FEASIBLE FLOWS IN MULTICOMMODITY GRAPHS

by

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ABSTRACT

This thesis establishes the minimal representation of the necessary conditions for feasible supplies and demands for a given multicommodity network. The fundamental theorem is an extension of the Wallace-Wets connectivity result for both directed and undirected graphs. A system of absolute value inequalities is developed for the undirected case, and special properties of this system are explored. Additional results include counting the number of nonredundant inequalities for specific classes of graphs.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication. Signed________________________

Harvey Greenberg

CONTENTS

1. Introduction

The set of feasible supplies and demands for a network flow problem can be described by a system of linear inequalities. Wallace and Wets have characterized the redundant inequalities in this system. The main theorem of this thesis is an extension of the Wallace-Wets Theorem to the case of undirected multicommodity graphs. In order to obtain the extension, a special system of inequalities is developed and several characteristics of this system are discussed. Following the extension, we prove several theorems on its usefulness.

In Chapter 2 we provide technical background on network flows. We discuss some differences between single and multicommodity network flows, as well as the pertinent literature on multicommodity flows. We also discuss the background literature for the Wallace-Wets Theorem.

In Chapter 3 we present the Wallace-Wets Theorem for single commodity networks along with a detailed proof of the theorem.

Chapters 4 and 5 contain the main results of the thesis. In Chapter 4 we show that the Wallace-Wets Theorem has a natural extension to directed multicommodity graphs. In Chapter 5 we consider the case of undirected multicommodity graphs, which is not as straightforward. A system of inequalities which gives a necessary condition for feasible supplies and demands for undirected multicommodity graphs is developed, and we explore properties of this system. Next we present our extension of the Wallace-Wets Theorem. We also present several theorems related to using the extension and the benefit realized.

In Chapter 6 we suggest avenues for further research. This includes further work related to the system of inequalities developed in Chapter 5, as well as evaluating the effects of preclassification of nodes as supply, demand, and transshipment nodes.

2. Technical Background

2.1 Terms and Concepts

Let $G = [V, E]$ be a finite graph with node set V and edge set E. For the problems we will consider in this thesis, there is no loss in generality to assume G does not contain any parallel edges. We therefore denote an edge by its endpoints: (i, j). This is an unordered pair in the context of undirected graphs, but an ordered pair in the context of directed graphs. We also assume, without loss in generality, that G is connected.

For any $Y \subseteq V$, let $\sim Y$ denote the complement of Y: $\sim Y = \{i \in V : i \notin Y\}$. Then the associated cut is the edge set: $< Y, \sim Y >= \{(i,j) \in E : i \in Y \text{ and } j \in \sim Y\}$. Note that for undirected graphs, $< Y, \sim Y >=<\sim Y, Y >$, but that this is not true for directed graphs. For each $(i, j) \in E$ there is a capacity U_{ij} which limits total flow across the edge. This extends naturally to subsets of E: $U(Y, \sim Y) \equiv \sum_{(i,j)\in<Y,\sim Y>} U_{ij}$. For each $Y \subseteq V$, we let $G(Y)$ denote the induced subgraph on the node set Y.

For each $i \in V$, we let the demand at node i be given by b_i. That is, if $b_i > 0$ then node i has a demand, if $b_i < 0$ then node i has a supply, and if $b_i = 0$ then node i has neither demand nor supply.

In the context of multicommodity graphs, s_k and t_k will denote the origin and destination, respectively, of commodity k, $k = 1, 2, \ldots, p$. We will let q_k represent the demand for commodity k. If the origin and destination of each commodity are not known, then b_i^k represents the demand for commodity k at node i.

The work in this thesis deals mainly with multicommodity graphs. It is well known that many of the "nice" properties associated with network flow problems do not extend to the multicommodity case. A complicating matter in multicommodity flows is that flows in opposite directions on an edge do not cancel if they are different commodities. This leads to a variety of problems. In a single commodity network flow problem, the maximum flow is equal to the minimum cut, and given integer capacities, the maximum flow is guaranteed to be integer. Neither of these properties extend to the general multicommodity case. Additionally, in single commodity network flow problems, the condition that the capacity of any cut is greater than or equal to the demand across the cut, is a necessary as well as sufficient condition for feasibility of the network. That is, if q is the required amount of flow to be sent through the network, the network is feasible if and only if

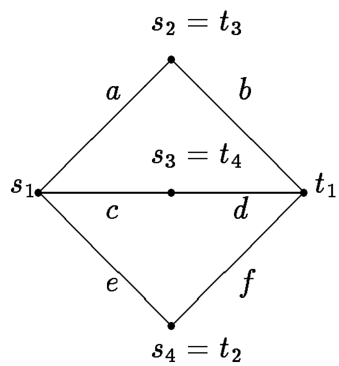
$$q \leq U(Y, \sim Y) \qquad \forall Y \subseteq V.$$

The analogous multicommodity condition

$$\sum (q_k : Y \text{ separates } s_k \text{ and } t_k\ ) \leq U(Y, \sim Y) \qquad \forall Y \subseteq V$$

is necessary for undirected graphs, but it is not sufficient for feasibility in general. Much work has been done to determine specific classes of graphs for which this condition is sufficient, and we will examine several of the results in the next section. We now present some examples of the above statements.

Example

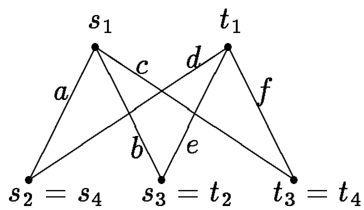


Assume a capacity of 1 on each edge. A maximum flow is given by the following, where x_i is the flow vector for edge i.

$$x_a = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right\}$$
$$x_b = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right\}$$
$$x_c = \left\{\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right\}$$
$$x_d = \left\{\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right\}$$
$$x_e = \left\{\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right\}$$
$$x_f = \left\{\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right\}$$

The flow is fractional, even though the capacities are integer. Also, the value of the maximum flow, which is the sum of the flows for each commodity, is $1 + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 3$. The notion of a cut in single commodity graphs translates to a disconnecting set in multicommodity graphs, that is, a set of edges whose removal will disconnect the origin and destination of each commodity. For more on disconnecting sets, see [3]. A minimum disconnecting set for this graph consists of the edges $\{a, c, d, f\}$. The capacity of this set is 4, which is not equal to the maximum flow.

<u>Example</u>



Suppose the capacity of each edge is 1, and the demand for each commodity is 1. That is, we wish to send one unit of commodity 1 from s_1 to t_1, and so on. It can be verified that the cut condition,

$$\sum(q_k : Y \text{ separates } s_k \text{ and } t_k\ ) \leq U(Y, \sim Y),$$

holds for every set Y. For example, if $Y = \{s_1\}$, then $< Y, \sim Y >= \{a, b, c\}$, so $U(Y, \sim Y) = 3$. The only commodity separated by Y is commodity 1, so the left-hand-side of the cut condition is 1. Thus the inequality holds, as could be shown for every Y. However, it is not possible to satisfy all of the demands simultaneously, so the problem is not feasible.

2.2 The Literature

As discussed in the previous section, a central issue in the study of multicommodity network flow problems is that the cut condition,

$$\sum(q_k : Y \text{ separates } s_k \text{ and } t_k\ ) \leq U(Y, \sim Y),$$

is not always a sufficient condition for feasibility. Here we will present several of the results which characterize graphs for which this condition is sufficient for feasibility. We refer the reader to the appropriate papers for the proofs. As the results are very different for the undirected and directed cases, we will discuss them separately, beginning with the undirected case.

Some of the pioneering work on this subject was done by T.C. Hu ([16],[17]). Hu proved that the cut condition is sufficient for feasibility if there are only two commodities. Additionally, he proved that in the two-commodity case, the maximum sum of the flows is equal to the minimum capacity of all cuts separating the origins and destinations of both commodities (the minimum disconnecting set). He provided an algorithm for constructing the flows. Finally, Hu showed that if the demands and the capacities are all even integers, then a two-commodity flow will have the property that the flows for each commodity on each edge are integer. This is commonly referred to as a half-integer property, and it arises in other cases as well.

Hu's result is the only one that relies on a specific number of commodities, and it does not generalize to three or more commodities. All other results rely on the graph having special characteristics.

A few definitions are in order before we discuss the next result. A planar graph is a graph that can be drawn in the plane so that no edges cross. Given a fixed drawing of a planar graph, we can identify various regions, which are simply the areas enclosed by the edges of the graph. The infinite region is the entire region outside the graph, and the boundary of the infinte region consists of the nodes and edges that separate the infinite

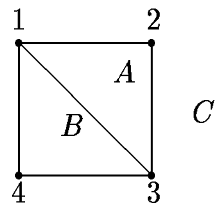


Figure 2.1: A Planar Graph and Its Regions

region from the "interior" of the graph. For example, in Figure 2.1, A is the region enclosed by the edges (1,2), (2,3), and (1,3); B is the region enclosed by the edges (1,3), (3,4), and (1,4); C is the infinite region. The boundary of the infinite region is 1, (1,2), 2, (2,3), 3, (3,4), 4, (4,1). Note that we could also draw the graph with edge (1,3) on the other side of node 2. This would still be a planar representation, but different regions would be defined. The boundary of the infinite region for that drawing would be 1, (1,3), 3, (3,4), 4, (4,1).

We are now ready for the theorem of Okamura and Seymour ([28]).

Theorem 2.1 (Okamura and Seymour, 1981) If G is planar and can be drawn in the plane so that $s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_p$ are all on the boundary of the infinite region, then

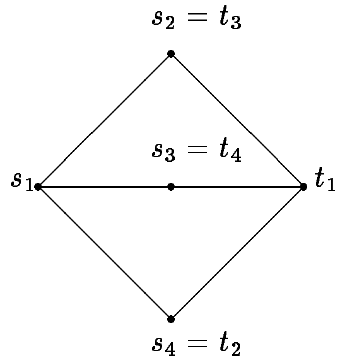
$$\sum(q_k : Y \text{ separates } s_k \text{ and } t_k\ ) \leq U(Y, \sim Y)$$

is a sufficient condition for feasibility. Further, if the demands and capacities are all integers, then there exists a set of feasible flows with the flow for each commodity on each edge being integer.

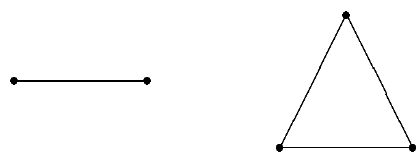
That is, if a graph can be drawn in the plane so that all the sources and sinks are on the boundary of the infinite region, then the cut condition is sufficient for feasibility. There have been some extensions of this theorem which relax slightly the condition of all the sources and sinks being on the boundary of the infinite region. Okamura ([27]) proved that the cut condition is sufficient if G is a planar graph and can be drawn with one set of sources and sinks on the boundary of any region, and the remainder of the sources and sinks on the boundary of the infinite region, with the source and sink for any one commodity on the same boundary. In the same paper, it is shown that the cut condition is also sufficient if G is planar and can be drawn with the sources and sinks for some commodities located on the boundary of the infinite region, and all other commodities share the same sink, also located on the boundary of the infinite region, but those sources may be located anywhere. Seymour ([39]) proved that the cut condition is sufficient if G is a planar graph and if the source and sink for each commodity may be joined by an edge without violating planarity. For a discussion of all of the above results plus algorithms for finding planar multicommodity flows, the reader is referred to [26].

