incident with v_k to $E(C'_{k-1})$. If σ is even, we create C_{k-1} by adding the 2-cycle $\overrightarrow{v_{k-1}v_k}$ and any $\frac{\sigma-2}{2}$ other 2-cycles containing v_k to $E(C'_{k-1})$. It is not difficult then to construct C_k from the remaining 2-cycles at

v_k and possibly the loop at v_k (if n_k is odd). This completes Subcase 2.2 and Case 2. With all cases exhausted, the proof is complete. ■

As a consequence of Lemma 7.5, several results follow:

Theorem 7.6 ([18]) *Let D be a union of t directed paths $P_{k_1}, P_{k_2}, ..., P_{k_t}$ where $k_t \geq 2$ for all k_t. Then $\vec{s}(D) = \max\{t, \left\lceil \sqrt{\sum_{i=1}^{i=t}(k_i - 2)} \right\rceil\}$.*

Proof: Let $n = \max\{t, \left\lceil \sqrt{\sum_{i=1}^{i=t}(k_i - 2)} \right\rceil\}$. Corollary 4.5 states that $\vec{s}(D) \geq n$. Lemma 7.5 implies that we can construct circuits $C_1, \ldots, C_t$ such that each C_i is rooted at vertex v_i in $\vec{K}_n$ and has length $k_i - 2$. For each i, let $s_1^i, s_2^i, \ldots, s_{k_i-2}^i$ with $s_1^i = i$ be the deBruijn-like sequence associated with C_i. If $a_1^i, \ldots, a_{k_i-1}^i$ denotes the arcs of P_{k_i} in order, we will label a_j^i with s_j^i for $1 \leq j \leq k_i - 2$ and label $a_{k_i-1}^i$ with i. Since the C_i were constructed to be arc-disjoint in $\vec{K}_n$ each of the vertices of degree $(1, 1)$ in D will have distinct weight. Additionally, the initial and terminal vertices of the i^{th} path will have weights $(0, i)$ and $(i, 0)$ respectively. It follows that this is an irregular labeling of D with maximum label at most n, and the result follows. ■

Using the Rooted Circuit Packing Lemma it is also possible to irregularly label some unions of cycles that were not addressed by Corollary 7.4.

Proposition 7.7 ([18]) *If D is a union of $k \leq n$ directed cycles, $C_{n_1}, C_{n_2}, ..., C_{n_k}$ where $n(n-1) < |D| \leq n^2$ then $\vec{s}(\bigcup_{i=1}^{i=k} C_{n_i}) = n$.*

7.2 Irregular Labelings of Antipaths

Recall that an antipath is an orientation of a path such that every vertex has either in-degree 0 or out-degree 0. Let A_n denote the antipath of order n whose first vertex has in-degree 0. In this section, we determine $\vec{s}(A_n)$.

Theorem 7.8 ([18]) *Let $n \geq 2$. Then*

$$\vec{s}(A_n) = \begin{cases} \lceil n/4 \rceil + 1, & \text{for} n \equiv 3 (mod\ 4) \\ \lceil n/4 \rceil, & otherwise. \end{cases}$$

Proof: The lower bound follows from the fact that the structure of A_n implies that in any irregular labeling there are $\lceil \frac{n}{2} \rceil$ vertices with weighted degrees of the form $(k, 0)$. Hence, some vertex v must have weight $(w, 0)$ where $w \geq \lceil \frac{n}{2} \rceil$. It follows that one of the (at most two) arcs adjacent to v has label $\ell \geq \lceil \frac{n}{4} \rceil$.

For the remainder of the proof let the vertices of A_n, in order, be $v_1, \ldots, v_n$ where by our assumption $deg^-(v_1) = 0$. We also allow $a_1, \ldots, a_{n-1}$ to denote the arcs of A_n in order. We proceed by considering the following cases.

Case 1: $n \equiv 1$ (mod 4),

For $n = 4k + 1$, construct an irregular labeling $\vec{w}_{4k+1}$ of A_{4k+1} that has the property that no arc receives a label greater than $k + 1 = \lceil \frac{n}{4} \rceil$. Additionally, $\vec{w}_{4k+1}$ will have the property that the arcs a_{3k-1}, a_{3k} and a_{3k+1} are labeled with $k + 1$ and these are the only arcs labeled $k + 1$. To construct $\vec{w}_5$, label the arcs of A_5 with the labels $1, 2, 2, 2$.

Proceeding recursively, define a labeling of A_{4k+5}, where $k \geq 1$ in the following way:

$$\vec{w}_{4k+5}(a_i) = \begin{cases} \vec{w}_{4k+1}(a_i) & 1 \leq i \leq 3k, \\ k+1 & i = 3k+1, \\ k+2 & 3k+2 \leq i \leq 3k+4, \\ \vec{w}_{4k+1}(a_{i-4}) & 3k+5 \leq i \leq 4k+4. \end{cases}$$

In other words, starting with a copy of A_{4k+1} labeled by $\vec{w}_{4k+1}$, create a copy of A_{4k+5} by inserting four appropriately directed arcs at v_{3k+1} and labeling them

$k+1,\ k+2,\ k+2$, and $k+2$. Let this arc-labeling be $\vec{w}_{4k+5}$. As an example, the figure demonstrates the process by $\vec{w}_9$ follows from $\vec{w}_5$. In this labeling

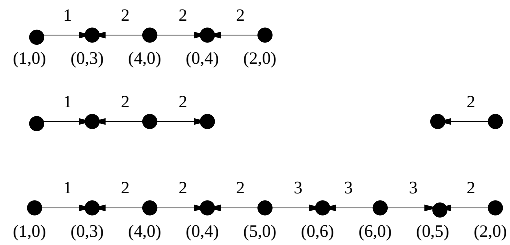


Figure 7.3: A minimal irregular labeling of A_5 and the resulting minimal labeling of A_9.

of A_{4k+5}, note that the condition that $a_{3(k+1)-1}$,$a_{3(k+1)}$ and $a_{3(k+1)+1}$ are all labeled with $k+2$ is satisfied. If k is odd, then $v_{3k}, \ldots, v_{3k+5}$ have induced weights $(2k+2,0), (0,2k+2), (2k+3,0), (0,2k+4), (2k+4,0)$ and $(0,2k+3)$, respectively. If k is even, then $v_{3k}, \ldots, v_{3k+5}$ have induced weights $(0,2k+2), (2k+2,0), (0,2k+3), (2k+4,0), (0,2k+4)$ and $(2k+3,0)$, respectively. In either case, by the assumptions that $\vec{w}_{4k+1}$ was an irregular weighting which assigned the weight $k+1$ to the arcs a_{3k-1}, a_{3k} and a_{3k+1} (in A_{4k+1}) it is not difficult to verify that the other vertex weights in A_{4k+5} are distinct and have in-weight or out-weight at most $2k+1$. Thus $\vec{w}_{4k+5}$ is an irregular labeling.

Case 2: $n \equiv 0 \pmod 4$,

This case is similar to Case 1: In the case that $n = 4$, label the antipath from left to right 1,1,1. In the case $n = 4k+4$, where $k \geq 1$, form an irregular labeling from the case $n = 4k$ by inserting four arcs at v_{3k} and labeling the arcs, in order, $k+1,\ k+1,\ k+1,\ k$.

Case 3: $n \equiv 2 \pmod 4$,

In the case $n = 2$, the digraph A_2 is a directed K_2 which has irregularity strength 1. In the case $n = 6$ label the arcs of A_6 from left to right 1,2,2,2,1. To label A_{4k+6}, for $k \geq 1$ we begin with the previous irregular labeling of A_{4k+2}, insert 4 arcs at v_{3k+1} and label these arcs, in order, $k+1$, $k+2$, $k+2$, $k+2$.

Case 4: $n \equiv 3 \pmod 4$

Let $n = 4k + 3$. First notice $\vec{s}(A_n) > k + 1$. In this orientation, there are $2k + 2$ vertices with in-degree 0. If the largest arc label was $k + 1$, then the vertices with in-degree 0 would have weighted degrees $(1, 0)$, $(2, 0)$,...,$(2k + 2, 0)$ and hence the out-weights on the arcs of A_n sum to $\frac{(2k+2)(2k+3)}{2}$. Now consider the fact that in any weighting of a digraph, the sum of the in-weights and out-weights are equal. There are only $2k + 1$ vertices with out-degree 0 and thus these vertices have $2k + 1$ weighted degrees from the set $(0, 1)$,...,$(0, 2k + 2)$. Since one weighted degree must be omitted, the sums of the in-weights must be strictly less than the sum of the out-weights. As such, the largest label used in an irregular labeling must be greater than $k + 1$.

With a recursive labeling as above, this new lower bound can be achieved. Label the arcs of A_3 from left to right with $1, 2$. Label the arcs of A_7 from left to right with $1, 2, 3, 3, 3, 2$. To label A_{4k+7}, where $k \geq 1$, begin with the previous irregular weighting of A_{4k+3}. Then insert four arcs at vertex v_{3k+2} and weight them, in order, $k + 2$, $k + 3$, $k + 3$, $k + 3$.

An argument similar to the one given in Case 1 suffices to show that the arc-labelings given are, in fact, irregular. ■

We now have examples of families of graphs G_n such that the value $disp(G_n)$ is constant with respect to n, (K_n, $K_{n,n}$) and a family, P_n, where $disp(G_n)$ grows in a nearly linear fashion with respect to n. There are other families of graphs where the behavior of $disp(G_n)$ is easy to describe such as $S_{1,n}$ and nK_2. We continue our investigation of the behavior of the function $disp$ in the next section.

7.3 More on Disparity

For planar graphs little is known about the the parameters $\vec{s}$ and $disp$, but there is one example where the disparity is large for large orders, P_k^3, which is maximal planar.

Definition 7.9 *The graph P_k^3 is the graph defined by vertex set $v_1, v_2, ..., v_k$ and edge set $\{v_iv_j : |i-j| \leq 3\}$.*

It is easy to show that P_k^3 is a maximal planar graph and that $\Delta(P_k^3) = 6$.

We will use the graph P_k^3 as an example where $disp(G)$ can be arbitrarily large.

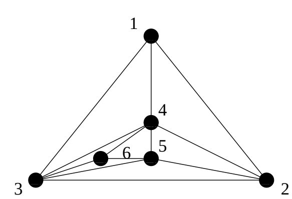


Figure 7.4: P_k^3.

Theorem 7.10 *The parameter $disp(P_k^3) \geq |\lceil \frac{k}{24} \rceil - \lceil \sqrt{k-2}+1 \rceil|$ for $k \geq 476$.*

Proof: By the fact that P_k^3 has a hamiltonian path, we can show that $s(D) \leq \sqrt{k-2}+1$ for some orientation D of P_k^3.

It remains to show that there is an orientation of P_k^3 that requires at least $k/24$ labels for $k \geq 2$.

To this end notice that there is a 4 coloring of P_3^k. We color v_i according to its modulus with respect to 4. (If $i \equiv 0 \ mod\ 4$ use color 4.) Now divide the vertex set of G into color classes and permute the colors so that the largest class, which must have size at least $|V(G)|/4$, be colored color 1. Color the other vertices in color classes 2-4 with colors 2-4 respectively. Now, orient P_k^3 in the following way: direct the arc from vertex x to vertex y whenever x and y are adjacent and the color of $x <$ the color of y. This leaves us with $\lceil k/4 \rceil$ vertices with 0 in-degree. In order for these vertices to be labeled differently, we need at least $(k/4)/\Delta(P_k^3)$ labels. For there are at most $\Delta(P_k^3)$ arcs leaving each vertex and if we suppose the labels of the vertices are consecutive, we get $\Delta(P_k^3)\vec{s} \geq k/4$, or rather, $\vec{s} \geq (k/4)/\Delta(P_k^3)$. But it is an easy exercise to show $\Delta(P_k^3)$ is 6 so then, $\vec{s} \geq k/24$. This proves the theorem because we have produced two orientations of P_k^3, one with a lower bound for $\vec{s} \geq \frac{k}{24}$, one with an upper bound for $\vec{s} \leq \sqrt{k-2}+1$, giving the desired lower bound for $disp(P_k^3)$. ■

While the disparity, *disp*, can grow arbitrarily large for arbitrarily large values of k, our theorem only guarantees around $k/24$ for large given values of k (order). This is much smaller than the conjectured largest $disp(G_n) = n - 1 - \lceil \frac{n-1}{2} \rceil$, where n indicates order.

7.3.1 The Band: $K_2 \times C_n$

Definition 7.11 *Let the arcs of a first $\vec{C}_n$ be $v_{1A}\vec{v}_{2A} = a_1, a_2, ..., a_n$ and the arcs of a second $\vec{C}_n$ be $v_{1B}\vec{v}_{2B} = b_1, b_2, ..., b_n$ where $v_{iA}\vec{v}_{iB} \in \vec{A}(D)$ for $1 \leq i \leq n$. We call this digraph with underlying graph $K_2 \times C_n$ the directed band $\vec{D} \equiv DB_n$.*

Theorem 7.12 *If $n \geq 38$, the irregularity strength of DB_n, has $\vec{s}(DB_n) = \lceil \sqrt{\frac{2n}{3}} \rceil$.*

The construction below achieves the value $s = \lceil\sqrt{\frac{2n}{3}}\rceil$ for $2n \geq 75$. An irregular labeling cannot be achieved with fewer labels.

Proof: Build an Eulerian circuit C_1 of maximal length $m_1 \geq \frac{2n}{3}$ in the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ vertices. (Because $h(x) = \lceil x\rceil$ and $g(x) = x^2$ are both increasing on $x \geq 1$ we get that $\lceil\sqrt{x}\ \rceil^2 \geq x$ so this construction can always be achieved.) Find a circuit $C_2 := v_1 = v_1^2 v_2^2, ..., v_{m_2}^2 v_1^2 = v_1$ of maximal length m_2 in the bipartite graph formed by the odd and even vertices of the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ vertices. Next, build an Eulerian circuit $C_3 := v_1 = v_1^3 v_2^3, ..., v_{m_3}^3 v_1^3 = v_1$ of maximal length m_3 on the odd vertices of the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ vertices including the loop at v_1. Finally, build an Eulerian circuit $C_4 := v_2 = v_1^4 v_2^4, ..., v_{m_4}^4 v_1^4 = v_2$ of maximal length m_4 on the even vertices of the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ vertices. Now the lengths of C_2, C_3, C_4 sum to the length of C_1. There are two cases depending on the parity of $\lceil\sqrt{\frac{2n}{3}}\rceil$. Notice that if the parity is even, the lengths of C_3 and C_4 sum to the length of C_2: $m_3 + m_4 = m_2$. In this case use the following labeling procedure. Let $m = |A(C_1)| + |A(C_2)| - n$. Build an Eulerian circuit of length $|A(C_1)| - m$ in the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ rooted at v_1; call it $C_1' := v_1 = v_1^1 v_2^1 \cdots v_{|A(C_1')|}^1 v_1^1 = v_1$. Let C_2' be an Eulerian circuit on $A(C_2) \cup \{v_3\vec{v}_3, v_2\vec{v}_2\} - \{v_1\vec{v}_2, v_2\vec{v}_1\}$. Let C_3' be an Eulerian circuit on $A(C_3) \cup A(C_4) \cup \{v_1\vec{v}_2, v_2\vec{v}_1\} - \{v_3\vec{v}_3, v_2\vec{v}_2\}$ with the loop at v_1 taken first. Arc label $a_i\vec{a}_{i+1}$ and $b_i\vec{b}_{i+1}$ with v_i^1 for $1 \leq i \leq |A(C_1')|$. Put labels v_i^3 on $b_{i+|A(C_1')|}\vec{b}_{i+|A(C_1')|+1}$ for $1 \leq i \leq |A(C_3')|$. Labels v_i^2 on $b_{i+|A(C_1')|}\vec{b}_{i+|A(C_1')|+1}$ for $1 \leq i \leq |A(C_2')|$. Finally arc label $v_{iA}v_{iB}$ with a 1-label for $1 \leq i \leq |A(C_1')|$ and a $\lceil\sqrt{\frac{2n}{3}}\rceil$-label for $|A(C_1)'| + 1 \leq i \leq n$.

