

BISHOP DOMINATION ON AN  $M \times N$   
CHESSBOARD

by

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This thesis for the Master of Science  
degree by  
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Bishop's Dominating Number on a  $M \times N$  Chessboard

Thesis directed by Associate Professor David C. Fisher

### ABSTRACT

In the game of Chess, a Bishop attacks diagonally any number of squares. A set of Bishops “dominate” an  $m$  by  $n$  chessboard if for every square there is either a Bishop on it or a Bishop attacking it. What is the minimum number of Bishops that are necessary to dominate a  $m$  by  $n$  chessboard? This thesis will make a conjecture (which will be partially proven) for the answer to this question.

When  $m = n$ , I have a new proof for a result that  $m$  Bishops are required to dominate the Chessboard fully given first by Yaglom and Yaglom. When  $m < n \leq 2m$ , I find a similar result that the total number of Bishops required is  $2 \lfloor \frac{n}{2} \rfloor$ . When  $n > 2m$ , I give a domination with  $2 \lfloor \frac{n+m}{3} \rfloor$  Bishops, and conjecture that this is the minimum number. This conjecture is verified when  $m = 2$  and  $m = 3$ . It is also verified when  $n = 2m + 1, 2m + 2, 2m + 3, 2m + 4$  and  $2m + 5$ . Further, an Integer Programming formulation failed to find a counterexample for values of  $m \leq 9$  and values of  $n \leq 110$ .

Signed \_\_\_\_\_  
David C. Fisher

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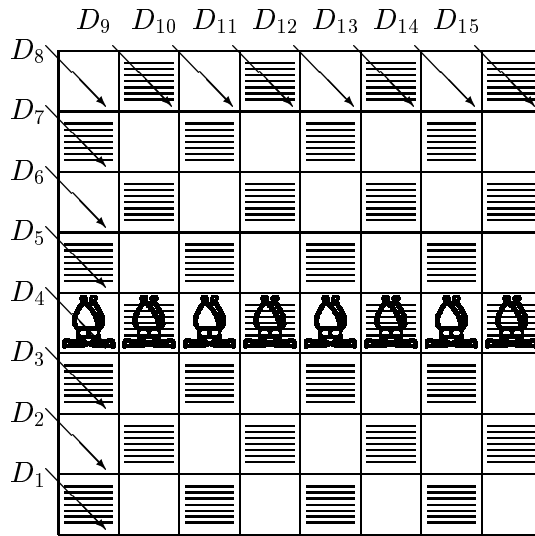
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# 1. Bishop Domination

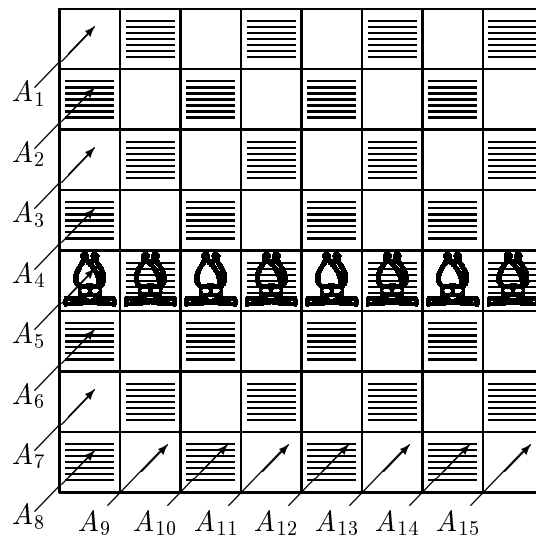


**Figure 1.1.** *Diagonal labeling for an  $8 \times 8$  Chessboard.*

The  $8 \times 8$  Chessboard has been often been the stage to much mathematical thought. A conventional game involves two players, where each player has a Queen and a King and two Rooks, two Knights and two Bishops. I am primarily concerned with the Bishops alone, and this paper concerns having any number of Bishops. Given that a Chessboard is a two dimensional array of squares in rows and columns, a single Bishop moves diagonally across them. Given also if a Chessboard is colored with two distinct colors as a Checkerboard, a Bishop moves on a single color alone. The domination, or attacking every square of this Chessboard by various chess pieces has been a topic of

some interest, and doing so with the use of minimal number of pieces of considerable interest.