The final class of undirected graphs that we will present here actually relies on the configuration of the sources and sinks of the commodities. We call this configuration the commodity graph. Given a connected undirected network $G = [V, E]$ with sources $s_1, s_2, \ldots, s_p$ and sinks $t_1, t_2, \ldots t_p$, let the *commodity graph*, $H = [T, U], (T \subset V)$ be the undirected graph whose edges correspond the the source-sink pairs of G. That is, $e \in U \iff e = (s_k, t_k)$. Note that U is not necessarily a subset of E.

Example Given the following graph



we construct the commodity graph by drawing each node and then connecting s_1 to t_1, s_2 to t_2, and so on. The resulting graph is



Karzanov ([19]) presents a result of Papernov regarding the commodity graph.

Theorem 2.2 (Papernov, 1976) (1) If H is K_4 (the complete graph on four nodes), C_5 (the cycle on five nodes), or a union of two stars, and if the cut condition is satisfied, then the problem is feasible.

(2) If H is a graph which does not belong to the collection above, then there exists a graph G, capacities U, and demands q, such that the cut condition is satisfied and yet the problem is not feasible.

In other words, the cut condition can be known to be sufficient based only on the commodity graph if and only if the commodity graph is a complete graph on four nodes, a cycle on five nodes, or a union of two stars. If the commodity graph is other than the graphs mentioned, then it is possible to construct a graph which has that commodity graph and for which the cut condition is not sufficient for feasibility.

We are now ready to discuss some results on feasibility in directed graphs. As mentioned earlier, the directed case is very different from the undirected case, and it has received much less attention in the literature. Note that in the case of directed graphs, the cut condition changes slightly to become

$$\sum(q_k : s_k \in Y, t_k \in \sim Y) \leq U(Y, \sim Y).$$

It is known that Hu's result for two-commodity graphs does not extend to the multicommodity case ([21]). However, the cut condition is sufficient for feasibility if the network is planar with all the sources on the left side and all the sinks on the right side (a transportation network) and if the sources and sinks appear in the same order (see [29]). The main work is by Nagamochi and Ibaraki, and we will give a brief overview of their results.

In several papers and a Ph.D. thesis ([22], [23], [24]), Nagamochi and Ibaraki studied directed multicommodity networks. They developed three classes of networks, capacity balanced (CB), capacity semi-balanced (CS), and capacity semi-balanced unilateral (CU). In a CB network, the capacity is "balanced" at each node. That is, for every node, the capacity on outgoing arcs plus the demand at the node is equal to the capacity on incoming arcs plus the supply at the node. It is shown that the max-flow min-cut theorem holds for multicommodity networks in CB, and also that the flows will be integer if the capacities and demands are integer. The class CS is shown to be a relaxation of class CB, and the same properties shown for CB networks also hold for CS networks. Finally, it is shown that the properties do not extend to CU networks.

For a more general survey of multicommodity network flows, see either [1] or [21].

The foundations needed for the Wallace-Wets Theorem come from the work done by Gale and Hoffman for single-commodity network flow problems. In [11], Gale gives a system of linear inequalities which are necessary and sufficient for feasibility of a set of supplies and demands in a network. Hoffman ([15]) extends this result to feasible circulations. The inequalities are those generated by considering all bipartitions of the vertices of the network. It is assumed that any node may be a supply node, demand node, or transshipment node. The resulting system of inequalities is

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y)$$

for all $Y \subset V$. This is the system which Wallace and Wets work with, and the inequalities will be referred to as the Gale-Hoffman inequalities.

3. The Wallace-Wets Theorem

In [43], [44], [45], Wallace and Wets examine the system of inequalities which describe the feasible supplies and demands for a network. The inequalities are the ones given by Gale and Hoffman, and Wallace and Wets determine the mimimum number of inequalities needed.

The Wallace-Wets Theorem gives a characterization of the redundant inequalities based on a connectedness criterion. The Gale-Hoffman inequalities can be generated from all possible partitions of the vertices into two sets. Wallace and Wets prove that the resulting inequality is redundant if and only if the induced subgraph on at least one of the partitions is not connected. Their theorem is the following.

Theorem 3.1 [Wallace-Wets Theorem] For all nonempty $Y \subset V$, the associated inequality,

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y)$$

is redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected.

It is important to note that this theorem applies to the case where all supplies must be used and all demands must be met exactly. Thus, the cut system also includes the equation $\sum_{i \in V} b_i = 0$, or supply equals demand. For more on this and other aspects of the Wallace-Wets Theorem, see [14].

We present a proof of the Wallace-Wets Theorem for completeness and to clarify several of their steps. The sufficiency part of the proof is done by construction. We assume that either $G(Y)$ or $G(\sim Y)$ is not connected, and we show how to construct the dependency set for the inequality corresponding to Y. For the necessity part of the proof, we first prove a lemma showing that we may restrict our consideration of the right-hand-sides to an arc membership function. We then show that the only sets which may be used in constructing a redundant inequality are those whose cutset arcs are a subset of the cutset arcs of the original set. If both $G(Y)$ and $G(\sim Y)$ are connected, then there are no sets which have that characteristic. Thus, the original inequality is not redundant.

Lemma 3.2 Let Y_l define a partition of the vertices. If the inequality corresponding to Y_l is redundant, then the arcs in the union of the cutsets for partitions corresponding to the inequalities in the dependency set must be exactly the arcs in $< Y_l, \sim Y_l >$.

Proof: Since the inequality corresponding to Y_l is redundant, there exist $\lambda_j, \lambda \geq 0$, $j = 1, 2, \ldots, 2^{|V|}$, such that

$$\sum_{i\, \in \sim\, Y_l} b_i = \sum_{j \neq l} \sum_{i\, \in \sim\, Y_j} \lambda_j b_i - \lambda \sum_{i \in V} b_i \leq \sum_{j \neq l} \lambda_j U(Y_j, \sim Y_j) \leq U(Y_l, \sim Y_l).$$

Let J be the set of j's such that $\lambda_j > 0$.

First, assume there exists $a \in < Y_l, \sim Y_l >$ such that $a \notin < Y_j, \sim Y_j > \forall j \in J$. Let $a = (\alpha, \beta)$ $(\alpha \in Y_l, \beta \in \sim Y_l)$. We will derive a contradiction. For any $j \in J$ if $\alpha \in Y_j$ then $\beta \in Y_j$ (or else $a \in < Y_j, \sim Y_j >$. Thus, β is in at least as many sets Y_j as α, so β is in at most as many sets $\sim Y_j$ as α. But since $\beta \in \sim Y_l$ and $\alpha \in Y_l$, we must have β

in exactly one more set $\sim Y_j$ than α in order for the left-hand-side of the inequality to work out. So we have a contradiction. Thus, there does not exist $a \in < Y_l, \sim Y_l >$ such that $a \notin < Y_j, \sim Y_j > \forall j \in J$. This tells us that the arcs of the cutset from Y_l are a subset of the union of all arcs in the cutsets from Y_j for $j \in J$.

What we have left to show is that there is no arc in any set $< Y_j, \sim Y_j >, j \in J$ such that the arc is not in $< Y_l, \sim Y_l >$. Let $a \in < Y_l, \sim Y_l >, a = (\alpha, \beta)$ where $\alpha \in Y_l, \beta \in \sim Y_l$. If the inequality from Y_l is redundant, then the following must be true:

$$\sum_{j\,:\,\beta \in \sim Y_j} \lambda_j - \lambda = 1$$

$$\sum_{j\,:\,\alpha \in \sim Y_j} \lambda_j = \lambda$$

where the first equation comes from setting $b_\beta = 1$ and all other b_i's to zero, and the second equation comes from setting $b_\alpha = 1$ and all other b_i's to zero. From the two equations above, we get that

$$\sum_{j\,:\,\beta \in \sim Y_j} \lambda_j - \sum_{j\,:\,\alpha \in \sim Y_j} \lambda_j = 1.$$

This can be rewritten as

$$\sum_{\substack{j\,:\,\beta \in \sim Y_j \\ \alpha \in Y_j}} \lambda_j + \sum_{\substack{j\,:\,\beta \in \sim Y_j \\ \alpha \in \sim Y_j}} \lambda_j - \sum_{\substack{\alpha \in \sim Y_j \\ \beta \in Y_j}} \lambda_j - \sum_{\substack{\alpha \in \sim Y_j \\ \beta \in \sim Y_j}} \lambda_j = 1.$$

The second and fourth terms cancel. Since the third term is nonnegative, we have

$$\sum_{j:(\alpha,\beta) \in < Y_j, \sim Y_j >} \lambda_j \geq 1.$$

So the capacity of each edge in $< Y_l, \sim Y_l >$ is added in to $\sum_{j \neq l} \lambda_j U(Y_j, \sim Y_j)$. Therefore, we cannot have $a \in < Y_j, \sim Y_j >, j \in J$, and $a \notin <$

$Y_l, \sim Y_l >$ because that would give us $\sum_{j \neq l} \lambda_j U(Y_j, \sim Y_j) > U(Y_l, \sim Y_l)$.
■

We are now ready to prove the Wallace-Wets Theorem. We restate the theorem for convenience. Note that this theorem holds for both the directed and undirected cases.

Theorem 3.1 For all nonempty $Y \subset V$, the associated inequality,

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y)$$

is redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected.

Proof: We begin with the "if" direction. First, suppose $G(Y)$ is not connected. Let $Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and there are no edges between Y_1 and Y_2 (See Figure 3.1). Since there are no edges between Y_1 and Y_2, we know that $< Y_1, \sim Y_1 > \cup < Y_2, \sim Y_2 >=< Y, \sim Y >$. The inequality from Y is

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y) = U(Y_1, \sim Y_1) + U(Y_2, \sim Y_2).$$

The inequality from Y_1 is

$$\sum_{i \in \sim Y_1} b_i = \sum_{i \in Y_2} b_i + \sum_{i \in \sim Y} b_i = U(Y_1, \sim Y_1).$$

The inequality from Y_2 is

$$\sum_{i \in \sim Y_2} b_i = \sum_{i \in Y_1} b_i + \sum_{i \in \sim Y} b_i = U(Y_2, \sim Y_2).$$

Summing, we get

$$\sum_{i \in Y_2} b_i + \sum_{i \in \sim Y} b_i + \sum_{i \in Y_1} b_i + \sum_{i \in \sim Y} b_i = \sum_{i \in Y} b_i + 2 \sum_{i \in \sim Y} b_i \leq U(Y, \sim Y).$$

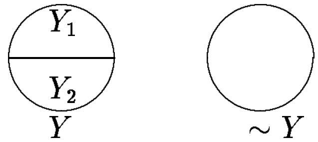


Figure 3.1: Illustration of Y and $\sim Y$

Subtracting $\sum_{i \in V} b_i = 0$ leaves us with

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y).$$

Therefore, the inequality from Y is redundant.