Meanwhile, if the parity is odd, the lengths of C_3 and C_4 sum to 1 more than the length of $C_2 : m_3 + m_4 = m_2 + 1$. In this case, let $\sqrt{\frac{2n}{3}} \geq 5$ and use the following labeling procedure. Let C_2' be an Eulerian circuit of the arcs from $C_2 \cup \{v_1\vec{v}_3, v_3\vec{v}_1, v_5\vec{v}_1, v_1\vec{v}_5\} - \{v_1\vec{v}_2, v_2\vec{v}_1\}$. Let C_3' be an Eulerian circuit of $C_3 \cup C_4 \cup \{v_1\vec{v}_2, v_2\vec{v}_1\} - \{v_1\vec{v}_3, v_3\vec{v}_1, v_1\vec{v}_5, v_5\vec{v}_1\}$ with the loop at v_1 taken first. It follows that $|A(C_2')| = |A(C_3')|$. Let $m = |A(C_1)| + |A(C_2')| - n$. Build an Eulerian circuit of length $|A(C_1)| - m$ in the complete symmetric digraph on $\lceil\sqrt{\frac{2n}{3}}\rceil$ rooted at v_1 : call it $C_1' := v_1 = v_1^1 v_2^1 \cdots v_{|A(C_1')|}^1 v_1^1 = v_1$. Now put labels v_i^1 on $a_i\vec{a}_{i+1}$ and $b_i\vec{b}_{i+1}$ for $1 \leq i \leq |A(C_1')|$. Put labels v_i^3 on $b_{i+|A(C_1')|}\vec{b}_{i+|A(C_1')|+1}$ for $1 \leq i \leq |A(C_3')|$. Finally, put labels v_i^2 on $b_{i+|A(C_1')|}\vec{b}_{i+|A(C_1')|+1}$ for $1 \leq i \leq |A(C_2')|$. Next, arc label $v_{iA}\vec{v}_{iB}$ with a 1 for $1 \leq i \leq |A(C_1')|$ and an $\lceil\sqrt{\frac{2n}{3}}\rceil$ for $|A(C_1)'| + 1 \leq i \leq n$. We need to show that the vertex weights are all distinct and that there does not exist an irregular labeling using fewer labels. It is clear that the vertices a_i where $1 \leq i \leq |A(C_1')|$ and b_i where $1 \leq i \leq |A(C_1')|$ are distinct from one another and distinct from values where the indices of the vertices are greater than $|A(C_1')|$. The remainder of the vertices in the a-cycle are distinct from one another because they are generated by the rooted circuit packing lemma and the arc label going from the a-cycle to the b-cycle acts on all the ordered weighted vertex pairs in the same way. Furthermore, all these ordered pairs have either two odd arguments or two even arguments except the vertices labeled by the arcs $\{v_1\vec{v}_3, v_3\vec{v}_1, v_5\vec{v}_1, v_1\vec{v}_5\}$ augmented by an arc labeled with a 1 going from the a-cycle to the b-cycle; there are no collisions because we exclude $\{v_1\vec{v}_3, v_3\vec{v}_1, v_1\vec{v}_5, v_5\vec{v}_1\}$ from C_3' and the vertices on the b-cycle consequently have one odd and one even argument with the exception of the vertices incident with a 1-label and the arcs labeled by $\{v_1\vec{v}_2, v_2\vec{v}_1\}$. Because we remove these last two arcs from C_2 when we form C_2' there are no collisions. Of course, there are no internal collisions because the arc labels on the cycles come from the rooted circuit packing lemma. All that remains is to show that there is no irregular labeling with $\lceil\sqrt{\frac{2n}{3}}\rceil - 1 \leq \sqrt{\frac{2n}{3}} = s'$ labels. Due to the structure of our digraph, given a largest label s, no vertices of the digraph can achieve degree pairs where both arguments are properly between $s+1$ and $2s+1$. Having established this condition, calculating the polynomial $(2s-1)^2 - s^2 = 3s^2 - 4s + 1$ at $s' := 2n - 4\sqrt{\frac{2n}{3}} + 1$ verifies that out labeling is of minimum strength. Thus if $2n$ is suitably large, we get $\vec{s}(DB_n) = \lceil\sqrt{\frac{2n}{3}}\rceil$. ■

Definition 7.13 *The antiband AB_n is a band $K_2 \times C_{2n}$ which is bipartite and has one partite set beating the other partite set.*

Theorem 7.14 *The irregularity strength of the antiband $\vec{s}(AB_n) = \lceil\frac{n}{3}\rceil$.*

The following construction achieves the bound $\overrightarrow{\lambda}(D)$ for the antiband AB_n. This demonstrates that $\vec{s}(AB_n) = \vec{\lambda}(AB_n)$.

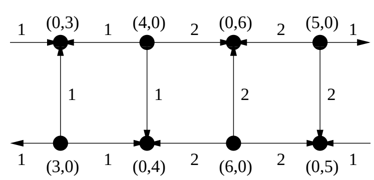


Figure 7.5: Case $n = 4$ in the construction of the anti-band.

Proof: Let AB_4 be given and arc labeled as in Figure 7.5. Let all the arcs be labeled with a 2 except $\vec{w_1}(a_1) = \vec{w_1}(a_2) = \vec{w_1}(b_1) = \vec{w_1}(b_2) = \vec{w_1}(c_1) =$

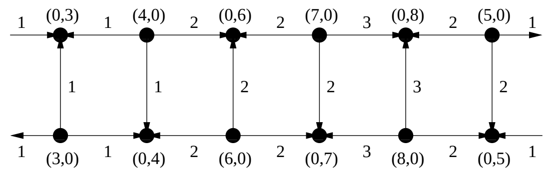


Figure 7.6: Case $n = 5$ of the construction of the anti-band.

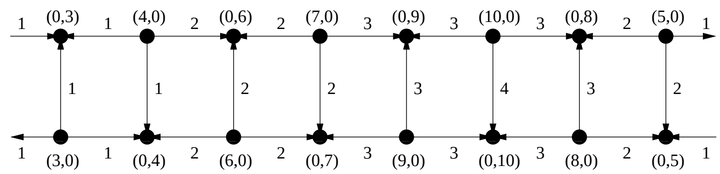


Figure 7.7: Case $n = 6$ of the construction of the anti-band.

$\vec{w_1}(c_2) = 1$ in the base case and see that the labeling is irregular. The rest of the construction is iterative and depends on the congruence of n with 3.

In the case $n \equiv 1 \mod 3$,

$$\begin{aligned} \vec{w}_{n+1}(a_i) &= \vec{w}_n(a_i) \quad \text{for } 1 \le i \le 2n \\ &= \vec{w}_n(a_{i-1}) \quad \text{for } i = 2n+1 \\ &= \vec{w}_n(a_n) + 1 \quad \text{for } i = 2n+2 \end{aligned}$$

In both cases $i = 2n+1, 2n+2$, $\vec{w}_{n+1}(a_i) = \vec{w}_{n+1}(b_i) = \vec{w}_{n+1}(c_i)$. When $n > 4$, all other arc labels stay fixed with respect to their indices. Because $\vec{w}_{n+1}(a_{2n}) = \vec{w}_n(a_{2n})$, it remains only to demonstrate that $\vec{w_{n+1}}(v_{(2n+1)A})$, $\vec{w_{n+1}}(v_{(2n+2)A})$, $\vec{w_{n+1}}(v_{(2n+1)B})$, $\vec{w_{n+1}}(v_{(2n+2)B})$ are all distinct and distinct from the weights on the vertex weights of AB_n. To see this notice that the largest total weight of any vertex in AB_n is $n+2$. But the labels incident $v_{(2n+1)A}$ and $v_{(2n+1)B}$ sum to

$n+3$; meanwhile the labels incident with $v_{(2n+2)A}$ and $v_{(2n+2)B}$ sum to $n+4$.

In the case $n \equiv 2 \mod 3$,

$$\begin{aligned} \vec{w}_{n+1}(a_i) &= \vec{w}_n(a_i) \quad \text{for} \quad 1 \le i \le 2n \\ &= \vec{w}_n(a_{i-1}) + 1 \quad \text{for} \quad i = 2n+1 \\ &= \vec{w}_n(a_{i-2}) \quad \text{for} \quad i = 2n+2 \end{aligned}$$

In the case $i = 2n+2$, $\vec{w}_{n+1}(a_i) = \vec{w}_{n+1}(b_i) = \vec{w}_{n+1}(c_i)$. In the case $i = 2n+1$, $\vec{w}_{n+1}(b_i) = \vec{w}_{n+1}(c_i) = \vec{w}_{n+1}(a_i) - 1$. In all cases, the labels on arcs stay fixed with respect to their indices. Because $\vec{w}_{n+1}(a_{2n+2}) = \vec{w}_n(a_{2n})$, it remains only to demonstrate that $\vec{w}_{n+1}(v_{(2n+1)A}), \vec{w}_{n+1}(v_{(2n+2)A}), \vec{w}_{n+1}(v_{(2n+1)B}), \vec{w}_{n+1}(v_{(2n+2)B})$ are all distinct and distinct from the weights on the vertex weights of AB_n.

The largest total weight of any vertex in AB_n is $n+2$. The labels incident either $v_{(2n+1)A}$ and $v_{(2n+1)B}$ sum to $n+3$. Meanwhile the labels incident with $v_{(2n+2)A}$ and $v_{(2n+2)B}$ sum to $n+4$.

In the case $n \equiv 0 \mod 3$,

$$\begin{aligned} \vec{w}_{n+1}(a_i) &= \vec{w}_n(a_i) \quad \text{for} \quad 1 \le i \le 2n, \\ &= \vec{w}_n(a_{i-1}) \quad \text{for} \quad i = 2n+1, \\ &= \vec{w}_n(a_{i-2}) - 1 \quad \text{for} \quad i = 2n+2. \end{aligned}$$

When $i = 2n+1$ $\vec{w_{n+1}}(b_i) = \vec{w_{n+1}}(c_i) = \vec{w_{n+1}}(a_i) - 1$. When $i = 2n+2$, $\vec{w_{n+1}}(b_i) = \vec{w_{n+1}}(c_i) = \vec{w_{n+1}}(a_i) + 1$. In all other cases, arc labels stay fixed with respect to their indices. Because $\vec{w}_{n+1}(a_{2n+2}) = \vec{w}_n(a_n)$, it remains only to demonstrate that $\vec{w}_{n+1}(v_{(2n+1)A}), \vec{w}_{n+1}(v_{(2n+2)A}), \vec{w}_{n+1}(v_{(2n+1)B}), \vec{w}_{n+1}(v_{(2n+2)B})$ are all distinct and distinct from the weights on the vertex weights of AB_n. To see this notice that the largest total weight of any vertex in AB_n is $n+2$. But the labels incident with $v_{(2n+1)A}$ and $v_{(2n+1)B}$ sum to $n+3$; meanwhile the labels incident with $v_{(2n+2)A}$ and $v_{(2n+2)B}$ sum to $n+4$. ■

7.3.2 The Wheel

To show that Lemma 7.5 has applications in the study of digraph irregularity strength beyond its utility in constructing irregular labelings of paths and cycles, we have considered an orientation of $K_2 \times C_n$. Now we consider an orientation of the wheel, and use the Rooted Circuit Packing Lemma to construct an irregular labeling of minimum strength. In general, Lemma 7.5 can be seen to generate irregular labelings of paths and cycles with $\vec{\lambda}$ labels, which can then be pieced together to span a particular orientation of graph. If the oriented graph is sparse enough, the remaining arcs can be weighted with the weights 1 and $\vec{\lambda}$ so that every vertex has a unique weighted degree pair, giving an irregular labeling of minimum strength.

Theorem 7.15 *The orientation of a wheel W_n with one vertex beating a directed cycle $\vec{C}_{n-1}$ has strength $\lceil\sqrt{\frac{n-1}{2}}\rceil$.*

Proof: Let the directed cycle be $x_1x_2\cdots x_{n-1}x_1$. Then we have arcs from x_n to x_i where $1 \le i \le n-1$. We build an Eulerian circuit in the complete symmetric digraph on $s = \lceil\sqrt{\frac{n-1}{2}}\rceil = v_1v_2...v_{\frac{n-1}{2}}v_1$. Label the arcs on the directed cycle $a_i = \vec{x_ix_{i+1}}$ where subscripts are in the group Z_{n-1} by the labeling function $f(a_i) = v_i$ where $1 \le i \le \frac{n-1}{2}$ and $f(a_i) = v_{i-\frac{n-1}{2}}$ and $\frac{n-1}{2}+1 \le i \le n-1$. Then label the arc $= \vec{x_nx_i}$ with an $s = \lceil\frac{n-1}{2}\rceil$ if $1 \le i \le \frac{n-1}{2}$. Label the arc $\vec{x_nx_i}$ with a 1 if $\frac{n-1}{2} + 1 \leq i \leq n-1$. It is clear that all of the vertex weights are distinct. It remains to show that there are no irregular labelings with fewer labels.

We have $s = (\lceil \sqrt{\frac{n-1}{2}} \rceil - 1)^2 \geq \frac{n-1}{2}$. So if we evaluate $q(x) = 2x^2 - x$ at s, we get a value smaller than $2\frac{n-1}{2} - \lceil \sqrt{\frac{n-1}{2}} \rceil < n-1$ for $n \geq 3$. ■

Theorem 7.16 *The orientation of* W_{2n+1} *in which every vertex on the rim with has in-degree 3 or out-degree 3 has irregularity strength* $\overrightarrow{s} = \lceil \frac{n+2}{3} \rceil$.

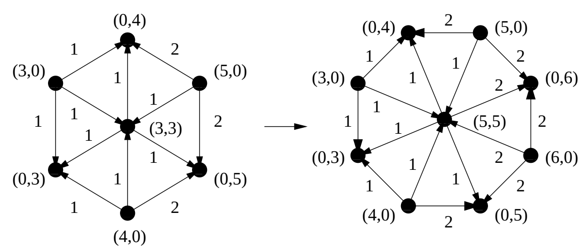


Figure 7.8: The labelings of ascending cases of the orientation of the wheel from Theorem 7.16 have similar vertex weights.

Proof: Clearly $\overrightarrow{\lambda}(D) = \lceil \frac{n+2}{3} \rceil$. But the construction below achieves this bound. In the case W_{2n+1} with $n = 3, 4$, a labeling for the near anti-directed orientation of the wheel is given in Figure 7.8. There are arcs from v_{2n+1} to x_i where i is odd and arcs from x_i to v_{2n+1} where i is even. Let the labeling for $n = 3$ serve as a base case.

In general, order the arcs on the outer rim by the element j if the arc has endpoints x_j and x_{j+1} for $1 \leq j \leq n$ and by j if the arc has endpoints x_{2n} and x_{2n-j} for $0 \leq j \leq n$. Order the arcs on the interior of the wheel by j if the arc has endpoints v_{2n+1} and x_j for $1 \leq j \leq n$ and x_{2n-j} for $0 \leq j \leq n$. Label the arcs ordered j with the label $\lceil \frac{j+2}{3} \rceil$. It is clear that the total weight of the vertices increase by one as their subscripts increase by one for x_i where

$1 \leq i \leq n$. Similarly, the total weight of the vertices increase by one as the subscripts decrease by one for x_i where $n \leq i \leq 2n$. This means that the total weight of x_n equals the total weight of x_{n+1} equals $n+2$. This pair of vertices and every pair of vertices with the same total weight have the property that one vertex is a sink and the other vertex is a source; this source-sink pairing verifies that the labeling is irregular. The largest label used was $\lceil \frac{n+2}{3} \rceil$. ■

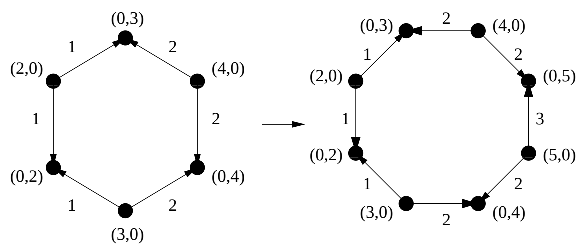


Figure 7.9: The ascending cases of the anti-cycle have similar labelings.