I now formally define our problem. A set of Bishops “dominate” an  $m$  by  $n$  chessboard if for every square there is either a Bishop on it or a Bishop attacking it. What is the minimum number of Bishops that are necessary to dominate a  $m$  by  $n$  chessboard? Figure 1.1 and Figure 1.2 show a domination of the  $8 \times 8$  Chessboard using 8 Bishops in a center-most row. Yaglom and Yaglom [3] first showed that Bishop domination for a general



**Figure 1.2.** *Antidiagonal labeling for an  $8 \times 8$  Chessboard.*

square  $m \times m$  Chessboard cannot be accomplished with fewer than  $m$  Bishops. Cockayne, Gamble and Shepard [1] also independently show these results as well as independent dominations for square Chessboards. We will also extend

these results to rectangular Chessboards. We simplify our problem by assuming that  $m \leq n$ , because the Bishops domination problem for a  $m$  by  $n$  chessboard is equivalent to the problems of  $n$  by  $m$  chessboard. Through further analysis, we have decomposed the rest of the problem into three different cases. The first case discusses the issues when the rows  $m$  are equal to the number of columns  $n$ . The second case's main topic is about when  $m < n \leq 2m$ , and the third case discusses the issues for when  $n > 2m$ .

The following are a few definitions that were used throughout to help clarify and simplify our problem.

Let  $B_{m,n}$  be the graph with nodes  $V_{m,n}$ , such that

$$V_{m,n} = \{(1, 1), (1, 2), \dots, (1, n), \dots, (m, 1), (m, 2), \dots, (m, n)\} .$$

The edge set  $E_{m,n}$  is a function of  $mn$  nodes, where if given two nodes  $(i, j)$  and  $(k, l)$  have an edge between them, then  $|k - i| = |j - l| \neq 0$ .

We also note that since  $B_{m,n}$  is isomorphic to  $B_{n,m}$ , then without loss of generality we will assume throughout that  $m \leq n$ .

The following are several subsets of  $V_{m,n}$  :

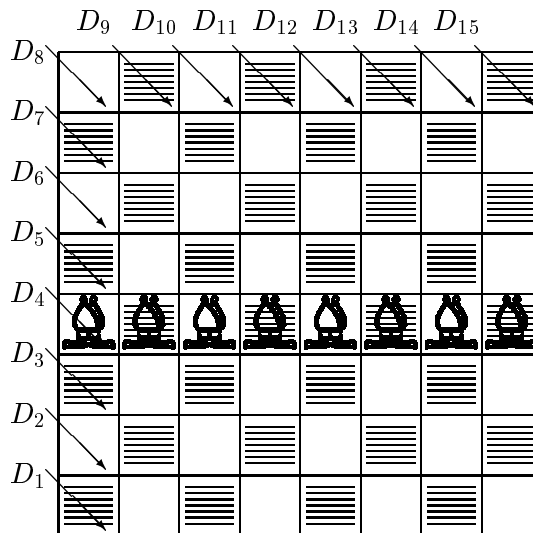
- (1) The  $i^{\text{th}}$  row is:  $R_i = \{(i, 1), (i, 2), \dots, (i, n)\}$ .
- (2) The  $j^{\text{th}}$  column is:  $C_j = \{(1, j), (2, j), \dots, (m, j)\}$ .
- (3) The  $p^{\text{th}}$  diagonal is:  $D_p = \{(i, j) \in V_{m,n} \mid p = m + j - i\}$ .
- (4) The  $p^{\text{th}}$  antidiagonal is denoted as  $A_p = \{(i, j) \in V_{m,n} \mid p = j + i - 1\}$ .
- (5) The set of white and black squares (denoted respectively as  $\mathcal{W}$  and  $\mathcal{L}$ ) are a subset of squares that have size  $\lceil \frac{mn}{2} \rceil$  and size  $\lfloor \frac{mn}{2} \rfloor$ , such that each set is denoted as