Next, suppose $G(\sim Y)$ is not connected. Let $\sim Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and there are no edges between Y_1 and Y_2. Thus, $< Y, \sim Y > = <\sim Y_1, Y_1 > \cup <\sim Y_2, Y_2 >$. Again the inequality from Y is

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y) = U(\sim Y_1, Y_1) + U(\sim Y_2, Y_2).$$

The inequality from $\sim Y_1$ is

$$\sum_{i \in Y_1} b_i \leq U(\sim Y_1, Y_1).$$

The inequality from $\sim Y_2$ is

$$\sum_{i \in Y_2} b_i \leq U(\sim Y_2, Y_2).$$

Summing, we get

$$\sum_{i \in Y_1} b_i + \sum_{i \in Y_2} b_i = \sum_{i \in \sim Y} b_i \leq U(Y, \sim Y).$$

Therefore, the inequality from Y is redundant.

We now prove the "only if" direction. We assume that $G(Y_l)$ and $G(\sim Y_l)$ are both connected, and we will show that the inequality corresponding to Y_l cannot be redundant.

Let $\underline{l}$ index the set that is the complement of Y_l, that is, $Y_{\underline{l}} = \sim Y_l$. Let c^j be the arc membership vector for the cut defined by Y_j ($c_a^j = 1 \iff a \in < Y_j, \sim Y_j >$). Also, let d^j be the node membership vector for the set Y_j ($d_i^j = 1 \iff i \in Y_j$). Let $J_l = \{1, 2, \ldots, 2^{|V|}\} \setminus \{l, \underline{l}\}$. Let e be the vector of ones. The term $\lambda_0 e$ will account for being able to subtract multiples of the equation that states that total supply equals total demand. We want to show that the following system has no solution:

$$c^l = \sum_{j \in J_l} \lambda_j c^j + \lambda_{\underline{l}} c^{\underline{l}}$$

$$d^l = \sum_{j \in J_l} \lambda_j d^j + \lambda_{\underline{l}} d^{\underline{l}} - \lambda_0 e,$$

$$\lambda_0, \lambda_{\underline{l}}, \lambda_j \geq 0 \qquad \forall j \in J_l.$$

Suppose the system is feasible, and let $(\bar{\lambda}, \bar{\lambda}_{\underline{l}}, (\bar{\lambda}_j, j \in J_l))$ be a solution.

Let

$$J_l^+ = \{j \in J_l | \bar{\lambda}_j > 0\}.$$

That is, J_l^+ indexes the inequalities which are used in constructing the redundancy of inequality l.

If $k \notin < Y_l, \sim Y_l >$, then $c_k^l = 0$ and so

$$0 = \sum_{j \in J_l^+} \bar{\lambda}_j c_k^j + \bar{\lambda}_{\underline{l}} c_k^{\underline{l}} \qquad \forall k \notin < Y_l, \sim Y_l > .$$

This equation tells us that $c_k^j = 0$ if $k \ni < Y_l, \sim Y_l >$ and if $j \in J_l^+$. Written another way, this says: $k \ni < Y_l, \sim Y_l > \rightarrow k \ni < Y_j, \sim Y_j >$ if

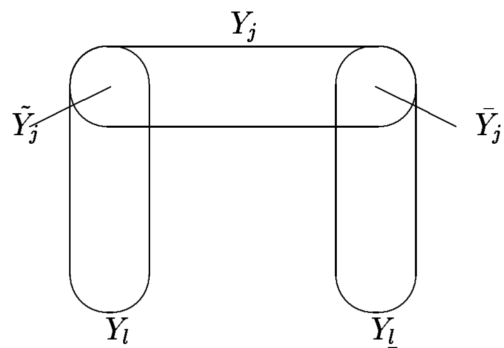


Figure 3.2: Illustration of $\tilde{Y_j}$ and $\bar{Y_j}$

$j \in J_l^+$ If we negate the implication, we get that $j \in J_l^+$ implies $k \in < Y_j, \sim Y_j > \rightarrow k \in < Y_l, \sim Y_l >$. This then is the same as saying

$$j \in J_l^+ \rightarrow < Y_j, \sim Y_j > \subset < Y_l, \sim Y_l > .$$

That is, we cannot use any sets whose cutset arcs are not a subset of the arcs in $< Y_l, \sim Y_l >$.

For all $Y_j \subset V$, we define two new sets (see Figure 3.2):

$$\tilde{Y_j} = Y_j \cap Y_l \text{ and } \bar{Y_j} = Y_j \cap Y_{\underline{l}}.$$

We will show that if Y_j is such that either $\tilde{Y_j}$ is a proper subset of Y_l ($\emptyset \neq \tilde{Y_j} \neq Y_l$) or $\bar{Y_j}$ is a proper subset of $Y_{\underline{l}}$ ($\emptyset \neq \bar{Y_j} \neq Y_{\underline{l}}$) then $j \notin J_l^+$. That is, then the inequality corresponding to Y_j is not used in constructing the redundancy of inequality l.

First we will work with $\tilde{Y_j}$. If we write the equation for d_i^l (referring back to the equation for d^l) we have

$$d_i^l = \sum_{j \in J_l} \bar{\lambda}_j d_i^j + \bar{\lambda}_{\underline{l}} d_i^{\underline{l}} - \bar{\lambda}_0 e_i.$$

When $i \in Y_l$, $d_i^l = 1$, so we have

$$1 = \sum_{j \in J_l} \bar{\lambda}_j d_i^j + 0 - \bar{\lambda}_0$$

$$1 + \bar{\lambda}_0 = \sum_{j \in J_l} \bar{\lambda}_j d_i^j \qquad \forall i \in Y_l.$$

If $q \in J_l^+$ and $\tilde{Y}_q$ is a proper subset of Y_l, then either there is an arc from $\tilde{Y}_q$ to $Y_l \setminus \tilde{Y}_q$, or there is an arc from $Y_l \setminus \tilde{Y}_q$ to $\tilde{Y}_q$. Since $< Y_q \sim Y_q > \subset < Y_l, \sim Y_l >$, there cannot be any arcs from $\tilde{Y}_q$ to $Y_l \setminus \tilde{Y}_q$. Hence, there must be an arc from $Y_l \setminus \tilde{Y}_q$ to $\tilde{Y}_q$. Let this arc be (i, i').

Now consider any set Y_j such that $i \in Y_j$ and $j \in J_l^+$. Since $< Y_j, \sim Y_j > \subset < Y_l, \sim Y_l >$, we must also have $i' \in Y_j$. Thus, any set Y_j that contains i must also contain i'. This means that the following must be true:

$$1 + \bar{\lambda}_0 = \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j < \bar{\lambda}_q + \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j \leq \sum_{j \in J_l^+} \bar{\lambda}_j d_{i'}^j = 1 + \bar{\lambda}_0,$$

which is a contradiction. Therefore, there is no $q \in J_l^+$ with $\tilde{Y}_q$ a proper subset of Y_l.

The same argument is used to show that there is not $q \in J_l^+$ with $\bar{Y}_q$ a proper subset of $Y_{\underline{l}}$. For this case, note that $i \in Y_{\underline{l}} \Rightarrow d_i^l = 0$, so we have the equation

$$0 = \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j + \bar{\lambda}_{\underline{l}} - \bar{\lambda}_0$$

$$\bar{\lambda}_{\underline{l}} - \bar{\lambda}_0 = \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j \qquad \forall i \in Y_{\underline{l}}.$$

If $q \in J_l^+$ and $\bar{Y}_q$ is a proper subset of $Y_{\underline{l}}$, then either there is an arc from $\bar{Y}_q$ to $Y_{\underline{l}} \setminus \bar{Y}_q$ or there is an arc from $Y_{\underline{l}} \setminus \bar{Y}_q$ to $\bar{Y}_q$. Since

$< Y_q \sim Y_q > \subset < Y_l, \sim Y_l >$, there cannot be any arcs from $\bar{Y}_q$ to $Y_{\underline{l}} \setminus \bar{Y}_q$. Hence, there must be an arc from $Y_{\underline{l}} \setminus \bar{Y}_q$ to $\bar{Y}_q$. Let this arc be (i, i').

Now consider any set Y_j such that $i \in Y_j$ and $j \in J_l^+$. Since $< Y_j, \sim Y_j > \subset < Y_l, \sim Y_l >$, we must also have $i' \in Y_j$. Thus, any set Y_j that contains i must also contain i'. This means that the following must be true:

$$\bar{\lambda}_{\underline{l}} - \bar{\lambda}_0 = \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j < \bar{\lambda}_q + \sum_{j \in J_l^+} \bar{\lambda}_j d_i^j \leq \sum_{j \in J_l^+} \bar{\lambda}_j d_{i'}^j = \bar{\lambda}_{\underline{l}} - \bar{\lambda}_0,$$

which is a contradiction. Therefore, there is no $q \in J_l^+$ with $\bar{Y}_q$ a proper subset of $Y_{\underline{l}}$.

Thus, the only index that could possibly belong to J_l^+ is the one corresponding to V, say $Y_p = V$. But then we would have

$$1 + \bar{\lambda}_0 = \bar{\lambda}_p$$

$$\bar{\lambda}_0 - \bar{\lambda}_{\underline{l}} = \bar{\lambda}_p$$

$$\bar{\lambda}_0, \bar{\lambda}_{\underline{l}}, \bar{\lambda}_p \geq 0$$

which is not possible. Thus, J_l^+ must be empty, but then we have

$$d^l = \bar{\lambda}_{\underline{l}} d^{\underline{l}} - \bar{\lambda}_0 e$$

$$\bar{\lambda}_0, \bar{\lambda}_{\underline{l}}, \geq 0$$

which is also not possible.

Therefore, when $G(Y)$ and $G(\sim Y)$ are both connected,

$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y)$$

is never redundant. ∎

4. Feasibility in Directed Multicommodity Graphs

4.1 Preliminaries

This thesis deals with linear multicommodity flows, ignoring integer restrictions. We begin by noting that a multicommodity network with multiple sources and sinks for each commodity can be transformed into a multicommodity network that has a single source and sink for each commodity. This transformation is analogous to the transformation from multiple sources/sinks to a single source/sink as given by Ford and Fulkerson ([9]).