7.3.3 Anticycles

Theorem 7.17 *Given a collection C of anti-cycles whose total order is n, we get that $\vec{s}(C) = \lceil \frac{n}{4} \rceil$.*

Proof: Given a desired smallest vertex weight k, the minimum strength of an anti-cycle whose vertex sums are consecutive $k, k+1, ..., n+k-1$, is $\lceil \frac{n+k-1}{2} \rceil$. This lower bound can be achieved. For the base case–a C_4 where k is given and even, the irregular labeling f_1 given by $f_1(a_1) = \frac{k}{2}$, $f_1(a_2) = \frac{k}{2}$, $f_1(a_3) = \frac{k+2}{2}$, and $f_1(a_4) = \frac{k}{2}$ gives an irregular labeling of minimum strength. Letting, $k = 2$, the standard labeling of an anti-directed C_4 arises. In the case where the given cycle is a C_4, k is odd, $f_1(a_1) = \frac{k-1}{2}$, $f_1(a_2) = \frac{k+1}{2}$, $f_1(a_3) = \frac{k+1}{2}$, $f_1(a_4) = \frac{k+1}{2}$. In the case where $k = 3$, we get a C_4 whose vertices are labeled $(3, 0)$, $(0, 3)$,

$(4, 0)$, and $(0, 4)$. Now define f_n given an irregular labeling of $\vec{C}_{2n-2}$, f_{n-1} in the following way:

$$f_n(a_i) = f_{n-1}(a_i) \quad \text{for} \quad 1 \leq i \leq n+1,$$
$$f_n(a_i) = \lceil \frac{n+k}{4} \rceil \quad \text{for} \quad i = n+2,$$
$$f_n(a_i) = f_{n-1}(a_{i-2}) \quad \text{for} \quad n+3 \leq i \leq 2n.$$

There are only two vertices which do not have vertex weights given by the previous weighting: v_{n+1} and v_{n+2}. All the other vertex weights are identical images of vertex weights that appeared under the previous labeling. If $1 \leq i \leq n$ the $\vec{w_n}(v_i) = \vec{w_{n-1}}(v_i)$. If $n+2 \leq i \leq 2n$, then $\vec{w_n}(v_i) = \vec{w_{n-1}}(v_{i-2})$. That is, all the vertex weights are distinct from each other if the subscripts are limited to $\{1, 2, ..., 2n\} - \{n+1, n+2\}$. These vertices have vertex weights distinct from one another and all the other vertices and it follows that f_n is an irregular labeling. Clearly $\vec{w}_n(v_{n+1}) \neq \vec{w}_n(v_{n+2})$ because we know that one vertex is a source and the other vertex is a sink. Furthermore, the total weight of vertex v_{n+1} is strictly greater than that of v_n under f_n because the total vertex weight of v_n is $= \lceil \frac{n+k-4}{4} \rceil + \lceil \frac{n+k-2}{4} \rceil < \lceil \frac{n+k-2}{4} \rceil + \lceil \frac{n+k}{4} \rceil$. Now, in the labeling of $\vec{C}_{2n-2}$ by f_{n-1}, v_n has the largest total weight; since v_{n+3} is a sink in $\vec{C}_{2n}$, it follows that all vertices have distinct vertex weights under the arc-labeling f_n and that v_{n+1} and v_{n+2} have the greatest total weight $= \lceil \frac{n+k-2}{4} \rceil + \lceil \frac{n+k}{4} \rceil$ in this labeling. The next step is to build our union of anti-cycles, each time picking a smallest vertex weight 1 larger than the largest total weight in the previous anti-cycle. So the labels are consecutive on the cycles and there are two copies of each

Authors	Graph	Disparity	Year
Ferrara, Gilbert, Jacobson [18]	K_n	1	2007
Gilbert	$K_{n,n}$	0	2008
FGJ	nK_2	0	2007
FGJ [18]	P_n p=3 mod 4	$\geq \lvert\lceil\sqrt{n-2}\rceil - \lceil n/4\rceil + 1\rvert$	2007
FGJ [18]	otherwise	$\geq \lvert\lceil\sqrt{n-2}\rceil - \lceil n/4\rceil\rvert$	
Gilbert, Jacobson	C_{2n}	$\geq \lvert\lceil\sqrt{2n}\rceil - \lceil\frac{n+1}{2}\rceil\rvert$	2008
FGJ	Type 1 and Type 2	1	2007
FGJ	W_{2n+1}	$\geq \lvert\lceil\sqrt{n}\rceil - \lceil\frac{n+2}{3}\rceil\rvert$	2007
FGJ	$P_2 \times C_{2n}$	$\geq \lvert\lceil\frac{n}{3}\rceil - \lceil\sqrt{\frac{4n}{3}}\rceil\rvert$	2007
GJ	$P_2 \times P_{2n}$	$\geq \lvert\lfloor\frac{n}{3}\rfloor - \lceil\sqrt{\frac{4n-4}{3}}\rceil\rvert$	2008
Gilbert [22]	$C_{4n} \times C_{4n}$	$\geq \lvert 2n^2+1-2n\rvert$	2007
FGJ	$K_{1,n-1}$	$n-1-\lceil\frac{n-1}{2}\rceil$	2007

Table 7.1: A table bounding the disparity of some graph classes from below and listing exact values where possible; notice some of the values have been proven to hold for large n only

total weight. That is, start with vertex weight $\sum = 2 + (n_1 - 2) + (n_2 - n_1) + \cdots + (n_{n-2} - n_{n-3}) + n_{n-1} - n_{n-2}$ on the final cycle and get a largest arc label $\lceil\frac{n-\sum+\sum}{4}\rceil$ for the union of the anti-cycles whose total order is n, as claimed. ■

7.3.4 Ladders

Theorem 7.18 *The orientation of the ladder $P_n \times K_2$, with one partite set beating the other partite set, $\vec{L}_n$, has irregularity strength $\vec{s}(\vec{L}_n) = \vec{\lambda} = \lfloor\frac{n}{3}\rfloor$.*

Proof: Given are three ladders with irregular labelings. They achieve $\vec{\lambda} = \lceil \frac{n-3}{3} \rceil = \lfloor \frac{n}{3} \rfloor$ in all cases. A proof by induction starts from here. Define an arc labeling f_n iteratively by the congruence of n mod 3.

If $n \equiv 1 \mod 3$, then

$$
\begin{aligned}
f_n(a_i) &= f_{n-1}(a_i) \quad 1 \leq \ i \ \leq \ \lceil \frac{n}{2} \rceil - 2 \\
f_n(a_i) &= f_{n-1}(a_i) - 1 \quad i \ = \ \lceil \frac{n}{2} - 1 \rceil, \\
f_n(a_i) &= f_{n-1}(a_{i-1}) \quad \lceil \frac{n}{2} \rceil \leq i \leq n-1, \\
f_n(b_i) &= f_{n-1}(b_i) \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil - 2, \\
f_n(b_i) &= f_{n-1}(a_{i-1}) \quad i = \lceil \frac{n}{2} \rceil - 1, \\
f_n(b_i) &= f_{n-1}(b_{i-1}) \quad \lceil \frac{n}{2} \rceil \leq i \leq n-1, \\
f_n(c_i) &= f_{n-1}(c_i) \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil - 2, \\
f_n(c_i) &= f_{n-1}(a_{i-1}) \quad i = \lceil \frac{n}{2} \rceil - 1, \\
f_n(c_i) &= f_{n-1}(c_{i-1}) \quad \lceil \frac{n}{2} \rceil \leq i \leq n-1.
\end{aligned}
$$

In this case, the image of each vertex v_{iA} and v_{iB} stays fixed under the insertion of a new rung on the ladder between $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. That is, $w_n(v_{iA}) = w_{n-1}(v_{iA})$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 2$ and $w_n(v_{iA}) = w_{n-1}(v_{(i-1)A})$ where $\lceil \frac{n}{2} \rceil \leq i \leq n-1$. But notice $v_{(\lceil \frac{n}{2} \rceil - 1)A}$ has total weight exactly one larger than $v_{(\lceil \frac{n}{2} \rceil - 2)A}$ under the arc labeling f_n and further that this vertex has the largest total weight among the vertices in the labeled ladder. (From the base case or the previous case in our induction; this case does not have the same congruency modulo 3.) In all these cases the total weight of v_{iB} equals the total weight of v_{iA} (but one vertex is a source while the other is a sink) and so this demonstrates all vertices have a distinct weighted vertex pair under the arc labeling f_n. If $n \equiv 2 \mod 3$, then

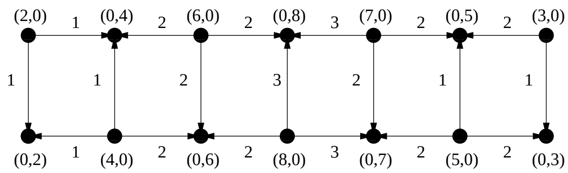


Figure 7.10: Case $n = 7$ of the construction of the anti-ladder.

$$f_n(a_i) = f_{n-1}(a_i) \quad \text{for} 1 \leq i \leq \lceil \frac{n}{2} \rceil$$
$$f_n(a_i) = f_{n-1}(a_{i-1}) \quad \text{for } \lceil \frac{n}{2} + 1 \rceil \leq i \leq n-1.$$
$$f_n(b_i) = f_{n-1}(b_i) \quad \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$$
$$f_n(b_i) = f_{n-1}(b_{i-1}) \quad \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-1.$$
$$f_n(c_i) = f_{n-1}(c_i) \quad \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$$
$$f_n(c_i) = f_{n-1}(c_{i-1}) \quad \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-1.$$

In this case, the image of each vertex v_{iA} stays fixed under the insertion of a

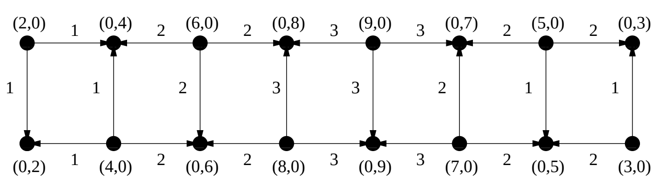


Figure 7.11: Case $n = 8$ of the construction of the anti-ladder.

new rung on the ladder between $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. That is, $w_n(v_{iA}) = w_{n-1}(v_{iA})$ if

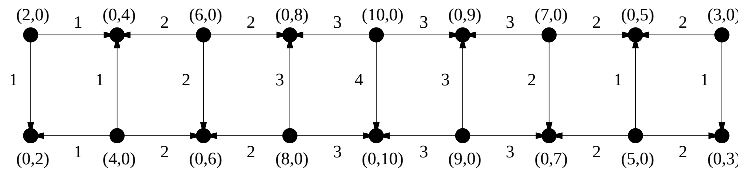


Figure 7.12: Case $n = 9$ of the construction of the anti-ladder.

$1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $w_n(v_{iA}) = w_{n-1}(v_{(i-1)A})$ where $\lceil \frac{n}{2} + 1 \rceil \leq i \leq n - 1$. But notice $v_{(\lceil \frac{n}{2} \rceil)A}$ has total weight exactly one larger than $v_{(\lceil \frac{n}{2} \rceil - 1)A}$ under the arc labeling f_n and this vertex attains the greatest total weight among all vertices in the arc-labeled ladder. In all these cases the total weight of v_{iB} equals the total weight of v_{iA} (but one vertex is a source while the other is a sink) and so this demonstrates all the vertices have a distinct weighted vertex pair under the arc labeling f_n.

If $n \equiv 0 \mod 3$, then the following array shows how to label the arcs of our ladder based on an iterative construction. In this case, the images of vertices v_{iA} and v_{iB} stay fixed under the insertion of a new rung on the ladder between $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. Again, the inserted vertices have total weight one larger than the largest weight from the previous labeling demonstrating that all the vertices have distinct weighted degree and maintaining the inductive hypothesis for the next congruence class of n.

$$f_n(a_i) = f_{n-1}(a_i) \;\; 1 \le i \le \lceil \frac{n}{2} \rceil - 1$$
$$f_n(a_i) = f_{n-1}(a_i) + 1 \;\; i = \lceil \frac{n}{2} \rceil$$
$$f_n(a_i) = f_{n-1}(a_{i-1}) \;\; \lceil \frac{n}{2} + 1 \rceil \le i \le n - 1.$$
$$f_n(b_i) = f_{n-1}(b_i) \;\; 1 \le i \le \lceil \frac{n}{2} \rceil - 1$$
$$f_n(b_i) = f_{n-1}(b_i) \;\; i = \lceil \frac{n}{2} \rceil$$
$$f_n(b_i) = f_{n-1}(b_{i-1}) \;\; \lceil \frac{n}{2} \rceil + 1 \le i \le n - 1.$$
$$f_n(c_i) = f_{n-1}(c_i) \;\; 1 \le i \le \lceil \frac{n}{2} \rceil - 1$$
$$f_n(c_i) = f_{n-1}(c_{i-1}) \;\; i = \lceil \frac{n}{2} \rceil$$
$$f_n(c_i) = f_{n-1}(c_{i-1}) \;\; \lceil \frac{n}{2} \rceil + 1 \le i \le n - 1.$$

■

Theorem 7.19 *The irregularity strength of a ladder* $\vec{D} \equiv \vec{P}_2 \times \vec{P}_n$ *is* $s = \lceil \sqrt{\frac{2n-2}{3}} \rceil$ *for* $n \ge 38$.

Proof: The lower bound is $\vec{\lambda} = \lceil \frac{n-1}{3} \rceil$. Consider the set of all vertices except the source and the sink. The set can have weighted degrees from the set $\{(k,k) : 1 \le k \le \vec{s}\} \cup \{(\vec{s}+k,k) : 1 \le k \le \vec{s}\} \cup \{(k,\vec{s}+k) : 1 \le k \le \vec{s}\}$, so that the minimum s positive integer such that $(2n-2) \le 3s^2$ i.e., $\lceil \sqrt{\frac{(2n-2)}{3}} \rceil \le s$ is a lower bound for $\vec{s}$. First, construct a directed band DB_{n-1} and label it with minimum strength as in Theorem 7.13, there is a pair of consecutive arcs on the rims of the band that are labeled by the loop at 1, i.e., labeled $f(a_i) = f(a_{i-1}) = f(b_i) = f(b_{i-1}) = 1$; in our algorithm, $i = |A(C_1')| + 1$. Similarly, the arc c_i between the vertices v_{iA} and v_{iB} is labeled with a 1. After we have produced this labeling for DB_n, label the ladder (which we decompose into an α-path, a β-path and a set of nK_2 which we call ρ-edges). Let $f'(\rho_{j-i+1}) = f(c_j)$, $f'(\alpha_{j-i+1}) = f(a_j)$, $f'(\beta_{j-i+1}) = f(b_j)$, for $1 \leq i \leq n$ and where all addition and subtraction is modulo n. Now notice that $f'(\rho_1) = f'(\rho_n) = 1$ since both these arcs have their pre-image at c_i. Change this label to a 2, giving v_{1B} weighted vertex degree $(1, k)$ where $(1, k)$ is identical to the weighted vertex degree of v_{iB} in our directed band DB_{n-1}, (i defined above) and vertex v_{nA} weighted degree pair $(2, 1)$. Furthermore, since all the vertex weights were distinct before the arc labels were mapped onto the ladder, the only vertices we need to check are the images of the end-points of the paths. One of these end-points is the only sink; one of these end-points is the only source. Furthermore, since $(2, 1)$ and $(1, k)$ appeared precisely at vertices v_{iA} and v_{iB} in the previous construction, the vertices v_{nA} and v_{1B} have weighted vertex degree pairs distinct from any of the other vertices and from each other, and the labeling is irregular. To see that the labeling is of minimum strength notice that the largest label used was $\lceil\sqrt{\frac{2n-2}{3}}\rceil$. ∎

8. Forests

8.1 Star Forests

In the graph theoretic cases, the first instance where it was discovered that there is a family of graphs where $s(G)$ exceeds $\lambda(G)+c$ for any constant c in the reals, was the case where $G = tP_3$, a disconnected collection of isomorphic stars. There is a similar counterexample, based on this counterexample for graphs, which shows that in some cases $\vec{s}(G)$ exceeds $\vec{\lambda}(\vec{G}) + c$ for any constant c in the reals for some instance of a digraph $\vec{G}$. Notice, that by contrast with the graph-theoretic example, directed stars can be out-directed stars, in-directed stars, or stars whose roots have mixed degree. In the case where all the stars in our collection have roots of mixed degree, the parameter $\vec{s}$ is identical to $\vec{\lambda}$.