Theorem 4.1 A capacitated multicommodity network $N = [V, E, U]$ with multiple sources and multiple sinks for each commodity can be transformed into an equivalent multicommodity network $N' = [V', E', U']$ that has a single source and single sink for each commodity.

Proof: Let $S_k \subseteq V$ and $T_k \subseteq V$ be the indices for the set of sources and sinks, respectively, for commodity k. Let $b^k : V \rightarrow \Re$ be such that $b_i^k = 0$ for $i \notin S_k \cup T_k$, $b_i^k < 0$ for $i \in S_k$, $b_i^k > 0$ for $i \in T_k$, and $\sum b_i^k = 0$ for each k.

Construct N' as follows. $V' = V \cup \{O^k, D^k\}$, and $E' = E \cup \{(O^k, i : i \in S_k), (i : i \in T_k, D^k)\}$. $U'(O^k, i) = -b_i^k$ for $i \in S_k$, $U'(i, D^k) = b_i$ for $i \in T_k$, and $U'(e) = U(e)$ for $e \in E$. ${b'}_{O^k} = \sum_{i \in S_k} b_i^k$, ${b'}_{D^k} = \sum_{i \in T_k} b_i^k$, $b'_i = 0$ for $i \in V \setminus (S_k \cup T_k)$.

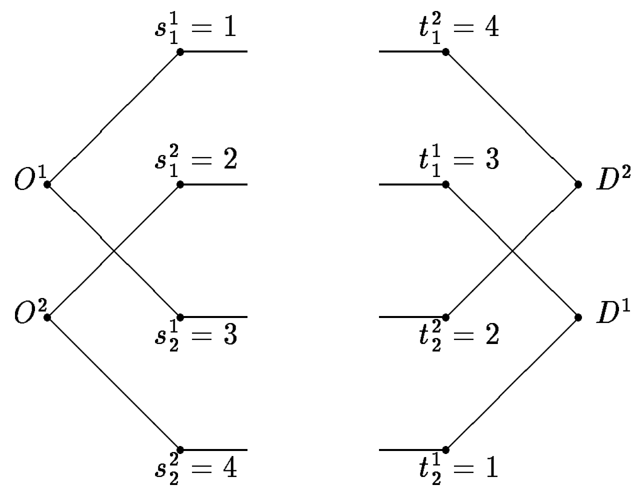
Let x be any feasible flow in N. We will construct a feasible flow x' in N'. Let $(x^k_e)' = x^k_e$ for $e \in E$, $(x^k_{O^k,i})' = U'(O^k, i)$ for $i \in S_k$, and $(x^k_{i,D^k})' = U'(i, D^k)$ for $i \in T_k$. The capacity constraints are satisfied for all edges by definition. We also need flow conservation at each node. $\sum_i (x^k_{i,j})' - \sum_i (x^k_{j,i})' = \sum_i x^k_{i,j} - \sum_i x^k_{j,i} = 0 \forall k$ for $i \in V \setminus (S_k \cup T_k)$. $\sum_{i \in S_k} (x^k_{O^k,i})' = \sum_{i \in S_k} -b^k_i = -b'_{O^k} \forall k$. $\sum_{i \in T_k} (x^k_{i,D^k})' = \sum_{i \in T_k} b^k_i = b'_{D^k} \forall k$. $\sum_j (x^k_{j,i})' - \sum_j (x^k_{i,j})' = (x^k_{O^k,i})' + \sum_j x^k_{j,i} - \sum_j x^k_{i,j} = -b^k_i + b^k_i = 0 \forall k$ for $i \in S_k$. $\sum_j (x^k_{j,i})' - \sum_j (x^k_{i,j})' = \sum_j x^k_{j,i} - (\sum_j x^k_{i,j} + (x^k_{i,D^k})') = b^k_i - b^k_i = 0 \forall k$ for $i \in T_k$. Therefore, x' is feasible in N'.

Conversely, let x' be any feasible flow in N'. We will construct a feasible flow x in N. Let $x^k_e = (x^k_e)'$ for $e \in E$. Since $U'(e) = U(e)$ for $e \in E$, $\sum_k x^k_e = \sum_k (x^k_e)' \leq U'_e = U_e$. So the capacity constraints are satisfied. $\sum_i x^k_{i,j} - \sum_i x^k_{j,i} = \sum_i (x^k_{i,j})' - \sum_i (x^k_{j,i})' = 0 \forall k$ for $i \in V \setminus (S_k \cup T_k)$. If $i \in S_k$ then $\sum_j (x^k_{j,i})' - \sum_j (x^k_{i,j})' = (x^k_{O^k,i})' + \sum_j x^k_{j,i} - \sum_j x^k_{i,j} = 0$ $\Rightarrow \sum_j x^k_{j,i} - \sum_j x^k_{i,j} = b^k_i$. If $i \in T_k$ then $\sum_j (x^k_{j,i})' - \sum_j (x^k_{i,j})' = \sum_j x^k_{j,i} - \sum_j x^k_{i,j} - (x^k_{i,D^k})' = 0 \Rightarrow \sum_j x^k_{j,i} - \sum_j x^k_{i,j} = b^k_i$. Therefore, x is feasible in N. ■

We illustrate this theorem with the following example.

Example

The following graph illustrates the addition of super-origins and super-destinations for each commodity.



For this example, node O^1 has a supply of 4 $(s_1^1 + s_2^1)$ and node O^2 has a supply of 6 $(s_1^2 + s_2^2)$. Placing a capacity of 1 on edge (O_1, s_1^1) and a capacity of 3 on edge (O^1, s_2^1) will insure that nodes s_1^1 and s_2^1 will send the proper amounts of commodity 1. Similarly, we place a capacity of 2 on edge (O^2, s_1^2) and a capacity of 4 on edge (O^2, s_2^2).

Node D^2 has a demand of 6 $(t_1^2 + t_2^2)$ and node D^1 has a demand of 4 $(t_1^1 + t_2^1)$. Placing a capacity of 4 on edge (t_1^2, D^2) and a capacity of 2 on edge (t_2^2, D^2) will insure that nodes t_1^2 and t_2^2 will receive the proper amounts of commodity 2. Similarly, we place a capacity of 3 on edge (t_1^1, D^1) and a capacity of 1 on edge (t_2^1, D^1).

Thus, we see that any flow that is feasible in the original network is feasible in the new network, and vice versa.

Note that this theorem holds whether the graph is directed or undirected. For the remainder of this thesis we will assume, without loss of generality, that each commodity has a single origin and a single destination.

4.2 The System

Wallace and Wets work with a system of inqualities which define feasibility for a set of supplies and demands when the supply and demand nodes are not known a priori. We wish to develop an analogous system for the multicommodity case, and then we will characterize the redundant inequalities in that system.

If the source and sink is known for each commodity, then a necessary condition for feasibility of a set of demands q is

$$\sum(q_k : 1 \leq k \leq p, s_k \in Y, t_k \in \sim Y) \leq U(Y, \sim Y) \qquad \forall Y \subset V.$$

This condition is sufficient for the classes of graphs identified by Nagamochi and Ibaraki (see Chapter 2). In order to place this condition in the Wallace and Wets setting, we need to generalize this condition for the case where the supply and demand nodes are not know a priori.

Looking at the system for a single commodity,

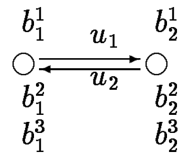
$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y) \qquad \forall Y \subset V,$$

one might guess that the system for multicommodities can be obtained by simply summing over each commodity. The resulting system is

$$\sum_{k=1}^{p} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall Y \subset V.$$

The following example shows that this is not enough.

Example



The inequalities are:

$$Y = \{1\} : b_2^1 + b_2^2 + b_2^3 \leq u_1$$

$$Y = \{2\} : b_1^1 + b_1^2 + b_1^3 \leq u_2$$

This system allows for $b_1^1 = -1$, $b_1^2 = -1$, $b_1^3 = 1$, $b_2^1 = 1$, $b_2^2 = 1$, $b_2^3 = -1$, and $u_1 = 1$, $u_2 = 1$. However, it is easily seen that this set of demands and capacities are not feasible. Thus, we try a stronger system.

We actually need to sum over all possible combinations of commodities. This results in

$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall Y \subset V, \forall K,$$

where K is a nonempty set of commodities. This condition gives us the desired system.

Theorem 4.2 The set of inequalities

$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall Y \subset V, \forall K$$

is a necessary condition for feasibility of b, the vector of demands.

Proof: We start with the system of constraints for a directed multicommodity flow problem,

$$\sum_{j|(j,i) \in A} x_{ji}^k - \sum_{j|(i,j) \in A} x_{ij}^k = b_i^k \qquad \forall i$$

$$\sum_{k=1}^{p} x_{ij}^k \leq u_{ij} \qquad \forall i,j$$
$$x_{ij}^k \geq 0 \qquad \forall k,i,j$$

and we want to show that this system implies the inequality

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k = U(Y,\sim Y).$$

Recall that $U(Y,\sim Y) = \sum_{(j,i):j\in Y, i\in\sim Y} u_{ji}$.

We begin by working with the first constraint in the multicommodity flow problem. First, we sum both sides of the equation over $i \in\sim Y$ to obtain

$$\sum_{i\in\sim Y} b_i^k = \sum_{i\in\sim Y}(\sum_{j|(j,i)\in A} x_{ji}^k - \sum_{j|(i,j)\in A} x_{ij}^k).$$

We next sum both sides over $k \in K$, obtaining

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k = \sum_{k\in K}\sum_{i\in\sim Y}(\sum_{j|(j,i)\in A} x_{ji}^k - \sum_{j|(i,j)\in A} x_{ij}^k).$$

$$= \sum_{k\in K}\sum_{i\in\sim Y}\sum_{j|(j,i)\in A} x_{ji}^k - \sum_{k\in K}\sum_{i\in\sim Y}\sum_{j|(i,j)\in A} x_{ij}^k$$

$$= \sum_{k\in K}(\sum_{\substack{i\in\sim Y\\ j\in Y}} x_{ji}^k + \sum_{\substack{i\in\sim Y\\ j\in\sim Y}} x_{ji}^k - \sum_{\substack{i\in\sim Y\\ j\in Y}} x_{ij}^k - \sum_{\substack{i\in\sim Y\\ j\in\sim Y}} x_{ij}^k)$$

$$= \sum_{k\in K}\sum_{\substack{i\in\sim Y\\ j\in Y}} x_{ji}^k - \sum_{k\in K}\sum_{\substack{i\in\sim Y\\ j\in Y}} x_{ij}^k.$$

Since $x_{ij}^k \geq 0$, we now have

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq \sum_{k\in K}\sum_{\substack{i\in\sim Y\\ j\in Y}} x_{ji}^k.$$

Now if we work with the second constraint in the multicommodity flow problem, again using the constraint that $x_{ij}^k \geq 0$, we have

$$\sum_{k\in K} x_{ji}^k \leq \sum_{k=1}^{p} x_{ji}^k \leq u_{ji}.$$

This gives us

$$\sum_{\substack{j\in Y\\ i\in\sim Y}}\sum_{k\in K} x_{ji}^k \leq \sum_{\substack{j\in Y\\ i\in\sim Y}} u_{ji} = U(Y,\sim Y).$$

This gives us our desired inequality,

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq U(Y,\sim Y).$$

■

The next lemma relates this system to the system of inequalities for a directed multicommodity graph when the sources and sinks are known a priori.