Definition 8.1 *Let* $SF_{k_1,k_2,...,k_n}$ *denote a star forest* $S_{1,k_1} \cup S_{1,k_2} \cup \cdots \cup S_{1,k_n}$*.*

Definition 8.2 *The directed star forest with* n *stars with in-degrees* l_k *and out-degrees* w_k *on the leaves of the* k^{th} *star is denoted* $SF((w_1, l_1), (w_2, l_2), ..., (w_n, l_n))$*.*

Definition 8.3 *Let* $OSF_{k_1,k_2,...,k_n}$ *denote an oriented star forest* $S_{1,k_1} \cup S_{1,k_2} \cup \cdots \cup S_{1,k_n}$ *such that all arcs are directed from the roots to the leaves. We distinguish between root vertices and leaf vertices.*

Definition 8.4 *Let* $ISF_{k_1,k_2,...,k_n}$ *denote an oriented star forest* $S_{1,k_1} \cup S_{1,k_2} \cup \cdots \cup S_{1,k_n}$ *such that all arcs are directed from the leaves to the roots. We distinguish between root vertices and leaf vertices.*

Definition 8.5 *Given a star forest $SF = ISF \cup OSF$ with $ISF_{k_1,\ldots,k_t}$ having $k_i = k$ for all $1 \le i \le t$ union $OSF_{k_1,\ldots,k_t}$ having $k_i = k$ for all $1 \le i \le t$, we write $SF = SF_{k^t}$.*

8.2 History of the Problem for Graphs

The problem for tP_3 was an exciting development in the problem. Due to the discovery that $s(G)$ may be bounded below by the parameter $\lambda(G)$, but that this function is not in all cases bounded by $\lambda(G) + c$ for an additive constant, authors have employed more constructive approaches and concrete examples [7].

Suppose the ith weights on the in-directed P_3s are p_i, q_i, and $p_i + q_i$.

Suppose further that the ith weights on the out-directed P_3s are p_i', q_i' and $p_i' + q_i'$. Then $\max\{p_i, q_i, p_i', q_i'\} \ge \vec{s}(SF_{2^k})$.

Let $A = \{i : p_i + q_i \le \vec{s}\}$, $B = \{i : p_i + q_i > \vec{s}\}$.

Now let $A' = \{i : p_i' + q_i' \le \vec{s}\}$, $B' = \{i : p_i' + q_i' > \vec{s}\}$.

Then $|A| + |B| = t$ and $|A'| + |B'| = t'$. We get that $\vec{s} \ge 3|A| + 2|B| = 2t + |A|$ and $\vec{s} \ge 3|A'| + 2|B'| = 2t' + |A'|$.

Now set $a_i = \vec{s} - p_i$, $b_i = \vec{s} - q_i$, $a_i' = \vec{s} - p_i'$, $b_i' - \vec{s} - q_i'$.

Without loss of generality, let $B = B' = \{1, 2, ..., k\}$ and

$$a_1 + b_1 < a_2 + b_2 < \cdots < a_k + b_k.$$

Then for all i we have that $a_{k-i} + b_{k-i} \le \vec{s} - 1 - i$ and $a_{k-i}' + b_{k-i}' \le \vec{s} - 1 - i$. The sum of the a_i and b_i and the a_i' and b_i' is $\ge k(2k-1)$ (one sum counts leaf labels, the other sum counts root labels). But notice then that $k(2k-1) \le \sum_{i=0}^{k-1} (\vec{s} - 1 - i)$ and so

$$k(2k-1) \le k(\vec{s} - 1) - \frac{k(k-1)}{2},$$

and

$$\vec{s} - 1 \geq \frac{5k-3}{2},$$

and thus

$$\vec{s} \geq \frac{5|B|-1}{2}.$$

There are two cases from here:

Case 1. If $|A| \geq \frac{t}{7}$ then $\vec{s} \geq 2t + |A| = \frac{15t}{7}$.

Case 2. If $|A| \leq \frac{t-1}{7}$ then $|B| \geq \frac{6t+1}{7}$. Or rather, $\vec{s} \geq \frac{15t-1}{7}$.

Theorem 8.6 *The strength of $\vec{s}(SF_{2^t}) = s(tP_3)$ in the case where $t \equiv 3 \mod 7$.*

In all cases it follows from the argument above that $\vec{s}(SF_{2^t}) \geq \frac{15t-1}{7}$. Then notice by the construction in [32], that there is an irregular $\vec{s}$-weighting of this strength in the case $t \equiv 3 \mod 7$, where $\vec{s}$ is the value above $= \frac{15t-1}{7}$. Take the irregular weighting from the tP_3 graph and copy it a second time, one with all tP_3s in-directed and one with all tP_3s out-directed. There are no vertex weight collisions because this would imply a vertex weight collision in the initial irregular weighting of the tP_3.

The following section will make it clear that while collections of in-directed and out-directed stars have $\vec{s} > \vec{\lambda}$ in some cases, collections of mixed stars have $\vec{s} = \vec{\lambda}$.

8.3 Mixed Stars

The case of star forests where every root is of mixed degree is easy to analyze because the labels of the pendant vertices cannot possibly conflict with the labels of the root vertices.

Theorem 8.7 *The irregularity strength of a directed star forest with no trivial components and the root of every star having at least one in-leaf and one out-leaf is given by* $\vec{s}(SF((w_1, l_1), (w_2, l_2), ..., (w_n, l_n))) = max\{\sum_{j=1}^{j=n} w_j, \sum_{j=1}^{j=n} l_j\}$.

Proof: We will proceed to show that

$$\vec{s}(SF((w_1, l_1), (w_2, l_2), ..., (w_n, l_n))) \geq max\{\sum_{j=1}^{j=n} w_j, \sum_{j=1}^{j=n} l_j\}.$$

And then show $\vec{s}(SF((w_1, l_1), (w_2, l_2), ..., (w_n, l_n))) \leq max\{\sum_{j=1}^{j=n} w_j, \sum_{j=1}^{j=n} l_j\}$. This will show $\vec{s}(SF((w_1, l_1), (w_2, l_2), ..., (w_n, l_n))) = max\{\sum_{j=1}^{j=n} w_j, \sum_{j=1}^{j=n} l_j\}$. For the first inequality, simply note that all in-leaves and out-leaves must have distinct weights.

Next we show the second inequality by construction. Without loss of generality, let $w = \sum_{j=1}^{j=n} w_j \geq \sum_{j=1}^{j=n} l_j$. Then let $w_1 \leq w_2 \leq \cdots \leq w_n$. Weight the out-arcs of $(w_1, l_1) : 1, 2, ..., w_1$. Weight the out-arcs of $(w_2, l_2) : w_1+1, ..., w_1+w_2$. Weight the out-arcs of $(w_3, l_3) : w_1 + w_2 + 1, ..., w_1 + w_2 + w_3$, and so on until we weight the out-arcs of $(w_n, l_n) : \sum_{j=1}^{j=n-1} w_j \ + \ 1, ..., \sum_{j=1}^{j=n} w_j$. Now let $l_{m_1} \leq l_{m_2} \leq l_{m_3}, ..., \leq l_{m_n}$. Weight the in-arcs of $(w_{m_1}, l_{m_1}) : 1, 2, ..., l_{m_1}$. Weight the in-arcs of $(w_{m_2}, l_{m_2}) : l_{m_1} + 1, ..., l_{m_1} + l_{m_2}$. Weight the in-arcs of $(w_{m_3}, l_{m_3}) : l_{m_1}+l_{m_2}+1, ..., l_{m_1}+l_{m_2}+l_{m_3}$, and so on until we weight the in-arcs of $(w_{m_n}, l_{m_n}) : \sum_{j=1}^{j=n-1} l_{m_j} \ + \ 1, ..., \sum_{j=1}^{j=n} l_{m_j}$. First of all, no two pendant vertices have the same weighted degree pair. Now, no two in-weights of the roots of any two stars are the same unless they both have in-degree zero. For consider, since $w_{k+1} \geq w_k$ for all $1 \leq k \leq n-1$ it follows from term by term comparison that $\sum_{t=1}^{t=w_{k+1}} \sum_{j=1}^{j=k} w_j \ + \ t \geq \sum_{t=1}^{t=w_k} \sum_{j=1}^{j=k-1} w_j + t$. By a similar argument, no two out-weights of the roots of any two stars are the same unless they both have out-degree zero. Now then, for two in-weight, out-weight pairs to match, it would require than both in-degree and out-degree of two roots of two stars be zero. Because no stars are trivial in our star forest, no two roots of any two stars have both in and out-degree zero and so no two roots of any two stars have the same in-weight, out-weight pair. That is, our construction weights our star forest irregularly with $w = \vec{\lambda} = \vec{s}$ weights. Thus, we have shown both inequalities and the theorem holds. ■

8.4 Conclusions

There are various upper bounds we can use in the context of oriented star forests.

Clearly $\vec{s}(SF_{m^k})$ will be larger than any collection of k in-stars and k-out stars with roots of degree $\leq m$.

But we have $\lambda(S\vec{F}_{m^k}) = mk$ and $\vec{\mu}(SF_{m^k}) = (2m+1)k$.

9. Irregular Orientations and Conclusions

We examine a constructive technique for orienting 2-irregular graphs irregularly.

Theorem 9.1 *A necessary condition for a graph to have an irregular orientation is that there be at most $k+1$ vertices of degree k.*

Proof: There are at most $k+1$ different degree pairs of total degree k. By the pigeonhole principle, if there are more than $k+1$ different degree pairs of total degree k, then there are two vertices with the same degree pair. ■

Conjecture The above necessary condition is also sufficient for a graph to have an irregular orientation.

A Counterexample

Let the 7 sets of vertices A,B,C,D,E,F,H have cardinalities $|A| = j+1, |B| = k+1, |C| = j, |D| = k, |E| = k-2, |F| = j-2, |H| = n$. We set the graph $G =< A \cup B \cup C \cup D \cup E \cup F \cup H, \{xy : (x \in A \wedge y \in C) \vee (x \in C \wedge y \in E) \vee (x \in E \wedge y \in H) \vee (x \in B \wedge y \in D) \vee (x \in D \wedge y \in F) \vee (x \in F \wedge y \in H) \vee (xy \in H)\} >$. The degree sequence of this graph is

$$\pi(G) = (j)^{j+1}, k^{k+1}, (j+k-1)^{j+k}, (j+n)^{k-2}, (k+n)^{j-2}, (j+k+n-5)^n$$

and that as long as $n > j > k \geq 3$ the graph is a counterexample to the above conjecture.

Notice also that in some cases where we can replace the complete graph on the set H with some other graph on the set H.

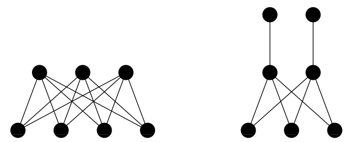


Figure 9.1: A counterexample.

There is a simpler counterexample included in Figure 9.1; notice the more general counterexample is connected.

Theorem 9.2 *If G is 2-irregular, then it has an irregular orientation.*

Proof: Suppose that G is 2-irregular. Partition the vertex set as follows so that it meets the following conditions:

i. X and Y are the partite sets

iii. 2 vertices of the same degree lie in opposing partite sets

iv. Maximize the number of edges between X and Y given that they meet the above 3 conditions.

Notice that though the construction partitions the vertex set the resulting graph is not necessarily bipartite.

Orient all edges from X to Y. Put a nearly regular orientation [5] on the graph induced by X and the graph induced by Y. The following two provisions apply. Given two vertices of degree 1, a vertex in each of X and Y and two vertices of degree 2, one in each of X and Y, the nearly regular orientation can be achieved so that if the vertices of degree 1 agree after the orientation, then reverse the oriented path entering or leaving the vertex of degree 1. Likewise, a path in the other partite set as well can be reversed as well. Since the only possibility for having the two vertices of total degree 2 agree is that each has one arc in the $[X, Y]$ bigraph and that the nearly regular orientation fixes the two vertices so that they both have degree (1,1), just reverse one of the paths incident one of our vertices of degree 2 in the partite set not fixed.

Claim: The resulting orientation is an irregular orientation.

Proof of Claim: By construction, $a > b \geq \lceil \frac{d}{2} \rceil$ where d is the degree of two vertices of the same total degree and a and b are the degrees of the vertices v and x respectively in $[X, Y]$. Then the degree pair of $v \in X$ is one of the following

$$(\frac{d+a}{2}, \frac{d-a}{2}), (\frac{d+a+1}{2}, \frac{d-a-1}{2}), (\frac{d+a-1}{2}, \frac{d-a+1}{2}).$$

Furthermore, we get that the degree pair of $x \in Y$ is one of the following

$$(\frac{d-b}{2}, \frac{d+b}{2}), (\frac{d-b+1}{2}, \frac{d+b-1}{2}), (\frac{d-b-1}{2}, \frac{d+b+1}{2}).$$

For the degree pair of v to equal the degree pair of w would have to have

$$\frac{d+a-1}{2} \leq \frac{d-b+1}{2}$$

or rather $a + b \leq 2$. But then the degree of our vertices is either 2 or 1. If our orientation has one or both of these problems, use the provisions above. ∎

APPENDIX A. Two Proofs By Cases

A.1 Proof by cases of 6.15: An overview

There are many cases in verifying there are only 2 forbidden initial condition-orientation combinations for C_4 for the argument in Chapter 6, Theorem 6.15. The cases follow: some are explained by a picture; others require more in-depth argument.

The first 11 plate diagrams verify that there is both an (n)-avoiding irregular 2-labeling for each orientation of C_4 under an initial condition and a $(2n)$-avoiding irregular 2-labeling for each orientation of C_4 under an initial condition; (with the exception of the 2 forbidden by 6.15) notice the proof for (n)-avoiding irregular 2-labelings also demonstrates there is a a $(0,1)$-labeling that is $(0,0)$ avoiding. In the base cases we have values for $[h + \text{def}]$ at each vertex over every orientation of C_4. Since there must be a vertex of greatest c_g value, we label that value at each vertex that achieves it. The remainder of the argument is showing that given the initial assignment of tiers, the vertices can be permuted so that each vertex has distinct weight. Notice that when we permute vertices, we keep the same label, even if the charges change in the problem cases. Therefore, $C_g \neq c_g$ is a label, not a deficiency, or weight. The labelings are standard in each case up to permutation of the vertices. In the $(2n)$-avoiding irregular 2-labelings put a 2 on the arc with endpoints v_{iA} and v_{jB} if $i + j > 3$ and label the arc with a 1 otherwise; in the $(0,1)$-labeling case this corresponds to using label 1 if the arc has endpoints i, j such that $i + j > 3$ and using the

0-label otherwise. In the (n)-avoiding case, use the 2 label on arcs with endpoints v_{iA} and v_{jB} if $i + j > 2$ and label the arc with a 1 otherwise; similarly in the $(0, 1)$-labeling case, this corresponds to using label 1 if the arc has endpoints v_{iA} and v_{jB} if $i + j > 2$ and the 0-label otherwise. Notice the labelings given for the irregular 2-labelings work for the $(0, 1)$-labeling case as well. Suppose a permutation of the vertices of an orientation of C_4 gives $[x_1, x_2, x_3, x_4]$ for the $[h + \text{def}]$ matrix. Then refer to that $[h + \text{def}]$ matrix in our table under the given orientation (as long as that matrix is not one of the 2 forbidden from the outset). The permutations $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ and $1 \leftrightarrow 2; 3 \leftrightarrow 4$ all have the same effect on the entries of the $[h + \text{def}]$ matrix in the $(0, 1)$-labeling as they do under a labeling with labels 1 and 2 allowed. So if the permutation gives an irregular 2-labeling, it also works to give an irregular $(0, 1)$-labeling as long as we use the same structured subdigraph of 1 labels, ie. an isomorphic digraph of 1 labels to the digraph of 2 labels we use in the irregular 2-labeling case. That is, given an orientation of C_4 and a charge matrix, we can find a $(0, 1)$ labeling f based on a labeling of C_4 with 2 labels. We simply use the same f given by the table with the following augmentation: $f' = f - 1$ gives a $(0, 1)$ labeling of C_4 where f is given by the corresponding entry in our table.

A.2 Cases of $K_{2,2}$ has $(2n)$-avoiding irregular 2-labelings

The charge matrices are recorded in the following way:

$$[x, w, y, z] = [c_g(v_{1A}), c_g(v_{1B}), c_g(v_{2A}), c_g(v_{2B})].$$

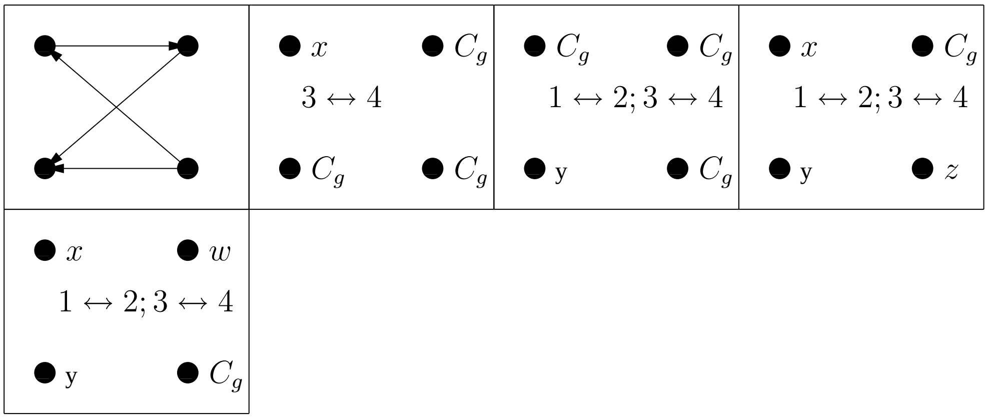
First notice that for every orientation, the matrix of all C_g is resolved by the permutation $1 \leftrightarrow 2; 3 \leftrightarrow 4$ in all arc-labeling cases under either of the tier systems we are using. The cases which require more explanation are dealt with case by case.