Lemma 4.3 The set of inequalities

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq U(Y,\sim Y) \qquad \forall Y\subset V, \forall K$$

contains the set of inequalities

$$\sum(q_k : 1\leq k\leq p, s_k\in Y, t_k \in\sim Y) \leq U(Y,\sim Y) \qquad \forall Y\subset V.$$

Proof: We know that $q_k = b_{t_k}^k = -b_{s_k}^k$ and $b_i^k = 0$ for $i \neq s_k, t_k$ for all k. We have a set of inequalities for each set of commodities; in particular, pick K' such that $\forall k\in K'$, $s_k\in Y$ and $t_k \in\sim Y$. So our system contains the inequalities

$$\sum_{k\in K'}\sum_{i\in\sim Y} b_i^k \leq U(Y,\sim Y) \qquad \forall Y\subset V.$$

The left side of this inequality has b_i^k for $i = t_k$ and $b_i^k = 0$ for all other $i \in\sim Y$. For $i = t_k$, $b_i^k = q_k$, so the summation yields $\sum_{k\in K'} q_k$. ■

We now have a system of linear inequalities which serve as a necessary condition for feasibility of multicommodity flows in a directed graph when the supply and demand nodes are not known a priori. In the next section, we will extend the Wallace-Wets Theorem to this system.

4.3 Redundancy Theorem

The Wallace-Wets Theorem extends quite naturally to the directed case. Using the system from the previous section, along with the requirement that supply equals demand for each commodity, we are able to characterize the redundancy in the system. This characterization is the same as the Wallace-Wets characterization for the single commodity system with one exception. For our system we need to separate the cases of $U(Y, \sim Y) = 0$ and $U(\sim Y, Y) = 0$.

Theorem 4.4 [Redundancy Theorem for Directed Graphs] Let G be a connected, capacitated, directed, multicommodity graph with p commodities, where each commodity has one origin and one destination.

(1) If $U(Y, \sim Y) > 0$ and $U(\sim Y, Y) > 0$, then the inequality corresponding to the cut $< Y, \sim Y >$,

$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall K,$$

is redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected.

(2) If $U(Y, \sim Y) = 0$ (connectedness of the graph then implies that $U(\sim Y, Y) > 0$), then all sets K with more than one element yield redundant inequalities. The inequalities

$$\sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall k$$

are redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected.

(3) If $U(\sim Y, Y) = 0$ (connectedness of the graph then implies that $U(Y, \sim Y) > 0$), then all sets K yield redundant inequalities except for $K = \{1, 2, \ldots, p\}$. The inequality

$$\sum_{k=1}^{p} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y)$$

is redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected.

Proof:

(1) We will first prove the "if" direction. Assume that one of $G(Y)$ and $G(\sim Y)$ is not connected.

Case 1: $G(Y)$ is not connected.

Let $Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and there are no edges between Y_1 and Y_2.

We examine the inequalities

$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall K.$$

The inequalities from Y_1 are

$$\sum_{k \in K} \sum_{i \in \sim Y_1} b_i^k = \sum_{k \in K} \sum_{i \in Y_2} b_i^k + \sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y_1, \sim Y_1) \qquad \forall K.$$

The inequalities from Y_2 are

$$\sum_{k \in K} \sum_{i \in \sim Y_2} b_i^k = \sum_{k \in K} \sum_{i \in Y_1} b_i^k + \sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y_2, \sim Y_2) \qquad \forall K.$$

Summing these inequalities gives

$$\sum_{k \in K} \sum_{i \in Y} b_i^k + 2 \sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y_1, \sim Y_1) + U(Y_2, \sim Y_2) = U(Y, \sim Y) \qquad \forall K.$$

Subtracting $\sum_{k\in K}\sum_{i\in V} b_i^k = 0$ gives

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq U(Y,\sim Y) \qquad \forall K.$$

Thus, those inequalities are redundant.

Case 2: $G(\sim Y)$ is not connected.

Let $\sim Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and there are no edges between Y_1 and Y_2. We examine the inequalities

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq U(Y,\sim Y) \qquad \forall K.$$

The inequalities from $\sim Y_1$ are

$$\sum_{k\in K}\sum_{i\in Y_1} b_i^k \leq U(\sim Y_1, Y_1) \qquad \forall K.$$

The inequalities from $\sim Y_2$ are

$$\sum_{k\in K}\sum_{i\in Y_2} b_i^k \leq U(\sim Y_2, Y_2) \qquad \forall K.$$

Summing these two inequalities gives

$$\sum_{k\in K}\Big(\sum_{i\in Y_1} b_i^k + \sum_{i\in Y_2} b_i^k\Big) = \sum_{k\in K}\sum_{i\in\sim Y} b_i^k \leq U(\sim Y_1, Y_1) + U(\sim Y_2, Y_2) = U(Y,\sim Y) \qquad \forall K.$$

Thus, the inequalities are redundant.

These proofs are sufficiently general to hold for $U(Y,\sim Y) \geq 0$.

Thus, we have also proved the "if" direction for (2) as well.

We now prove the "only if" direction. We assume that an inequality is redundant, and we will show that at least one of the corresponding graphs is not connected. If

$$\sum_{k\in K'}\sum_{i\in\sim Y_q} b_i^k \leq U(Y_q,\sim Y_q)$$

is redundant, then it is a nonnegative combination of linear inequalities from sets other than Y_q. To see this, assume that the above inequality is a nonnegative combination of linear inequalities in the cut system, some of which may be generated by the set Y_q. Clearly the only way the inequality can be a nonnegative combination of inequalities generated only by the set Y_q is if $U(Y, \sim Y) = 0$. This exception is treated in part (2) of the theorem. So the dependency set must contain at least one inequality generated by a set other than Y_q. However, it may not contain any edges not in $< Y_q, \sim Y_q >$. We assume that $G(Y_q)$ and $G(\sim Y_q)$ are both connected, and we will derive a contradiction. First we will show that an inequality from Y_s, $Y_s \cap \sim Y_q \neq \emptyset$, $Y_s \cap \sim Y_q \neq \sim Y_q$, cannot be in the dependency set. Since $G(\sim Y_q)$ is connected, there is an edge in $< Y_j, \sim Y_q/Y_j >$ or there is an edge in $<\sim Y_q/Y_j, Y_j >$. If the former is true, we are done with this case. If the former is false, then the latter is true. Call the edge (α, β). Sine α and β must be counted an equal number of times, there must be some set, containing α and not containing β, whose corresponding inequality is in the dependency set. But then this cutset contains the edge (α, β), which is not possible. Thus, an inequality corresponding to Y_s with $Y_s \cap \sim Y_q \neq \emptyset$, $Y_s \cap \sim Y_q \neq \sim Y_q$, cannot be in the dependency set. The same can be shown for a set Y_t with $Y_t \cap Y_q \neq \emptyset$, $Y_s \cap Y_q \neq Y_q$. The only remaining possibility is that an inequality corresponding to $\sim Y_q$ is in the dependency set. This can only occur if $U(\sim Y_q, Y_q) = 0$, which is handled in part (3) of the theorem. Thus, we have shown that the redundant inequality

$$\sum_{k \in K'} \sum_{i \in \sim Y_q} b_i^k \leq U(Y_q, \sim Y_q)$$

must be a nonnegative combination of linear inequalities from sets other than Y_q. That is, $\exists \lambda \in \Re_+^{2^{|V|}}$ such that

$$\sum_{k \in K'} \sum_{i \in \sim Y_q} b_i^k = \sum_{s \neq q} \lambda_s \sum_{k \in K'} \sum_{i \in \sim Y_s} b_i^k = \sum_{k \in K'} \sum_{s \neq q} \lambda_s \sum_{i \in \sim Y_s} b_i^k.$$

Note that the above equation holds for *all* b. In particular, for any choice of k, we can set b_i^j to be zero for all i and for all $j \neq k$. Therefore,

$$\sum_{i \in \sim Y_q} b_i^k = \sum_{s \neq q} \lambda_s \sum_{i \in Y_s} b_i^k$$

for each k. This gives us that

$$\sum_{i \in \sim Y_q} b_i^k \leq U(Y_q, \sim Y_q)$$

is redundant in the cut system for each commodity k. Thus, each inequality corresponding to Y_q is redundant for each commodity k. We can now apply the Wallace-Wets Theorem to conclude that either $G(Y_q)$ or $G(\sim Y_q)$ is not connected.

(2) Assume $U(Y, \sim Y) = 0$. We have the inequalities

$$\sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) = 0 \qquad \forall k.$$

Summing over $k \in K : |K| > 1$ gives

$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq 0 = U(Y, \sim Y) \qquad \forall K : |K| > 1.$$

Thus, the inequalities

$$\sum_{k\in K}\sum_{i\in\sim Y} b_i^k \le U(Y,\sim Y) \qquad \forall K : |K| > 1$$

are always redundant when $U(Y,\sim Y) = 0$.

(3) Assume $U(\sim Y, Y) = 0$. Let K' be any proper subset of the commodities. We will show that the inequality

$$\sum_{k\in K'}\sum_{i\in\sim Y} b_i^k \le U(Y,\sim Y)$$

is redundant.

From the set $\sim Y$, we have the inequalities

$$\sum_{k\in K}\sum_{i\in Y} b_i^k \le U(\sim Y, Y) = 0 \qquad \forall K.$$

In particular, if K^p is the set of all commodities, we have the inequality

$$\sum_{k\in K^p/K'}\sum_{i\in Y} b_i^k \le 0.$$

Since $\sum_{i\in Y} b_i^k = -\sum_{i\in\sim Y} b_i^k$, this gives us

$$-\sum_{k\in K^p/K'}\sum_{i\in\sim Y} b_i^k \le 0.$$

Adding this to the inequality

$$\sum_{k\in K^p}\sum_{i\in\sim Y} b_i^k \le U(Y,\sim Y)$$

results in the original inequality.

■

We note that when $p = 1$ (one commodity) this system reduces to the system used by Wallace and Wets. The inequality for the set Y becomes $\sum_{i\in\sim Y} b_i \le U(Y,\sim Y)$.

We proceed now to the undirected case, which is not as straightforward as the directed case.