Case 1: Consider the orientation from Base Cases: Part 1:

$$A(D) = \{v_{1A}\vec{v}_{1B}, v_{1B}\vec{v}_{2A}, v_{2B}\vec{v}_{2A}, v_{2B}\vec{v}_{1A}\}.$$

Case 1.1: Consider the $(2n)$-avoiding case. Case 1.1.1: There is 1 value not $= C_g$ in the charge matrix. Case 1.1.1.1: The charge matrix $= [C_g, w, C_g, C_g]$. Case 1.1.1.1.1: If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.1.1.2: If $w < C_g - 1$, then permute $3 \leftrightarrow 4$. Case 1.1.1.2: The charge matrix $= [C_g, C_g, C_g, w]$. Case 1.1.1.2.1: If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.1.2.2: If $w < C_g - 1$, then permute $1 \leftrightarrow 2$. Case 1.1.2: There are 2 values not $= C_g$ in the charge matrix. Case 1.1.2.1: Given $[C_g, C_g, y, z]$ there are 2 subcases. Case 1.1.2.1.1: If $y \leq z$ then permute $1 \leftrightarrow; 3 \leftrightarrow 4$. Case 1.1.2.1.2: If $z < y$, then $1 \leftrightarrow 2$. Case 1.1.2.2: Given $[x, w, C_g, C_g]$ there are 2 subcases. Case 1.1.2.2.1: If $w < C_g - 1$, then permute $3 \leftrightarrow 4$. Case 1.1.2.2.2: If $w = C_g - 1$, then $1 \leftrightarrow 2$. Notice this will leave the vertex initially charged w with the greatest charge on the top tier. Case 1.1.3 There are 3 values not $= C_g$ in the charge matrix. Case 1.1.3.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 1.1.3.1.1: If $w = y$, then permute $3 \leftrightarrow 4$. Case 1.1.3.1.2: If $w < y$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.3.1.3: If $y < w$, then $3 \leftrightarrow 4$. Case 1.1.3.2: Given $[x, C_g, y, z]$ there are 3 subcases. Case 1.1.3.2.1: If $x = y$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.3.2.2: If $x < y$, then $3 \leftrightarrow 4$. Case 1.1.3.2.3: If $y < x$, then $3 \leftrightarrow 4$. Case 1.1.3.3: Given $[x, w, C_g, z]$ there are 3 subcases. Case 1.1.3.3.1: If $x = z$, then permute $1 \leftrightarrow 2$. Case 1.1.3.3.2: If $x < z$, then $1 \leftrightarrow 2$. Case 1.1.3.3.3: If $z < x$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.3.4: Given $[x, w, y, C_g]$ there are 3 subcases. Case 1.1.3.4.1: If $x = y$, then permute

(2n)-avoiding



(n)-avoiding

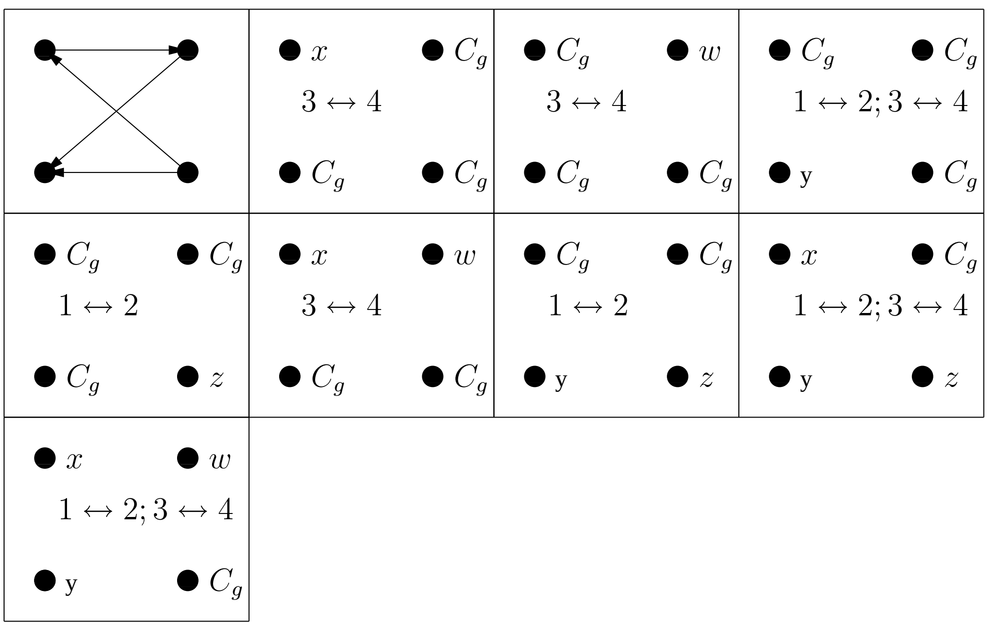


Figure A.1: Theorem 6.15 Base Cases: Part 1

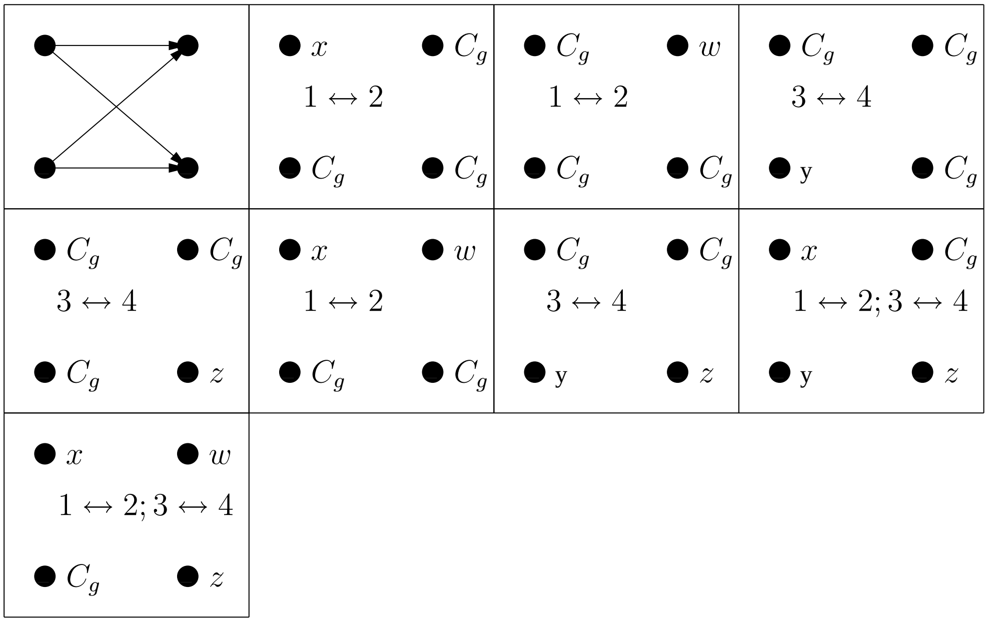
$3 \leftrightarrow 4$. Case 1.1.3.4.2: If $x < y$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.1.3.4.3: If $y < x$, then $3 \leftrightarrow 4$. Case 1.2: Consider the (n)-avoiding case where there are 3 values not $= C_g$ in the charge matrix. Case 1.2.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 1.2.1.1: If $w = y$, then permute $1 \leftrightarrow 2$. Case 1.2.1.2: If $w < y$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 1.2.1.3: If $y < w$, then $1 \leftrightarrow 2$. Case 1.2.2: Given $[x, w, C_g, z]$ there are 3 subcases. Case 1.2.2.1: If $x = z$, then permute $3 \leftrightarrow 4$. Case 1.2.2.2: If $x < z$, then $3 \leftrightarrow 4$. Case 1.2.2.3: If $z < x$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$.

Case 2: Consider the orientation from Base Cases: Part 2:

$$A(D) = \{v_{1A}\vec{v}_{1B}, v_{2A}\vec{v}_{1B}, v_{2A}\vec{v}_{2B}, v_{1A}\vec{v}_{2B}\}.$$

Case 2.1: Consider the $(2n)$-avoiding case. There are 3 values not $= C_g$ in the charge matrix. Case 2.1.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 2.1.1.1: If $w = y$, then permute $3 \leftrightarrow 4$. Case 2.1.1.2: If $w < y$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 2.1.1.3: If $y < w$, then there are 2 subcases. Case 2.1.1.3.1: If $w = C_g - 2$ and $y = z = C_g - 1$ the case cannot be resolved. This case does not have a feasible initial condition. Case 2.1.1.3.2: If $w \neq C_g - 2$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 2.1.2: Given $[x, w, y, C_g]$ there are 3 subcases. Case 2.1.2.1: If $x = y$, then permute $1 \leftrightarrow 2$. Case 2.1.2.2: If $x < y$, then $1 \leftrightarrow 2$. Case 2.1.2.3: If $y < x$, then there are 2 subcases. Case 2.1.2.3.1: If $x = w = C_g - 1$ and $y = C_g - 2$ the case cannot be resolved. This case does not have a feasible initial condition. Case 2.1.2.3.2: If $y \neq C_g - 2$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 2.2: Consider the (n)-avoiding case. Case 2.2.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 2.2.1.1: If $w = y$, then permute $3 \leftrightarrow 4$. Case 2.2.1.2: If $y < w$, then $3 \leftrightarrow 4$. Case 2.2.1.3: If $w < y$, then there are 2 subcases. Case 2.2.1.3.1: If $w = C_g - 2$ and $y = z = C_g - 1$ the case cannot be resolved. This case does not have a feasible initial condition.

(2n)-avoiding



(n)-avoiding

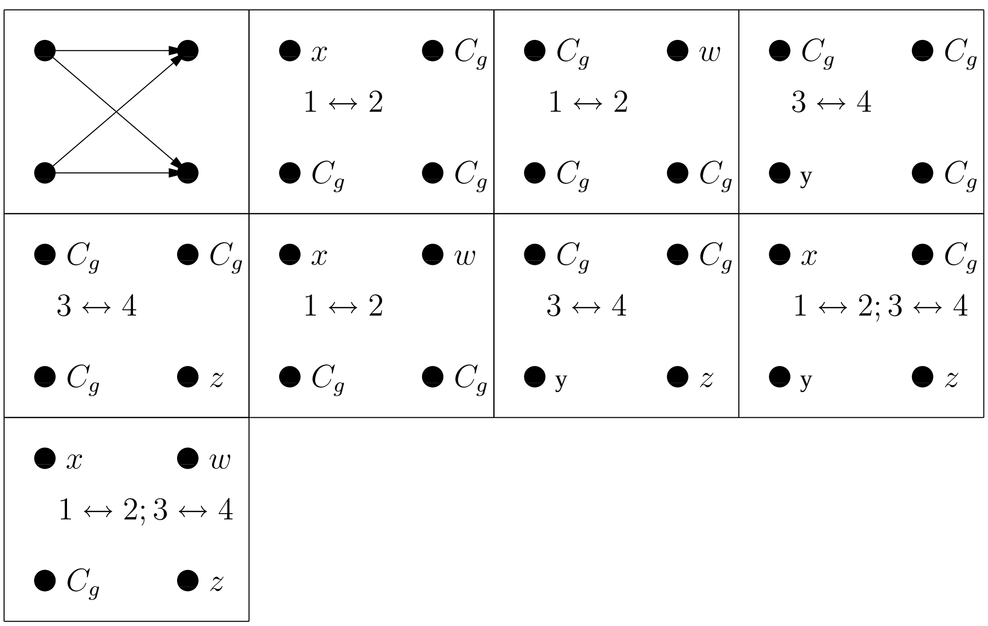


Figure A.2: Theorem 6.15 Base Cases: Part 2

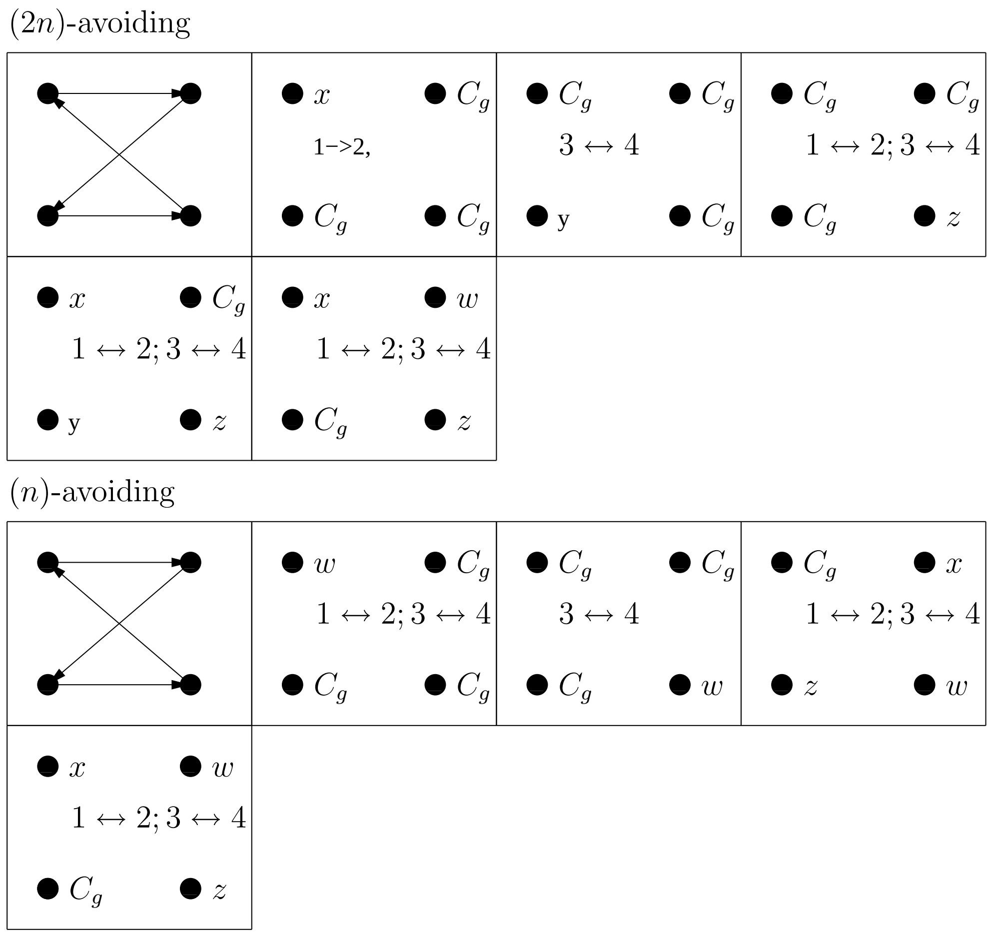


Figure A.3: Theorem 6.15 Base Cases: Part 3

Case 2.2.1.3.2: If $w \neq C_g - 2$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 2.2.2: Given $[x, w, y, C_g]$ there are 3 subcases. Case 2.2.2.1: If $x \leq y$, then permute $1 \leftrightarrow 2$. Case 2.2.2.2: If $y < x$, then there are 2 subcases. Case 2.2.2.2.1: If $y = C_g - 2$ the case cannot be resolved. This case does not have a feasible initial condition. Case 2.2.2.2.2: If $y < C_g - 2$, then the permutation $1 \leftrightarrow 2; 3 \leftrightarrow 4$ labels the C_4

irregularly. Case 2.2.2.3: If $y < x$, then $3 \leftrightarrow 4$.