5. Feasibility in Undirected Multicommodity Graphs

5.1 The Absolute Value System

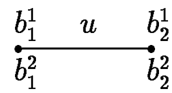
We will now work with undirected multicommodity graphs, meaning that the edges are not oriented and flow may travel in either direction on an edge. In particular, flow from different commodities may travel in opposite directions on the same edge. In working towards an extension of the Wallace-Wets Theorem to the undirected multicommodity case, we need to have a system of inequalities which is a necessary condition for feasible demands (and which will be sufficient for the cases discussed in Chapter 2). The traditional system of cut inequalities,

$$\sum(q_k : 1 \leq k \leq p, Y \text{ separates } s_k \text{ and } t_k) \leq U(Y, \sim Y) \qquad \forall Y \subseteq V,$$

assumes that the origin and destination nodes for each commodity are known a priori. We need to develop a system in which the origin and destination nodes do not need to be known a priori. As a first attempt, one might try generalizing the single commodity cut system by summing over all commodities, as in the directed case. If we use the system from the directed case, we have:

$$\sum_{i \in V} b_i^k = 0 \qquad \forall k$$
$$\sum_{k \in K} \sum_{i \in \sim Y} b_i^k \leq U(Y, \sim Y) \qquad \forall Y \subset V, \forall K.$$

However, this system is not restrictive enough in the undirected case. Consider the following graph.



The above system would give rise to the following set of inequalities:

$$
\begin{aligned}
& & b_1^1 + b_2^1 &= 0 \\
& & b_1^2 + b_2^2 &= 0 \\
Y = \{1\}: \quad & K = \{1\}: & b_2^1 &\leq u \\
& K = \{2\}: & b_2^2 &\leq u \\
& K = \{1,2\}: & b_2^1 + b_2^2 &\leq u \\
Y = \{2\}: \quad & K = \{1\}: & b_1^1 &\leq u \\
& K = \{2\}: & b_1^2 &\leq u \\
& K = \{1,2\}: & b_1^1 + b_1^2 &\leq u
\end{aligned}
$$

This system allows for $b_1^1 = 1, b_2^1 = -1, b_1^2 = -1, b_2^2 = 1$, and $u = 1$. However, we can easily see that this set of demands, along with the given capacity, is not feasible. In order to satisfy the demands, we would have to send a total of 2 units of flow across the edge, when only 1 unit is allowed. The problem is that at each node, the demand for one commodity cancels with the supply for the other commodity.

This problem is alleviated by taking the absolute value of the sum of the b_i's for each commodity. That is, we use the system

$$\sum_{i \in V} b_i^k = 0 \qquad \forall k$$

$$\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| \le U(Y, \sim Y) \qquad \forall Y \subset V.$$

Now the system for the above graph is:

$$b_1^1 + b_2^1 = 0$$

$$b_1^2 + b_2^2 = 0$$

$$Y = \{1\} : |b_2^1| + |b_2^2| \le u$$

$$Y = \{2\} : |b_1^1| + |b_1^2| \le u$$

We see that the set of b_i^k's that was feasible in the old system is not feasible in this system. In fact, this gives us the correct system for every multicommodity graph, as the next theorem shows.

Note that we do not need to sum over $k \in K$ because the inequalities corresponding to sets K other than the full set of commodities will be dominated by the inequality above. For example, for the graph in the above example, we would get

$$Y = \{1\} : K = \{1\} : |b_2^1| \le u,$$

which is dominated by the inequality obtained by summing over all commodities.

Theorem 5.1 The set of inequalities

$$\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| \le U(Y, \sim Y) \qquad \forall Y \subset V$$

is equivalent to the set of inequalities

$$\sum (q_k : 1 \le k \le p, Y \text{ separates } s_k \text{ and } t_k) \le U(Y, \sim Y) \qquad \forall Y \subset V.$$

Proof: Since $q_k = b^k_{t_k} = -b^k_{s_k}$ and $b^k_i = 0$ for $i \neq s_k, t_k$ for all k, we will treat separately the cases when Y does and does not separate s_k and t_k.

Case 1: Y does not separate s_k and t_k.

Then s_k and t_k are either both in Y or both in $\sim Y$. If s_k and t_k are both in Y, then

$$|\sum_{i \in Y} b^k_i| = |b^k_{s_k} + b^k_{t_k}| = 0.$$

If s_k and t_k are both in $\sim Y$, then $b^k_i = 0 \quad \forall i \in Y$. Therefore, if s_k and t_k are not separated, then $\sum_{i \in Y} b^k_i = 0$.

Case 2: Y does separate s_k and t_k.

Then either $s_k \in Y$ or $t_k \in Y$, but not both. $b^k_i = 0$ for all other $i \in Y$. Then

$$|\sum_{i \in Y} b^k_i| = b^k_{t_k} = q_k.$$

Thus, the absolute value inequality is equivalent to limiting Y to sets that separate s_k and t_k, in which case the two inequalities are the same. ■

This system of inequalities, which we will refer to as the cut system, has many properties which are not common to a general system of absolute value inequalities. First, we know that every inequality in the cut system contains absolute values. Further, for a given inequality, the number of terms within each absolute value is the same, and it is equal to the cardinality of the set Y. So, for example, we cannot have $|x_1| + |x_2 + x_3| \leq u$ in a cut system. Similarly, each inequality in the cut system has the same number of absolute value terms, which is equal to the number of commodities. That is, we cannot have both $|x_1| \leq u$ and

$|x_1|+|x_2| \leq u$ in the same system. Additionally, a term that appears inside of one absolute value will not appear anywhere else in that inequality. That is, we cannot have $|x_1+x_2|+|x_2+x_3| \leq u$ in a cut system. Finally, we note that the coefficients on all terms within the absolute values, as well as on the absolute value quantities themselves, are 0-1, and the right-hand side is always nonnegative. These observations lead us to some lemmas about the cut system.

An absolute value inequality from the cut system is equivalent to a system of 2^p linear inequalities. We can express the jth linear inequality as

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in Y} b_i^k \leq U(Y, \sim Y),$$

where $\delta_{kj} \in \{-1, 1\}$. In the proof of the next lemma, we will simply write U for $U(Y, \sim Y)$. To illustrate the equivalent system of linear inequalities, consider the following absolute value inequality:

$$|x_1+x_2|+|x_3+x_4| \leq u.$$

The corresponding system of linear inequalities is:

$$x_1+x_2+x_3+x_4 \leq u$$

$$-(x_1+x_2)+x_3+x_4 \leq u$$

$$x_1+x_2-(x_3+x_4) \leq u$$

$$-(x_1+x_2)+-(x_3+x_4) \leq u.$$

We will sometimes refer to this system as a *family* of linear inequalities. We next show that we cannot obtain an inequality in this system as a nonnegative combination of only other inequalities in this system.

Lemma 5.2 The family of linear inequalities corresponding to a given absolute value inequality from the cut system is internally nonredundant.

Proof: Consider the jth linear inequality corresponding to a given absolute value inequality:

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in Y} b_i^k \leq U.$$

Reverse the sense of the inequality and make it strict:

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in Y} b_i^k > U.$$

The system with this new inequality in place of the old one is feasible if and only if the old inequality is nonredundant. We will construct a feasible solution to the reversed system:

$$\forall i \in Y, b_i^k = \begin{cases} \frac{U}{|Y| \cdot p - 1} & \text{if } \delta_{kj} = 1 \\ \frac{-U}{|Y| \cdot p - 1} & \text{if } \delta_{kj} = -1 \end{cases}$$

Then

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in Y} b_i^k = \frac{|Y| \cdot p}{|Y| \cdot p - 1} U > U.$$

Further, changing any δ_{kj} from 1 to -1 or from -1 to 1 decreases the sum by $\frac{U}{|Y| \cdot p - 1}$, making it equal to U. Thus, this b satisfies all of the linear inequalities in this family, so the jth inequality is not implied by the others. ■

This lemma does not hold for the linear system corresponding to a general absolute value inequality. The following example will illustrate this.

Example

Consider the inequality

$$|x_1| + |x_2| + |x_1 + x_2| \leq u$$

A partial enumeration of the corresponding linear inequalities gives:

$$x_1 + x_2 + x_1 + x_2 \leq u \Rightarrow 2x_1 + 2x_2 \leq u$$

$$-x_1 + x_2 + x_1 + x_2 \leq u \Rightarrow 2x_2 \leq u$$

$$-x_1 - x_2 + x_1 + x_2 \leq u \Rightarrow 0 \leq u$$

$$x_1 + x_2 - x_1 - x_2 \leq u \Rightarrow 0 \leq u$$

The last two inequalities are copies of each other. Thus, this system is not internally nonredundant.

We can also make some observations about an inequality which is known to be redundant.

Lemma 5.3 A redundant linear inequality cannot be formed as a nonnegative combination of linear inequalities which are all from one family, different from the family of the redundant linear inequality.

Proof: Let a_{iq} be the node membership function for the set Y_q: $a_{iq} = 1$ iff $i \in Y_q$. Let c_{eq} be the edge membership function for the cut defined by Y_q: $c_{eq} = 1$ iff $e \in < Y_q, \sim Y_q >$. We want to show that

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in V} a_{iq} b_i^k \leq U(Y_q, \sim Y_q) = \sum_{e \in E} c_{eq} U_e$$

cannot be formed as

$$\sum_r \lambda_r \sum_{k=1}^{p} \delta_{kr} \sum_{i \in V} a_{it} b_i^k \leq U(Y_t, \sim Y_t) = \sum_{e \in E} c_{et} U_e, (\lambda_r \geq 0, \sum_r \lambda_r = 1, \lambda_j = 0).$$

Assume the two inequalities are equal. Then $c_t \sum_r \lambda_r = c_q$. This gives us that $\sum_r \lambda_r = 1$ and $c_t = c_q$. If the network is connected then either $Y_t = Y_q$ or $Y_t = \sim Y_q$. Since this is not possible, the two inequalities cannot be equal. ■

Lemma 5.4 If a linear inequality from the cut system is redundant, then it is redundant for each commodity.

Proof: If

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in V} a_{iq} b_i^k \leq U(Y_q, \sim Y_q)$$

is redundant, then it is a nonnegative combination of linear inequalities from sets other than q. That is, $\exists \lambda \in \Re_+^{2p} \times 2^{|V|}$ such that

$$\sum_{k=1}^{p} \delta_{kj} \sum_{i \in V} a_{iq} b_i^k = \sum_r \sum_{s \neq q} \lambda_{rs} \sum_{k=1}^{p} \delta_{kr} \sum_{i \in V} a_{is} b_i^k = \sum_{k=1}^{p} \sum_r \sum_{s \neq q} \lambda_{rs} \delta_{kr} \sum_{i \in V} a_{is} b_i^k.$$

Note that the above equation holds for *all* b. In particular, for any choice of k, we can set b_i^j to be zero for all i and for all $j \neq k$.