Case 3: Consider the orientation from Base Cases: Part 3:

$$A(D) = \{v_{1A}\vec{v}_{1B}, v_{1B}\vec{v}_{2A}, v_{2A}\vec{v}_{2B}, v_{2B}\vec{v}_{1A}\}.$$

Case 3.1: Consider the $(2n)$-avoiding case. Case 3.1.1: There is 1 value not $= C_g$ in the charge matrix: Case 3.1.1.1: The charge matrix is $[C_g, w, C_g, C_g]$. Case 3.1.1.1.1: If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.1.1.1.2: If $w < C_g - 1$, then permute $3 \leftrightarrow 4$. Case 3.1.2: There are 2 values not $= C_g$ in the charge matrix. Case 3.1.2.1: The charge matrix is $[C_g.C_g, y, z]$. Case 3.1.2.1.1: If $z < y$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.1.2.1.2: If $y < z$, then there are 2 subcases. Case 3.1.2.1.2.1: If $z = C_g - 1$, then $3 \leftrightarrow 4$. Case 3.1.2.1.2.2: If $z < C_g - 1$, then $1 \leftrightarrow 2$. Case 3.1.2.2: The charge matrix is $[x, w, C_g, C_g]$. Case 3.1.2.2.1: If $y < z$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.1.2.2.2: If $z < y$, then $3 \leftrightarrow 4$. Case 3.1.2: There are 3 values not $= C_g$ in the charge matrix. Case 3.1.2.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 3.1.3.1.1: If $w = y$, then there are 2 subcases. Case 3.1.3.1.1.1: If $w = y = C_g - 1$, then permute $3 \leftrightarrow 4$. Case 3.1.3.1.1.2: If $w = y < C_g - 1$, then permute $1 \leftrightarrow 2$. Case 3.1.3.1.2: If $w < y$, then 2 subcases. Case 3.1.3.1.2.1: If $y = C_g - 1$, then $3 \leftrightarrow 4$. Case 3.1.3.1.2.2: If $y < C_g - 1$, then $1 \leftrightarrow 2$. Case 3.1.3.1.3: If $y < w$, then $1 \leftrightarrow 2$. Case 3.1.3.4: Given $[x, w, y, C_g]$ there are 3 subcases. Case 3.1.3.4.1: If $x = y$, then permute $1 \leftrightarrow 2$. Case 3.1.3.4.2: If $x < y$, then $3 \leftrightarrow 4$. Case 3.1.3.4.3: If $y < x$, then $1 \leftrightarrow 2$. Case 3.2: Consider the (n)-avoiding case. Case 3.2.1: There is 1 non-C_g entry in the charge matrix. Case 3.2.1.1: The charge matrix is $[C_g, w, C_g, C_g]$. Case 3.2.1.1.1: If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.2.1.1.2: If $w < C_g - 1$, then permute $1 \leftrightarrow 2$. Case 3.2.1.2: The charge matrix is $[C_g, C_g, w, C_g]$. Case 3.2.1.2.1:

If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.2.1.2.2: If $w < C_g - 1$, then permute $3 \leftrightarrow 4$. Case 3.2.2: There are 2 non-C_g entries in the charge matrix. Case 3.2.2.1: The charge matrix is $[C_g, C_g, y, z]$. Case 3.2.2.1.1: If $y < z$, then permute $3 \leftrightarrow 4$. Case 3.2.2.1.2: If $z < y$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 3.2.2.2: The charge matrix is $[x, w, C_g, C_g]$. Case 3.2.2.2.1: If $w < x$, then there are cases. Case 3.2.2.2.1.1: If $x = C_g - 1$, or $x = C_g - 2$ then permute $1 \leftrightarrow 2$. Case 3.2.2.2.1.2: If $x \leq C_g - 3$, then permute $3 \leftrightarrow 4$. Case 3.2.2.2.2: If $x < w$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$ Case 3.2.3: There are 3 non-C_g entries in the charge matrix. Case 3.2.3.1: Consider the matrix $[C_g, w, y, z]$. Case 3.2.3.1.1: If $w = y$, there are 2 cases. Case 3.2.3.1.1.1: If $x = C_g - 1$, then $3 \leftrightarrow 4$. Case 3.2.3.1.1.2: If $x < C_g - 1$, then $1 \leftrightarrow 2$. Case 3.2.3.1.2: If $w < y$, then permute $1 \leftrightarrow 2$. Case 3.2.3.1.3: If $y < w$, then there are 2 cases. Case 3.2.3.1.3.1: If $x = C_g - 1$, then permute $3 \leftrightarrow 4$. Case 3.2.3.1.3.2: If $x < C_g - 1$, then permute $1 \leftrightarrow 2$.

Case 4: Consider the orientation from Bases Cases: Part 4.

$$A(D) = \{v_{1A}\vec{v}_{1B}, v_{1B}\vec{v}_{2A}, v_{2B}\vec{v}_{2A}, v_{1A}\vec{v}_{2B}\}.$$

Case 4.1: Consider the $(2n)$-avoiding case. Case 4.1.1: There is 1 value not $= C_g$ in the charge matrix: $[C_g, w, C_g, C_g]$. Case 4.1.1.1: If $w = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.1.1.2: If $w < C_g - 1$, then permute $3 \leftrightarrow 4$. Case 4.1.2: There are 2 values not $= C_g$ in the charge matrix, $[x, w, C_g, C_g]$, there are 2 subcases. Case 4.1.2.1: If $x < w$ then permute $1 \leftrightarrow; 3 \leftrightarrow 4$. Case 4.1.2.2: If $w < x$, then $1 \leftrightarrow 2$. Case 4.1.3: There are 3 values not $= C_g$ in the charge matrix. Case 4.1.3.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 4.1.3.1.1: If $w = y$, then permute $3 \leftrightarrow 4$. Case 4.1.3.1.2: If $w < y$, then $3 \leftrightarrow 4$. Case 4.1.3.1.3: If $y < w$, then there are 2 cases. Case 4.1.3.1.3.1: If $w = C_g - 2$,

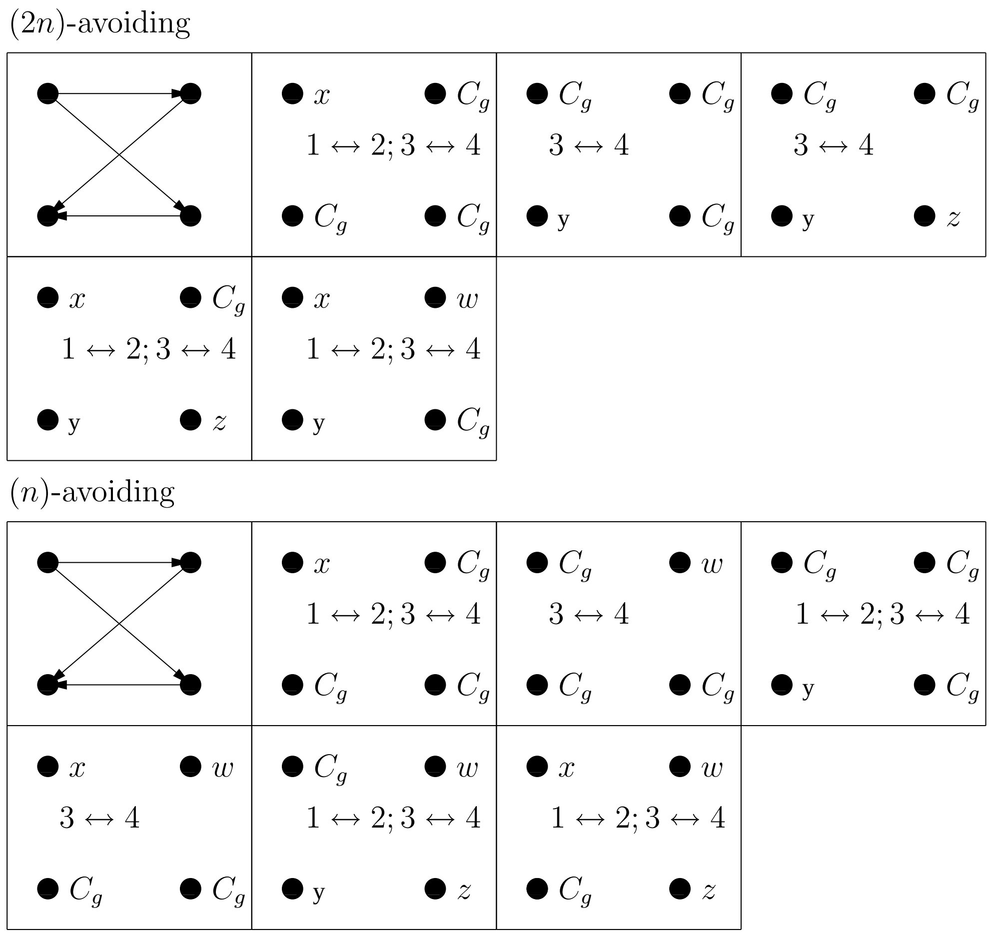


Figure A.4: Theorem 6.15 Base Cases: Part 4

and $y = C_g - 1$, the case cannot be resolved (see 6.17). Case 4.1.3.1.3.2: If $w < C_g - 2$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.1.3.3: Given $[x, w, C_g, z]$ there are 3 subcases. Case 4.1.3.3.1: If $z = C_g - 1$, then $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.1.3.3.2: If $z = C_g - 2$, then there are 3 cases. Case 4.1.3.3.2.1: If $z = x$, then either permutation $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$ resolves the case. Case 4.1.3.3.2.2: If $z < x$, then $x = w = C_g - 1$ and the case cannot be resolved. Case 4.1.3.3.2.3: If $x < z$, then permute $3 \leftrightarrow 4$. Case 4.1.3.3.3: If $z < C_g - 2$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.2: Consider the (n)-avoiding case. Case 4.2.1: Consider $[C_g, C_g, C_g, z]$. There are 2 cases. Case 4.2.1.1: If $z = C_g - 1$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.2.1.2: If $z < C_g - 1$, then permute $1 \leftrightarrow 2$. Case 4.2.2: Consider $[C_g, C_g, y, z]$. Case 4.2.2.1: If $z < y$, then permute $1 \leftrightarrow 2$. Case 4.2.2.2: If $y < z$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$. Case 4.2.3: Suppose there are 3 entries in the charge matrix $\neq C_g$. Case 4.2.3.1: Given $[C_g, w, y, z]$ there are 3 subcases. Case 4.2.3.1.1: If $w = y$, then permute $3 \leftrightarrow 4$. Case 4.2.3.1.2: If $w < y$, then permute $3 \leftrightarrow 4$. Case 4.2.3.1.3: If $y < w$, then permute $1 \leftrightarrow 2$. Case 4.2.3.2: Given $[x, w, C_g, z]$ there are 3 subcases. Case 4.2.3.2.1: If $x = z$, then permute $3 \leftrightarrow 4$. Case 4.2.3.2.2: If $x < z$, then permute $3 \leftrightarrow 4$. Case 4.2.3.2.3: If $z < x$, then there are 2 cases. Case 4.2.3.2.3.1: If $z = C_g - 2$ and $x = C_g - 1$, then the case cannot be resolved. Case 4.2.3.2.3.2: If $z < C_g - 2$, then permute $1 \leftrightarrow 2; 3 \leftrightarrow 4$.

A.3 Cases of $K_{3,3}$ has $(0,0)$-avoiding $(0,1)$-labelings

We present irregular labelings of $K_{3,3}$ with the standard labeling by tiers. In the $(0,1)$-labeling case we label $v_{iA}v_{jB}$ with a 1 if $i + j > 3$, 0 otherwise. In the (n)-avoiding case we label $v_{iA}v_{jB}$ with 2 if $i + j > 3$, 1 otherwise. In the $(2n)$-avoiding case we label $v_{iA}v_{jB}$ with a 2 if $i + j > 4$, and a 1 otherwise.

The permutations of the vertices relative to the feedback graphs are given in the following figures. The tiers are listed bottom up from 1 to 3.