Therefore,

$$\delta_{kj} \sum_{i \in V} a_{iq} b_i^k = \sum_r \sum_{s \neq q} \lambda_{rs} \delta_{kr} \sum_{i \in V} a_{is} b_i^k \text{ for each } k.$$

Which gives us that

$$\delta_{kj} \sum_{i \in V} a_{iq} b_i^k \leq U(Y_q, \sim Y_q)$$

is redundant in the cut system for commodity k. ■

5.2 Redundancy Theorem

We have seen many characteristics of single commodity flows that do not extend to the case of multicommodity flows. The question which motivated this thesis was whether the Wallace-Wets Theorem holds in the multicommodity case. In this section we prove that using the absolute value system, the Wallace-Wets Theorem can be extended to undirected multicommodity graphs.

In what follows, we assume the cut system to be

$$\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| \leq U(Y, \sim Y) \qquad \forall Y \subseteq V$$
$$\sum_{i \in V} b_i^k = 0 \quad (k = 1, \ldots, p).$$

Theorem 5.5 [Redundancy Theorem for Undirected Graphs] Let G be a connected, capacitated, undirected, multicommodity graph with p commodities, where each commodity has one origin and one destination.

(1) Only one of the pair of inequalities

$$\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| \leq U(Y, \sim Y)$$

$$\sum_{k=1}^{p} |\sum_{i \in \sim Y} b_i^k| \leq U(Y, \sim Y)$$

is ever necessary. That is, we need only consider the inequality from Y or $\sim Y$, but not both.

(2) The inequality corresponding to the cut $< Y, \sim Y >$,

$$\sum_{k=1}^{p} |\sum_{i \in \sim Y} b_i^k| \leq U(Y, \sim Y),$$

is redundant in the cut system if and only if either $G(Y)$ or $G(\sim Y)$ is not connected $(Y \neq \emptyset, \sim Y \neq \emptyset)$.

Proof:

(1) Since $\sum_{i \in V} b_i^k = 0 \quad (k = 1, \ldots, p)$, we have that $\sum_{i \in Y} b_i^k = -\sum_{i \in \sim Y} b_i^k$. Therefore,

$$\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| = \sum_{k=1}^{p} |-\sum_{i \in \sim Y} b_i^k| = \sum_{k=1}^{p} |\sum_{i \in \sim Y} b_i^k|.$$

So only one of each pair of inequalities is needed.

(2) We will first prove the "if" direction. If $G(Y)$ and $G(\sim Y)$ are not both connected, we may assume, without loss of generality, that $G(\sim Y)$ is not connected. Let $\sim Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \emptyset$ and there are no edges between Y_1 and Y_2. We will show that the inequality from Y is dominated by the sum of the inequalities from $\sim Y_1$ and $\sim Y_2$.

The inequality from Y is

$$\sum_{k=1}^{p} |\sum_{i \in \sim Y} b_i^k| = \sum_{k=1}^{p} |\sum_{i \in Y_1} b_i^k + \sum_{i \in Y_2} b_i^k| \le U(Y, \sim Y) = U(Y, Y_1) + U(Y, Y_2).$$

The inequality from $\sim Y_1$ is

$$\sum_{k=1}^{p} |\sum_{i \in Y_1} b_i^k| \le U(Y_1, \sim Y_1).$$

The inequality from $\sim Y_2$ is

$$\sum_{k=1}^{p} |\sum_{i \in Y_2} b_i^k| \le U(Y_2, \sim Y_2).$$

Summing the inequalities from $\sim Y_1$ and $\sim Y_2$, we get

$$\sum_{k=1}^{p} (|\sum_{i \in Y_1} b_i^k| + |\sum_{i \in Y_2} b_i^k|) \le U(Y_1, \sim Y_1) + U(Y_2, \sim Y_2).$$

However, since there are no edges between Y_1 and Y_2 we get that $U(Y_1, \sim Y_1) + U(Y_2, \sim Y_2) = U(Y_1, Y) + U(Y_2, Y)$. Thus

$$\sum_{k=1}^{p} |\sum_{i \in \sim Y} b_i^k| \le \sum_{k=1}^{p} (|\sum_{i \in Y_1} b_i^k| + |\sum_{i \in Y_2} b_i^k|) \le U(Y, \sim Y).$$

We next prove the "only if" direction. If the qth inequality

$$\sum_{k=1}^{p} |\sum_{i \in \sim Y_q} b_i^k| \leq U(Y_q, \sim Y_q)$$

is redundant, then some linear inequality (of the 2^p corresponding to it) is redundant. Assume it is the jth linear inequality. By Lemma 5.4, if the jth inequality is redundant, then it is redundant for some single commodity. At this point, we can apply the Wallace and Wets Theorem (Theorem 3.1) to this inequality, and we conclude that either $G(Y_q)$ or $G(\sim Y_q)$ is not connected. ■

We note that when $p = 1$ (one commodity), the system of absolute value inequalities reduces to the same system of linear inequalities that is used in the Wallace-Wets Theorem. We get that the inequality from set Y is

$$|\sum_{i \in \sim Y} b_i| \leq U(Y, \sim Y).$$

This is equivalent to the following two linear inequalities,

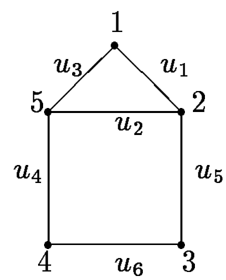
$$\sum_{i \in \sim Y} b_i \leq U(Y, \sim Y)$$

$$-\sum_{i \in \sim Y} b_i = \sum_{i \in Y} b_i \leq U(Y, \sim Y),$$

which are the inequalities that Wallace and Wets would generate from the sets Y and $\sim Y$. Thus, if $p = 1$, our theorem is equivalent to the Wallace-Wets Theorem.

Example

We illustrate the preceeding theorem using the following graph.



Assume there are p commodities. The inequalities from $Y = \{1,2,3\}$ and $Y = \{1,2,5\}$, as well as others, are nonredundant. Both $G(Y)$ and $G(\sim Y)$ are connected. The inequality from $Y = \{1,2,3\}$ is

$$\sum_{k=1}^{p} |b_4^k + b_5^k| \leq u_2 + u_3 + u_6.$$

$Y = \{2,4,5\}$ is an example of $G(Y)$ connected but $G(\sim Y)$ not connected. The (redundant) inequality is

$$\sum_{k=1}^{p} |b_1^k + b_3^k| \leq u_1 + u_3 + u_5 + u_6.$$

To construct the inequalities which dominate this one, let $Y_1 = \{1\}$ and $Y_2 = \{3\}$. The inequality from $\sim Y_1$ is

$$\sum_{k=1}^{p} |b_1^k| \leq u_1 + u_3.$$

The inequality from $\sim Y_2$ is

$$\sum_{k=1}^{p} |b_3^k| \leq u_5 + u_6.$$

Summing these gives

$$\sum_{k=1}^{p}(|b_1^k| + |b_3^k|) \leq u_1 + u_3 + u_5 + u_6$$

which dominates the inequality from Y.

Finally, $Y = \{3, 5\}$ is an example of $G(Y)$ and $G(\sim Y)$ both not connected, so the inequality from Y is redundant.

5.3 Some Building Blocks

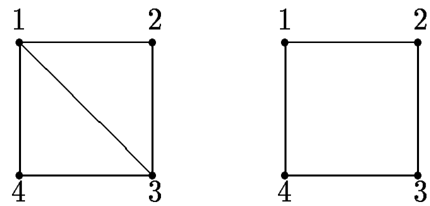
This section contains two types of results. The first several theorems demonstrate how our knowledge of redundant inequalities for one graph may be used to determine redundant inequalities for a related graph. The last several theorems count the number of redundant or nonredundant inequalities in general and for some specific graphs. These theorems give a measure of the usefulness of the Redundancy Theorem.

As a corollary to the Redundancy Theorem, we see that any inequality that is redundant in a graph is redundant in all graphs formed by removing only edges from the original graph.

Theorem 5.6 [Corollary to Redundancy Theorem] If $G' = [V, E']$, $E' \subset E$, then any inequalities that are redundant for G are redundant for G'.

Proof: If either $G(Y)$ or $G(\sim Y)$ is not connected in G, then it is also not connected in G'. ■

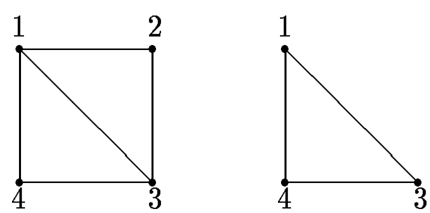
<u>Example</u> To illustrate, consider the following graphs G and G'.



G G′

We see that G and G' have the same vertex set, and that the edge set of G' is a subset of the edge set of G. The set $Y = \{1, 3\}$ gives a redundant inequality in G because $G(\sim Y)$ is disconnected. The same set also gives a redundant inequality in G'.

However, if we allow G' to be any subgraph of G, that is, if we allow the vertex set of G' to be a proper subset of the vertex set of G, then the corollary is not true. To illustrate, consider the following graphs G and G'.



G G′

Now the set $Y = \{1, 3\}$ gives a redundant inequality for G, but not for G'.

Corollary 5.7 If $G = [V, E]$, $G' = [V, E'], G'' = [V, E''], \ldots$ is a sequence of graphs such that $E \supset E' \supset E'' \supset \ldots$, then the number of nonredundant inequalities for G is at least as large as the number of nonredundant inequalities for G', which is at least as large as the number of nonredundant inequalities for G''.

Proof: By Theorem 5.6, every inequality which is redundant for G is also redundant for G'. Every inequality which is redundant for G' is also redundant for G'', and so on. ■

In order to use the Redundancy Theorem, we must find all vertex partitions Y of V such that $G(Y)$ and $G(\sim Y)$ are both connected. However, if we have already done this for two graphs and we then form a new graph by joining the first two in a specific way, then we may not have to recompute which sets give necessary inequalities. We may be able to find all the necessary inequalities through appropriate combinations of the sets we already have. One specific way of combining two graphs is with a bridge, that is, an edge joining one vertex of the first graph to one vertex of the second graph. Then we can find all the necessary inequalities for the new graph by combining the entire vertex set of each original graph with one of the vertex partitions from the other original graph which is already known to give a necessary inequality for that graph.

Theorem 5.8 Let $G_1 = [V_1, E_1]$, $G_2 = [V_2, E_2]$ be connected graphs, $V_1 \bigcap V_2 = \emptyset$. Let $G = G_1 \bigcup G_2 \bigcup (v_1, v_2) = [V, E]$, where $v_1 \in V_1$ and $v_2 \in V_2$ (See Figure 5.1). Let $\mathcal{N}_1 = \{N_1^1, N_1^2, \ldots, N_1^{n_1}\}$ be the collection of vertex partitions which give nonredundant inequalities for the graph G_1. Let $\mathcal{N}_2 = \{N_2^1, N_2^2, \ldots, N_2^{n_2}\}$ be the collection of vertex

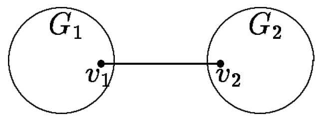


Figure 5.1: Graph with a bridge

partitions which give nonredundant inequalities for the graph G_2. The vertex partitions which give nonredundant inequalities for the graph G are

$$V_1 \bigcup \{N_2^i \in \mathcal{N}_2 : v_2 \in N_2^i\} \text{ and}$$

$$V_2 \bigcup \{N_1^i \in \mathcal{N}_1 : v_1 \in N_1^i\} \text{ and } V_1.$$

Proof: By construction, the induced subgraphs on each of the sets listed above and each of their complements are connected. We need to show that there are no other vertex partitions of V $(Y \bigcup \sim Y = V, Y \bigcap \sim Y = \emptyset)$ such that $G(Y)$ and $G(\sim Y)$ are both connected. We have already used all partitions which completely contain either V_1 or V_2. The only other possibility is that Y contains a proper subset of vertices from both V_1 and V_2. That is, $Y = Y_1 \bigcup Y_2$, where $Y_1 \subset V_1, Y_1 \neq \emptyset$, and $Y_2 \subset V_2, Y_2 \neq \emptyset$. If $v_1 \in Y$ and $v_2 \notin Y$, then $G(Y)$ and $G(\sim Y)$ must both be disconnected since neither contains the edge (v_1, v_2), the only edge joining vertices of V_1 and V_2. The same is true if $v_1 \notin Y$ and $v_2 \in Y$. If $v_1, v_2 \in Y$ then it is possible that $G(Y)$ is connected, but $G(\sim Y)$ is necessarily disconnected. If $v_1, v_2 \in \sim Y$, then it is possible that $G(\sim Y)$ is connected, but $G(Y)$ is necessarily disconnected. Thus, there are no other partitions of G with $G(Y)$ and $G(\sim Y)$ both connected. ■

A similar result holds if G has an articulation vertex. That is, G is the union of two graphs G_1 and G_2 that share a common vertex v, but no common edges. The one difference is that in this case V_1 gives a nonredundant inequality only if $G_2 \setminus v$ is connected.

Theorem 5.9 Let $G_1 = [V_1, E_1]$, $G_2 = [V_2, E_2]$ be connected graphs with $E_1 \bigcap E_2 = \emptyset$, $V_1 \bigcap V_2 = \{v\}$. Let $G = G_1 \bigcup G_2$. Let $\mathcal{N}_1 = \{N_1^1, N_1^2, \ldots, N_1^{n_1}\}$ be the collection of vertex partitions which give nonredundant inequalities for the graph G_1. Let $\mathcal{N}_2 = \{N_2^1, N_2^2, \ldots, N_2^{n_2}\}$ be the collection of vertex partitions which give nonredundant inequalities for the graph G_2. The vertex partitions which give nonredundant inequalities for the graph G are

$$V_1 \bigcup \{N_2^i \in \mathcal{N}_2 : v \in N_2^i\} \text{ and}$$

$$V_2 \bigcup \{N_1^i \in \mathcal{N}_1 : v \in N_1^i\}.$$

Proof: By construction, the induced subgraphs on each of the sets listed above and each of their complements are connected. We need to show that there are no other vertex partitions of V $(Y \bigcup \sim Y = V, Y \bigcap \sim Y = \emptyset)$ such that $G(Y)$ and $G(\sim Y)$ are both connected. We have already used all partitions which completely contain either V_1 or V_2. The only other possibility is that Y contains a proper subset of vertices from both V_1 and V_2. That is, $Y = Y_1 \bigcup Y_2$, where $Y_1 \subset V_1, Y_1 \neq \emptyset$, and $Y_2 \subset V_2, Y_2 \neq \emptyset$. If $v \in Y$, then $G(\sim Y)$ is necessarily disconnected. If $v \notin Y$, then $G(Y)$ is necessarily disconnected. Thus, there are no other partitions of G with $G(Y)$ and $G(\sim Y)$ both connected. ■

We now know that when checking the necessary conditions for feasibility of a multicommodity graph, we need only check inequalities corresponding to sets Y which have both $G(Y)$ and $G(\sim Y)$ connected. Do we really see much reduction when we discard inequalities corresponding to sets Y which have either $G(Y)$ or $G(\sim Y)$ (or both) disconnected? The following theorem gives a lower bound on the amount of reduction.

Theorem 5.10 Let G be a graph on n vertices, and let the degree of the minimum degree vertex be $\delta(G)$. Then using the Redundancy Theorem results in a reduction of at least

$$\sum_{i=1}^{n-1-\delta(G)} \binom{n-1-\delta(G)}{i} = 2^{n-1-\delta(G)} - 1$$

absolute value inequalities that must be checked for feasibility.

Proof: We want to count the number of disconnected induced subgraphs of G. However, if $G' = [V', E']$ is a disconnected induced subgraph and $G'' = [V'', E'']$ is also an induced subgraph, with $V'' = V \setminus V'$, then we only want to count one of that pair. A lower bound on this number can be computed by forming all sets consisting of the minimum degree vertex, v, and at least one vertex not adjacent to v. We count that by taking combinations of one or more of the $n - 1 - \delta(G)$ vertices that are neither v nor adjacent to v. ■

Corollary 5.11 If G is a complete graph, then no reduction is provided by the Redundancy Theorem.

We note that the above bound is tight for the class of graphs that have an induced subgraph on $n - 1$ nodes which is complete.

Sometimes it makes more sense to look at how many inequalities are left to be checked after applying the Redundancy Theorem, rather than how much reduction was realized. Note that without this theorem, we would always have to check an exponential number of inequalities. For some special cases, we can show that after applying the theorem, we only need to check a linear number of absolute value inequalities. That is, there are a linear number of partitions of the vertices so that the induced subgraphs on both partitions are connected.

Theorem 5.12 If G is a tree on n vertices, then $n-1$ absolute value inequalities are needed.

Proof: Removing any edge from a tree disconnects the graph. Removing any more than one edge separates the graph into more than two components. Thus, the only partitions of the vertices so that the induced subgraphs on both partitions are connected are those that consist of all the vertices on one side of a particular edge. Since a tree on n vertices has $n-1$ edges, there are $n-1$ partitions which meet the stated criteria. Hence, there are $n-1$ necessary inequalities. ∎

We also know of a case when a quadratic number of inequalities need to be checked.

Theorem 5.13 If G is a cycle on n vertices, then $\frac{1}{2}n(n-1)$ absolute value inequalities are needed.

Proof: There are connected subgraphs on $1, 2, \ldots, n-1$ nodes, and there are n connected subgraphs of each size. The subgraph on the complement of each of these node sets is also connected. However, since we want to count each graph and its "complement" only once, we must divide the total number of connected subgraphs by two. Thus, there are $\frac{1}{2}n(n-1)$

partitions of V such that both $G(Y)$ and $G(\sim Y)$ are connected. ■

If a graph contains a bridge or an articulation vertex, as in Theorem 5.8 and Theorem 5.9 then we can find the number of necessary inequalities just by looking at the subgraphs on either side of the bridge or articulation vertex.

Theorem 5.14 Let G be a graph with a bridge, such as described in Theorem 5.8. The number of nonredundant inequalities is $n_1 + n_2 + 1$, where n_1 and n_2 are the number of nonredundant inequalities for graphs G_1 and G_2, respectively.

Proof: In Theorem 5.8, we construct $|\mathcal{N}_1| + |\mathcal{N}_2| + 1$ vertex partitions of G which give nonredundant inequalities. If we let $|\mathcal{N}_1| = n_1$ and $|\mathcal{N}_2| = n_2$ then this number is $n_1 + n_2 + 1$. ■

Theorem 5.15 Let G be a graph with an articulation vertex, such as described in Theorem 5.9. The number of nonredundant inequalities is n_1+n_2, where n_1 and n_2 are the number of nonredundant inequalities for graphs G_1 and G_2, respectively.

Proof: In Theorem 5.9, we construct $|\mathcal{N}_1| + |\mathcal{N}_2|$ vertex partitions of G which give nonredundant inequalities. If we let $|\mathcal{N}_1| = n_1$ and $|\mathcal{N}_2| = n_2$ then this number is $n_1 + n_2$. ■

Using combinations of the above theorems, we can compute the number of necessary inequalities for several graphs with special structures. For example, if G has a bridge, and G_1 is a cycle and G_2 is a tree, then we know that $n_1 = \frac{1}{2}n'(n'-1)$ and $n_2 = n'' - 1$, where n' and n'' are the number of vertices of G_1 and G_2, respectively. Therefore, the number of nonredundant inequalities for G is $\frac{1}{2}n'(n'-1)+n''-1+1 = \frac{1}{2}n'(n'-1)+n''$.

6. Avenues for Further Research

This thesis opens several avenues for further research. We believe that there are some further special properties of the absolute value inequalities used in Chapter 5. We have not yet been able to find counterexamples or proofs of the following conjectures.

Conjecture 6.1 The dependency set for a redundant linear inequality contains at most one linear inequality from each family corresponding to a given absolute value inequality.

The proof of this conjecture would lead us to a proof for the "if" direction of the following conjecture. The "only if" direction of the next conjecture is readily apparent.

Conjecture 6.2 The inequality $\sum_{k=1}^{p} |\sum_{i \in Y} b_i^k| \leq U(Y, \sim Y)$ is nonredundant in the cut system if and only if the 2^p linear inequalities corresponding to it are all nonredundant in the cut system.

Another direction would be to consider the classification of nodes as supply, demand, or transshipment nodes. The work in this thesis assumes that no node classifications are known a priori. Greenberg ([14]) has demonstrated that the Wallace-Wets Theorem does not extend immediately to the situation when nodes are pre-classified as supply, demand, and transshipment nodes. It is not known how the theorem must be changed for this case. Classification of nodes can also be looked at in the multicommodity setting. Additionally, one could consider the effects of a partial classification of the nodes, which in the multicommodity setting may mean that different nodes are classified for different commodities.

Some further work is also possible in the area of counting the number of necessary inequalities for special graph structures. There are many special cases that have not been covered in this thesis. One particular possibility is to find a bound on the number of necessary inequalities for an arbitrary l-connected graph.

Glossary

articulation vertex A vertex whose removal increases the number of components of the graph

bridge An edge whose removal increases the number of components of the graph

complete graph A graph with an edge between every pair of vertices

connected graph A graph in which there is a path between every pair of vertices.

dependency set A system of inequalities which implies a redundant inequality.

feasible network A network in which the required flow can be shipped without violating the capacities of any edges.

multicommodity graph A graph in which flow must be sent from specific origins to specific destinations.

redundant inequality An inequality which is implied by other inequalities in the system.

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