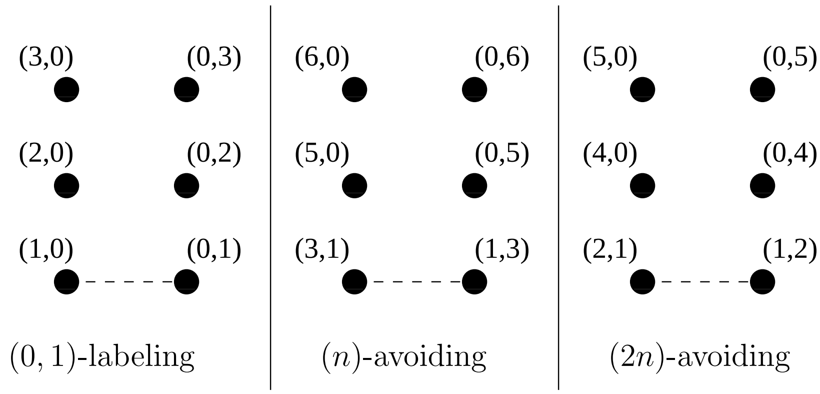


Figure A.5: Theorem 6.13: Case 1

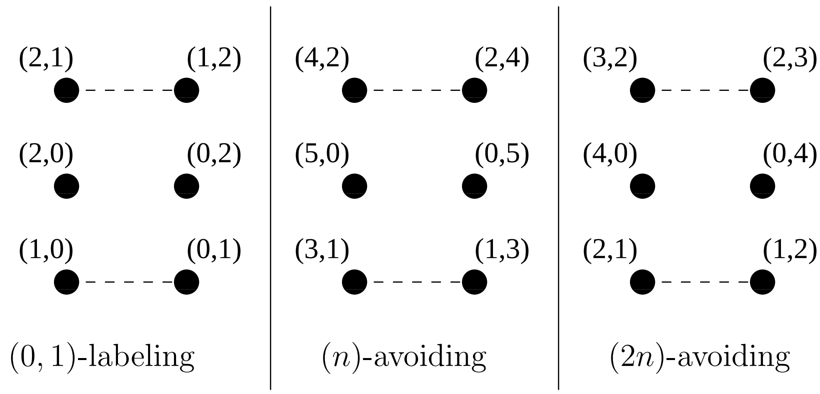


Figure A.6: Theorem 6.13: Case 2

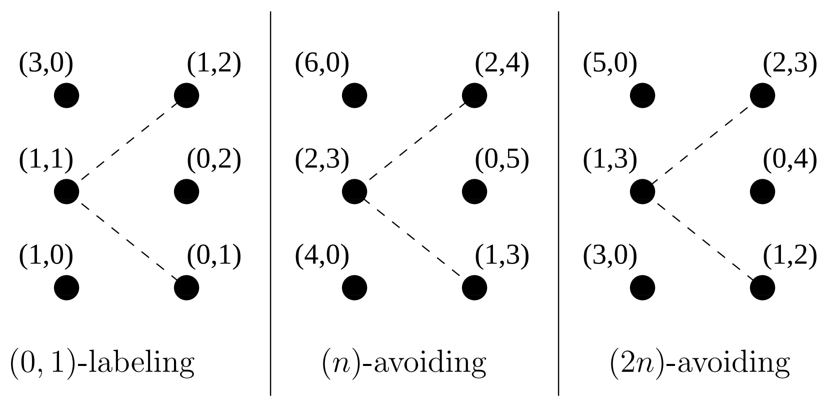


Figure A.7: Theorem 6.13: Case 3

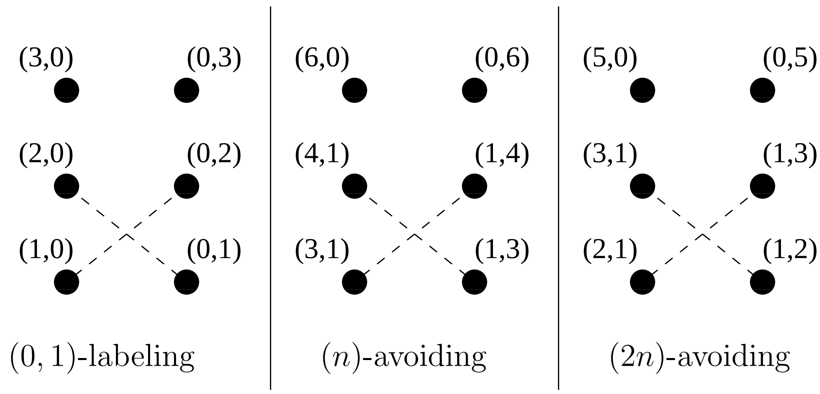


Figure A.8: Theorem 6.13: Case 4

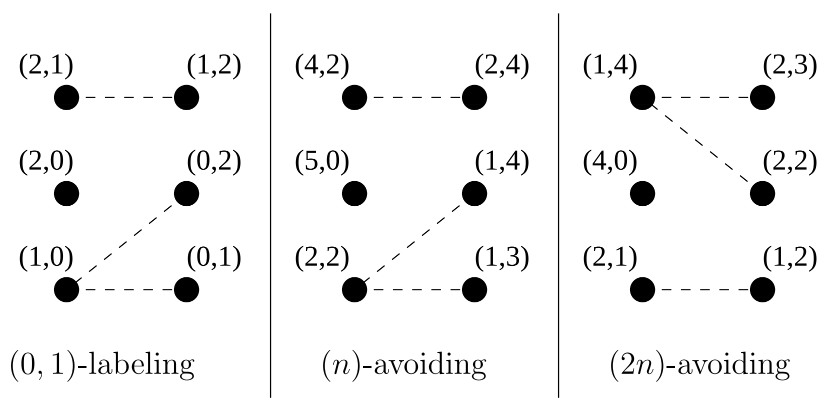


Figure A.9: Theorem 6.13: Case 5

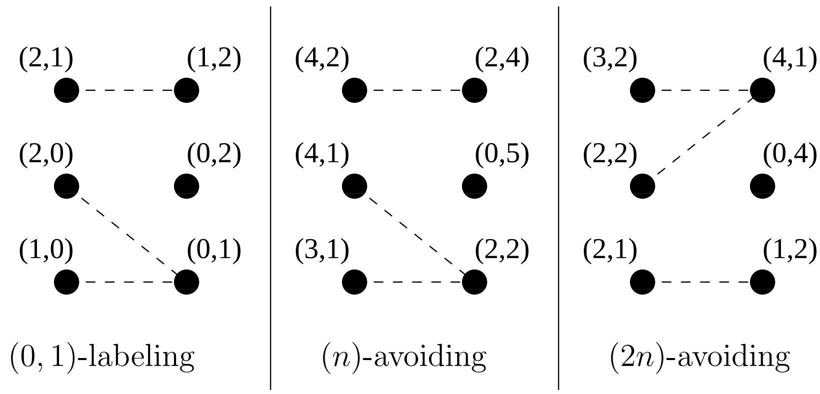


Figure A.10: Theorem 6.13: Case 6

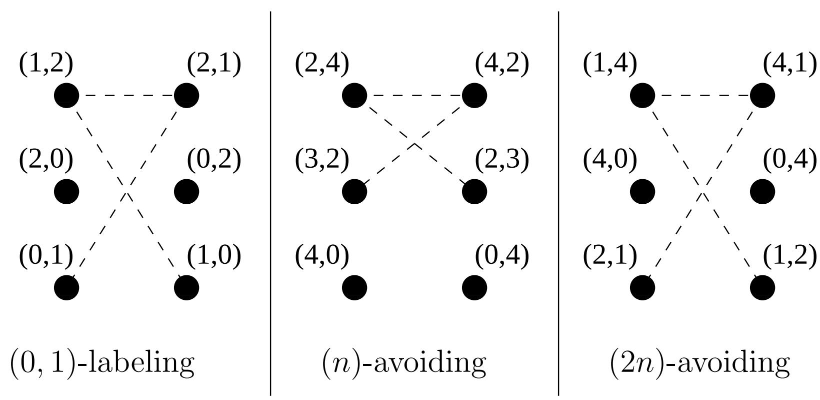


Figure A.11: Theorem 6.13: Case 7

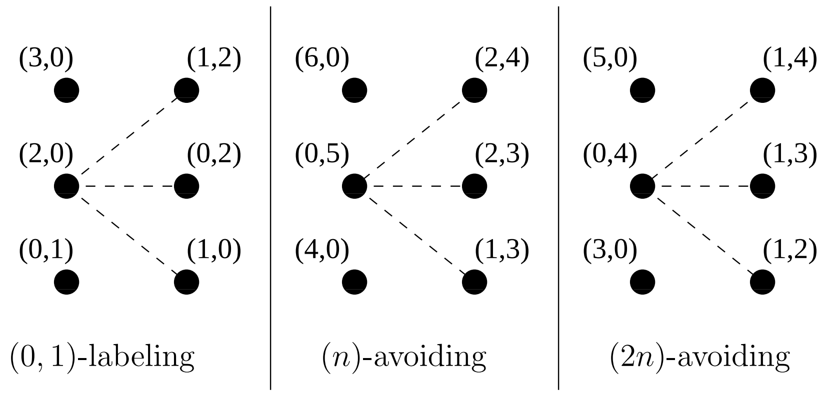


Figure A.12: Theorem 6.13: Case 8

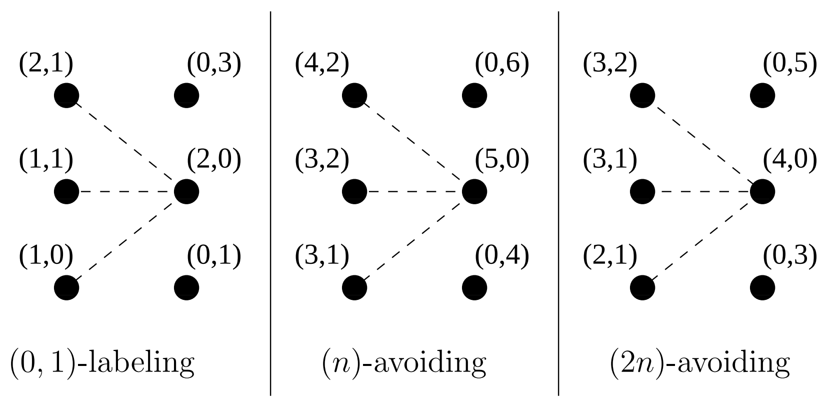


Figure A.13: Theorem 6.13: Case 9

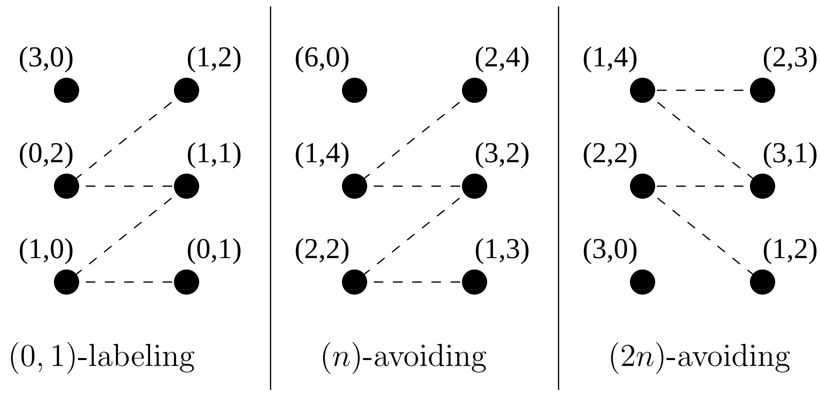


Figure A.14: Theorem 6.13: Case 10

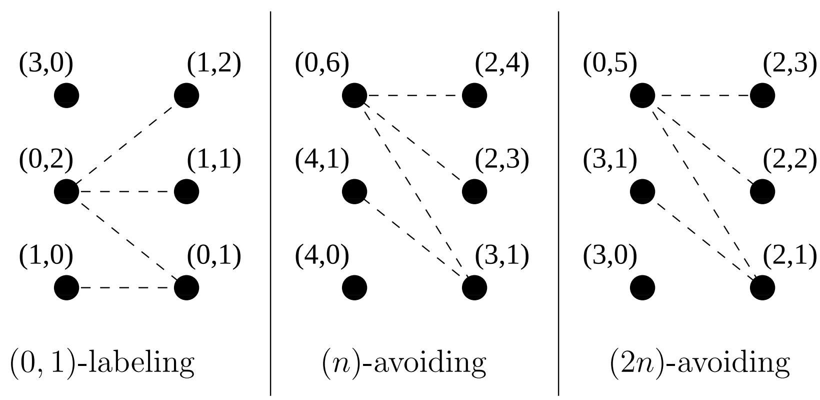


Figure A.15: Theorem 6.13: Case 11

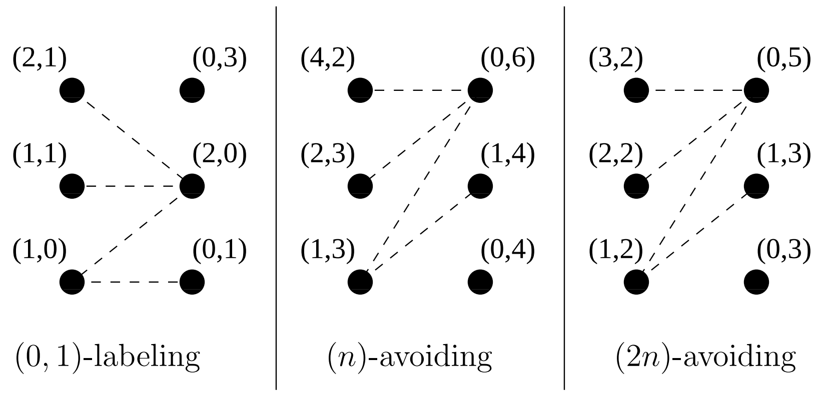


Figure A.16: Theorem 6.13: Case 12

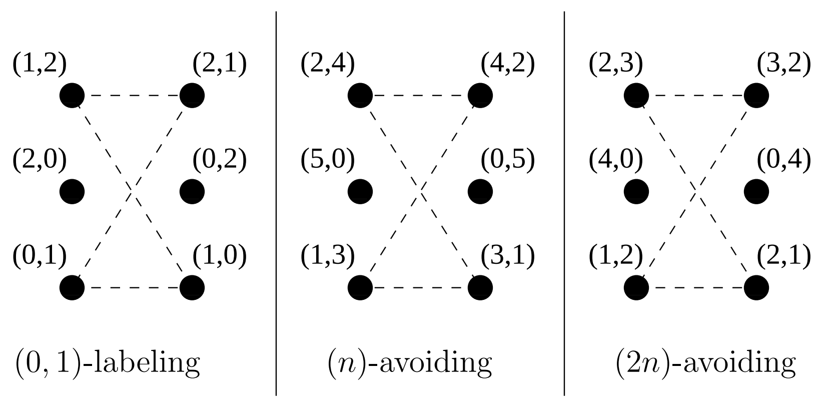


Figure A.17: Theorem 6.13: Case 13

APPENDIX B. A Proof of Theorem 3.3

In [12] the authors give a general construction and leave some details to the reader in order to show that Theorem 3.3 follows.

Here is a proof of Theorem 3.3 based on the general ideas in [12].

Proof of Theorem 3.3: We will show that the four categories of sequences in Theorem 3.3 are necessary and sufficient to cover all the arc-minimal digraphs of a given order. Furthermore, all the degree sequences of types (i.)-(iv.) can actually be realized. The necessity of types (i.) and (ii.) is clear, because from term by term comparison with any other irregular degree sequence, they are seen to have the fewest arcs. The third and fourth criteria are clearly necessary because we cannot have an odd number of odd degree vertices in the underlying graph of our realizations of arc-minimal degree sequences. Term by term comparison shows these two types have a minimal number of arcs when compared with any other type of degree sequence.

Suppose we have a sequence S which does not fall into one of the four criteria, and has a realization with a fewer or equal number of arcs than a realization of a sequence S' in the criteria from (i.)-(iv.) appropriate to the number of terms in S. If this were the case, a realization of S could not have arc sum less than a realization of S', from term by term comparison: we match the smallest terms of S with the smallest terms of S' in total degree. We get that if S' is of type (iii.), then S must have an odd number of odd total degrees, contradiction. If S' is of type (iv.), then S must have an odd number of odd total degrees, contradiction.

Similarly, if S' is of type (ii.), we get a contradiction. If S has a realization of minimal size, then it must have the form (ii), (iii), or (iv). If S does not have the form (ii) or (iii) then it must have the form (iv).

Next, we can show that any degree sequence from the four categories can be realized as an oriented graph.

Type (i.) The first type of degree sequence is realized by an oriented graph.

Type (ii.) The second type of degree sequence falls into several cases.

Assume τ_n is even and suppose $(\frac{\tau_n}{2}, \frac{\tau_n}{2}) \notin B_{\tau_n}[m_n]$.

Let $S_1 = \{(a,b) : a \geq b\}$. Take $S_2 = \{(a,b) : a \leq b\}$. Realize B_0 through $B_{\tau_n - 1}$ by a set of ascending order transitive tournaments. Label Hamiltonian paths of these tournaments $P_1, ...P_{\tau_n}$ where the subscript indicates order. Queue the arcs of the Hamiltonian paths so that a_i precedes a_j if a_i there is a directed path from a_i to a_j when $\{a_i, a_j\} \subset P_k$ or if $a_i \in P_k$ and $a_j \in P_m$ and $k < m$. Notice this is a linear order. Now queue the degree pairs in S_1 in any order and the degree pairs in S_2 in any order.

For each degree pair $(a,b) \in B_{\tau_n}[m_n]$ we will associate a vertex $v_{(a,b)}$. Now, there are $\binom{\tau_n}{2}$ arcs in our queue of Hamiltonian paths and at most τ_n null vertices we have associated with our degree pairs. The null vertices have at most $(\tau_n)^2$ arcs incident with them in any realization of the full degree set since $B_{\tau_n}[m_n]$ is not a complete block. We have 2 cases, one where $B_{\tau_n}[m_n] = B_{\tau_n} - \{(\frac{\tau_n}{2}, \frac{\tau_n}{2})\}$; otherwise there are at least 3 degree pairs missing from the complete block because the degree sets are balanced, we are assuming τ_n is even, and $(\frac{\tau_n}{2}, \frac{\tau_n}{2}) \notin B_{\tau_n}[m_n]$.

Suppose we are in the latter case. At the top of the queue, begin replacing arcs $a_i = \vec{v_i v_{i+1}}$ with arcs $\vec{v_i v_{(a,b)}}$ and $\vec{v_{(c,d)} v_{i+1}}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$. Iterate this process until $deg^+(v_{(a,b)}) = a$ for all $(a,b) \in S_1$ and $deg^-(v_{(c,d)}) = d$ for all $(c,d) \in S_2$. Because the set of null vertices was divided into 2 sets and because $\Sigma_{(a,b)\in S_1} a = \Sigma_{(c,d)\in S_2} d$ in this subcase the process exhausts $\leq \frac{(\tau_n)(\tau_n-2)}{2} < \frac{(\tau_n)(\tau_n-1)}{2}$, half of the arcs in our queue of Hamiltonian paths. Start at the bottom of the queue and begin replacing arcs $a_i = \vec{v_i v_{i+1}}$ with arcs $\vec{v_{(a,b)} v_{i+1}}$ and $\vec{v_i v_{(c,d)}}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$, this time picking our vertices from S_1 and S_2 in the reverse order from which they were previously selected. Iterate this process until $deg^+(v_{(c,d)}) = c$ for all $(c,d) \in S_2$ and $deg^-(v_{(a,b)}) = b$ for all $(a,b) \in S_2$. A realization of the sequence as an oriented graph results, unless $\tau_n \geq \binom{\tau_n}{2}$ or rather, $\tau_n \leq 3$. In this case, $m_n \leq 2$. Thus, since the set $B_{\tau_n}[m_n]$ is balanced, $m_n = 2$ since it is non-zero. We have thath $B_{\tau_n}[m_n]$ is either $(1,2);(2,1)$ or $(3,0);(0,3)$. In either case put an arc between two null vertices associated with the given pair of degree pairs. In the realization of B_0, B_1, B_2 we have transitive tournaments of order 3,2,1. If we trace these tournaments we get 3 arcs. Use these arcs to build up the degrees of the 2 null vertices and get the desired realization of our sequence.

Now suppose the former subcase above, namely, $B_{\tau_n}[m_n] = B_{\tau_n} - \{(\frac{\tau_n}{2}, \frac{\tau_n}{2})\}$. In this case, realize a tournament of order τ_n such that the degrees of the tournament are majorized by the degree set $B_{\tau_n}[m_n]$, then perform $\frac{\tau_n}{2}$ arc swaps. (It is clear that τ_n is even in this subcase; because the final degree set is balanced as is the tournament it is possible to pair the vertices so that one vertex in the pair is short one in-degree, the other vertex is short one out-degree.) The result is an oriented graph as long as $\tau_n < \binom{\tau_n}{2}$, as long as $3 < \tau_n$. Because τ_n is even, it remains to check all the cases for $\tau_n = 2$; the only case is $m_n = 2$ which gives the degree sequence $(0,0);(1,0);(0,1);(2,0);(0,2)$ and is realized as the oriented graph

$$\{v_1\vec{v}_2, v_3\vec{v}_2, v_3\vec{v}_4\}$$

with vertex set v_i where $1 \leq i \leq 5$.

Assume now that $(\frac{\tau_n}{2}, \frac{\tau_n}{2}) \in B_{\tau_n}[m_n]$.

Take $S_1 = \{(a,b) : a \geq b\}$. Take $S_2 = \{(a,b) : a \leq b\}$. Let $S_3 = \{(\frac{\tau_n}{2}, \frac{\tau_n}{2})\}$. Realize B_0 through $B_{\tau_n - 1}$ by a set of ascending order transitive tournaments. Label Hamiltonian paths of these tournaments $P_1, ...P_{\tau_n}$. Queue the arcs of the Hamiltonian paths so that a_i preceded a_j if a_i has a directed path to a_j when $\{a_i, a_j\} \subset P_k$ or if $a_i \in P_k$ and $a_j \in P_m$ and $k < m$. Notice this is a linear order. Now queue the degree pairs in S_1 in any order and the degree pairs in S_2 in any order.

For each degree pair $(a,b) \in B_{\tau_n}[m_n]$ associate a vertex $v_{(a,b)}$. Now, there are $\binom{\tau_n}{2}$ arcs in the queue of Hamiltonian paths and at most $\tau_n - 1$ (because the degree set is balanced and not full we must remove at least 2 entries) null vertices associated with degree pairs having at most $(\tau_n)(\tau_n - 1)$ arcs incident with those vertices in any realization of the full degree set.

Now, at the top of the queue begin replacing arcs $a_i = v_i\vec{v}_{i+1}$ with arcs $v_i\vec{v}_{(a,b)}$ and $v_{(c,d)}\vec{v}_{i+1}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1 \cup S_3$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$. In the queue of null vertices associated with vertex pairs, queue the unique element of S_3 before any of the elements of S_1. Iterate this process until $deg^+(v_{(a,b)}) = a$ for all $(a,b) \in S_1 \cup S_3$ and $deg^-(v_{(c,d)}) = d$ for all $(c,d) \in S_2$.

Start at the bottom of the queue and begin replacing arcs $a_i = v_i\vec{v_{i+1}}$ with arcs $v_{(a,b)}\vec{v}_{i+1}$ and $v_i\vec{v_{(c,d)}}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1 \cup S_3$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$. Iterate this process until $deg^+(v_{(c,d)}) = c$ for all $(c,d) \in S_2$ and $deg^-(v_{(a,b)}) = b$ for all $(a,b) \in S_2$. Again, reverse the order of vertex selection from $S_1 \cup S_3$ and S_2.

There are no multiple arcs unless the same condition as above $\tau_n = 2$, $[m_n] = 2$ results; this is a contradiction because then $(\frac{\tau_n}{2}, \frac{\tau_n}{2}) \notin B_{\tau_n}[m_n]$.

Finally assume that τ_n is odd. In this case, realize a tournament of order $[m_n]$ such that the degrees of the tournament are majorized by the degree set $B_{\tau_n}[m_n]$, then perform arc swaps. The result is an oriented graph as long as $m_n(\tau_n - m_n) < \binom{\tau_n}{2}$. The function $c - c^2$ is maximized at the midpoint of the interval $[0,1]$, so the only cases that remain to check are when $\frac{\tau_n^2}{4} \geq \binom{\tau_n}{2}$, or rather when $\frac{\tau_n}{2} > \tau_n - 1$, or rather when $\tau_n \geq 2$. This is the trivial case (since τ_n is odd).

In the case that $m_n = 0$, realize the degree set as a union of transitive tournaments.

Type (iii.) Realize B_0 through $B_{\tau_n - 2}$ by a set of ascending order transitive tournaments $T_1 ... T_{\tau_n - 1}$ and add a tournament of order $\tau_n - 1$ called $T'_{\tau_n - 1}$. Label Hamiltonian paths of the first non-trivial $\tau_n - 2$ tournaments other than $T'_{\tau_n - 1}$ by $P_2, ... P_{\tau_n - 1}$ where here subscript indicates order. Queue the arcs of the Hamiltonian paths P_2 through $P_{\tau_n - 1}$ so that a_i preceded a_j if a_i has a directed path to a_j when $\{a_i, a_j\} \subset P_k$ or if $a_i \in P_k$ and $a_j \in P_m$ and $k < m$.

Associate every ordered pair $(a,b) \in B_{\tau_n-1} - \{(\alpha,\beta)\}$ with a vertex $v_{(a,b)}$. Similarly, associate every pair in $(a,b) \in B_{\tau_n}[m_n+1]$ with a vertex $v_{(a,b)}$. Realize the vertices of $B_{\tau_n-1} - \{(\alpha,\beta)\}$ and $B_{\tau_n}[m_n+1]$ as appropriately sized tournaments such that all the vertices have the property that $deg(v_{(a,b)})$ is majorized by (a,b). Call this realization R.

Divide the vertices which do not realize the degree on their subscripts into 3 sets. Let $S_1 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a > deg_R^-(v_{(a,b)}) - b\}$. Let $S_2 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a < deg_R^-(v_{(a,b)}) - b\}$. Let $S_3 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a = deg_R^-(v_{(a,b)}) - b\}$. Now, at the top of the queue of arcs in Hamiltonian paths, begin replacing arcs $a_i = v_i\vec{v_{i+1}}$ with arcs $v_i\vec{v_{(a,b)}}$ and $v_{(c,d)}\vec{v}_{i+1}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1 \cup S_3$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$. Iterate this process until $deg^+(v_{(a,b)}) = a$ for all (a,b) and $deg^-(v_{(c,d)}) = d$ for all (c,d) for ordered pairs in $B_{\tau_n}[m_n+1]$. Less than $\frac{1}{2}$ of the arcs in our queue of Hamiltonian paths are exhausted because the tournaments are balanced as are the degree sets and a quantity less than $(\tau_n+1)+(\tau_n-1)$ divided by $2 = \tau_n$. Start at the bottom of the queue and begin replacing arcs $a_i = v_i\vec{v_{i+1}}$ with arcs $v_{(a,b)}\vec{v}_{i+1}$ and $v_i\vec{v_{(c,d)}}$ where $v_{(a,b)}$ is associated with $(a,b) \in S_1 \cup S_3$ and $v_{(c,d)}$ is associated with $(c,d) \in S_2$. Iterate this process until $deg^+(v_{(c,d)}) = c$ for all $(c,d) \in S_2$ and $deg^-(v_{(a,b)}) = b$ for all $(a,b) \in S_1 \cup S_3$, giving realization R'. It remains to show that $|A(R')| - |A(R)| < (\tau_n-1)(\tau_n-2)$.

To show this inequality notice $|A(R')| - |A(R)| = [(m_n+1)(\tau_n) + (\tau_n-1)(\tau_n-1)] - [(\tau_n-1)(\tau_n-2) + (m_n+1)(m_n)]$. The inequality reduces to $(m_n+1)(\tau_n - m_n) + 2\tau_n - 4 < (\tau_n-1)(\tau_n-2)$. Of course, $m_n \leq \tau_n - 2$ so that there are always enough arcs to swap and transform R to R'. Regardless of the value of m_n, the inequalities $(\tau_n - m_n) + 2 < \tau_n - 1$ or $(\tau_n - m_n) + 3 < \tau_n - 2$ occur. Consider the latter inequality and get $5 < m_n$. If $m_n \leq 5$ then $6(\tau_n - 5) + 2\tau_n - 4 < (\tau_n - 1)(\tau_n - 2)$ or $8\tau_n - 34 < (\tau_n - 1)(\tau_n - 2)$. Since the function is quadratic, test the point $\tau_n = 4$ and get $32 - 34 < 6$. The realization is an oriented graph. For notice that a counterexample can only occur if $\tau_n - m_n \geq \binom{\tau_n - 1}{2} + \tau_n - 2$ or rather $\tau_n^2 - 4\tau_n + m_n \leq 0$. Because both τ_n and m_n are odd, it follows we need only consider the cases $\tau_n = 1$ and $\tau_n = 3$. In the first case we need only consider the arc $v_1\vec{v}_2$ with degree sequence $(1,0);(0,1)$. In the second case there are 5 subcases: 1 of the first type, 4 of the second type and 2 of the second type symmetric up to arc reversal, $B_0, B_1, B_2 - \{(\alpha, \beta)\}, B_3[m_n + 1]$. Since m_n is odd there are 2 possibilities: $m_n + 1 = 4$ (there is one example of this first type) or $m_n + 1 = 2$ (there are 4 examples of this second type). In the first type, we realize the sequence $(0,0);(1,0);(0,1);(2,0);(0,2);(3,0);(2,1);(1,2);(0,3)$ by performing an arc swap on $T_1, T_2, T_4 \cup \{v\vec{x}\}$. Swap an arc $u\vec{w} \in T_4$ so that we replace $u\vec{w}$ with $v\vec{w} \cup u\vec{x}$. The desired degree sequence (above) results. The first case of the second type is the sequence $(0,0);(1,0);(0,1);(2,0);(0,2);(3,0);(0,3)$ and is realized by the oriented graph

$$\{v_1\vec{v}_2, v_1\vec{v}_3, v_1\vec{v}_4, v_5\vec{v}_3, v_5\vec{v}_4, v_6\vec{v}_3\}$$

with the vertex set

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

The second case is the sequence $(0,0);(1,0);(0,1);(2,0);(0,2);(2,1);(1,2)$ and is realized by the oriented graph

$$\{v_1\vec{v}_2, v_3\vec{v}_1, v_1\vec{v}_4, v_5\vec{v}_3, v_5\vec{v}_4, v_6\vec{v}_3\}$$

with the vertex set

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

The third case is the sequence $(0,0); (1,0); (0,1); (1,1); (0,2); (3,0); (1,2)$ and is realized by the oriented graph

$$\{v_1\vec{v}_2, v_1\vec{v}_3, v_1\vec{v}_4, v_2\vec{v}_3, v_5\vec{v}_4, v_3\vec{v}_6\}$$

with the vertex set

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

The fourth case is symmetric to the third up to arc reversal.

Type (iv.) Realize B_0 through B_{τ_n-1} by a set of ascending order transitive tournaments. Label Hamiltonian paths of these tournaments $P_1, ...P_{\tau_n-1}$. Queue the arcs of the Hamiltonian paths so that a_i preceded a_j if a_i has a directed path to a_j when $\{a_i, a_j\} \subset P_k$ or if $a_i \in P_k$ and $a_j \in P_m$ and $k < m$. Notice this is a linear order. Now queue the degree pairs in S_1 in any order and the degree pairs in S_2 in any order. For each degree pair S_1 associate a vertex $v_{(a,b)}$. Now, there are $\binom{\tau_n-1}{2}$ arcs in our queue of Hamiltonian paths.

Before we begin swapping arcs from the queue of arcs in the Hamiltonian paths, begin by associating a vertex $v_{(\alpha,\beta)}$ with degree (α, β); furthermore, for all degree pairs (a, b) in $B_{\tau_n}[m_n - 1]$ associate a vertex $v_{(a,b)}$. Notice that since m_n and τ_n have the same parity it is clear that in any realization, there are at most $(\tau_n)(\tau_n - 2)$ arcs incident to or from the set of vertices associated with $B_{\tau_n}[m_n - 1]$. Build a transitive tournament of order $m_n - 1$ on the null vertices associated with $B_{\tau_n}[m_n - 1]$ without inducing a degree on a vertex that cannot be majorized by its degree pair. Notice that $B_{\tau_n}[m_n - 1] \cup \{(\alpha, \beta)\}$ is balanced.

Now $m_n(\tau_n - m_n) + \tau_n + 1$ is maximized when $m_n = \frac{\tau_n}{2}$. We get that unless $m_n(\tau_n - m_n) + \tau_n + 1 \geq \frac{(\tau_n)(\tau_n - 1)}{2}$ our realization will be an oriented simple graph. That is, when $\tau_n^2 + 4\tau_n + 4 \geq 2\tau_n(\tau_n - 1)$, when $3\tau_n + 2 \geq \tau_n^2$, our realization will be an oriented simple graph. The only exceptions are when $\tau_n = 1, 3$.

Let the realization up to this point be called R. Divide the set of null vertices into 3 sets. Let $S_1 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a > deg_R^-(v_{(a,b)}) - b\}$. Let $S_2 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a < deg_R^-(v_{(a,b)}) - b\}$. Let $S_3 = \{v_{(a,b)} : deg_R^+(v_{(a,b)}) - a = deg_R^-(v_{(a,b)}) - b\}$. Now, at the top of the queue of arcs in Hamiltonian paths, begin replacing arcs $a_i = \vec{v_i v_{i+1}}$ with arcs $\vec{v_i v_{(a,b)}}$ and $\vec{v_{(c,d)} v_{i+1}}$ where $v_{(a,b)} \in S_1 \cup S_3$ is associated with (a, b) and $v_{(c,d)}$ is associated with $(c, d) \in S_2$. Iterate this process until $deg^+(v_{(a,b)}) = a$ for all (a, b) and $deg^-(v_{(c,d)}) = d$ for all (c, d) for ordered pairs in $B_{\tau_n}[m_n - 1] \cup (\alpha, \beta)$. We necessarily exhaust less than $\frac{1}{2}$ of the arcs in our queue of Hamiltonian paths. Start at the bottom of the queue and begin replacing arcs $a_i = \vec{v_i v_{i+1}}$ with arcs $\vec{v_{(a,b)} v_{i+1}}$ and $\vec{v_i v_{(c,d)}}$ where $v_{(a,b)}$ is associated with $(a, b) \in S_1$ and $v_{(c,d)}$ is associated with $(c, d) \in S_2$. Iterate this process until $deg^+(v_{(c,d)}) = c$ for all $(c, d) \in S_2$ and $deg^-(v_{(a,b)}) = b$ for all $(a, b) \in S_2$. This exhausts less than half the arcs in the queue and so none of the augmented representative vertices have multiple arcs to or from any of the vertices in our collection of vertices that formed the vertex set of transitive tournaments at the beginning of the arc-swapping process. That is, the sequence is realized as an oriented graph. The cases that are not covered by the preceding algorithm are limited to when $\tau_n = 3$ and $\tau_n = 1$. For the tournament used to build the realization of the vertices with degrees in $B_{\tau_n}[m_n - 1] \cup (\alpha, \beta)$ has order m_n.

By the description of the algorithm the desired degree sequence is realized as an oriented graph as long as $\tau_n > 3$. In the case that $m_n = 1, \tau_n = 3$ we get the digraph

$$\{v_1\vec{v}_2, v_1\vec{v}_3, v_4\vec{v}_3, v_4\vec{v}_1, v_5\vec{v}_1\}$$

on the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. If $3 \leq m_n$ we get the following 4 cases, 2 of which are symmetric up to arc reversal. Given the sequence

$$(0,0); (1,0); (0,1); (2,0); (1,1); (0,2); (2,1); (1,2); (2,2)$$

we get the following realization on vertex set v_i, $1 \leq i \leq 9$:

$$\{v_1\vec{v}_2, v_3\vec{v}_4, v_3\vec{v}_5, v_4\vec{v}_6, v_4\vec{v}_5, v_6\vec{v}_7, v_5\vec{v}_7, v_7\vec{v}_8, v_5\vec{v}_8\}.$$

The next case has the sequence

$$(0,0); (1,0); (0,1); (2,0); (1,1); (0,2); (3,0); (0,3); (2,2)$$

and the following realization on vertex set v_i, $1 \leq i \leq 9$ is a simple oriented graph

$$\{v_1\vec{v}_2, v_3\vec{v}_4, v_3\vec{v}_6, v_8\vec{v}_3, v_7\vec{v}_3, v_8\vec{v}_5, v_5\vec{v}_4, v_8\vec{v}_6, v_7\vec{v}_4\}.$$

The next case is the sequence

$$(0,0); (1,0); (0,1); (2,0); (1,1); (0,2); (3,0); (1,2); (1,3)$$

we get the following realization on vertex set v_i, $1 \leq i \leq 9$:

$$\{v_1\vec{v}_2, v_4\vec{v}_7, v_3\vec{v}_4, v_5\vec{v}_4, v_6\vec{v}_4, v_6\vec{v}_5, v_3\vec{v}_5, v_3\vec{v}_8, v_7\vec{v}_8\}.$$

The final case is

$$(0,0); (1,0); (0,1); (2,0); (1,1); (0,2); (0,3); (2,1); (3,1)$$

and is symmetric to the preceding case up to arc-reversal. ■

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