$$\mathcal{W} = \left\{ (i, j) \mid i + j \text{ is even} \right\}$$

$$\mathcal{L} = \left\{ (i, j) \mid i + j \text{ is odd} \right\}$$

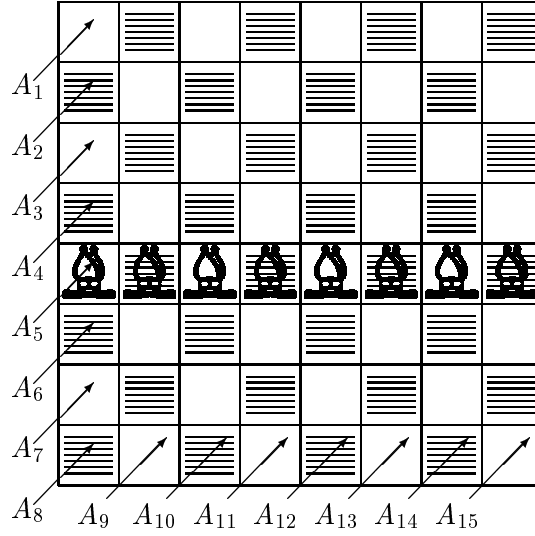
## 2. Square Chessboards, where $1 \leq m = n$

Yaglom and Yaglom [3] show that  $m$  Bishops are required to dominate all Chessboards with  $m$  rows and  $m$  columns. Cockayne, Gamble and Shepard [1] show that the independent domination number for the Bishops in square Chessboards is no more than the domination number. Here we will give a little different proof.



**Figure 2.1.** Here 8 Bishops dominate an  $8 \times 8$  Chessboard.

**Theorem 1** For all  $m \geq 1$ , we have  $\gamma(B_{m,m}) = m$ .



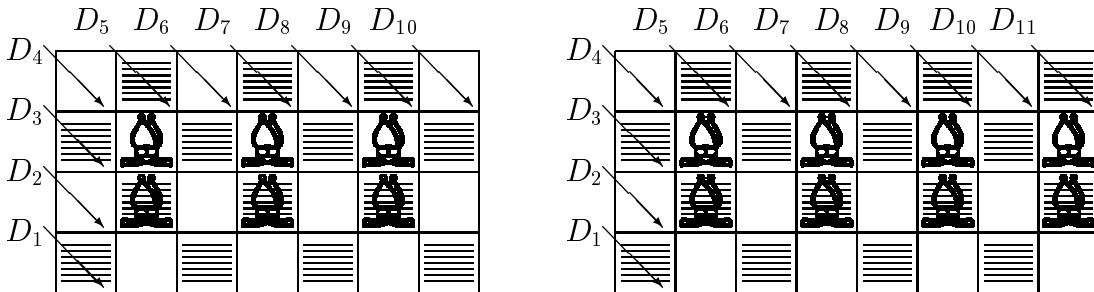
**Figure 2.2.** Here 8 Bishops dominate an  $8 \times 8$  Chessboard.

**Proof:** We can dominate  $B_{m,m}$  with  $m$  Bishops in row  $\lceil \frac{m}{2} \rceil$  (see Fig.2.2 and Fig. 2.1 ). So  $\gamma(B_{m,m}) \leq m$ .

Suppose we can dominate  $B_{m,m}$  with less than  $m$  Bishops. Let a diagonal be “large” if it contains at least  $\lceil \frac{m}{2} \rceil$  squares. Since there are at least  $m$  large diagonals, at least one of these diagonals must be bishopless. So to attack the squares of this large diagonal, we need at least  $\lceil \frac{m}{2} \rceil$  Bishops attack these squares via antidiagonals of the same color as the large diagonal. Thus there are fewer than  $m - \lceil \frac{m}{2} \rceil = \lfloor \frac{m}{2} \rfloor$  Bishops of the other color, leaving a large diagonal of this other color to be bishopless. Since the squares corresponding to this diagonal cannot be covered with less than  $\lceil \frac{m}{2} \rceil$  Bishops in corresponding antidiagonals, we have a contradiction. Therefore  $\gamma(B_{m,m}) = m$ .  $\square$

### 3. $m \times n$ Chessboards, where $1 \leq m < n \leq 2m$

In this section, we will be the first to prove in a domination of  $B_{m,n}$ , where  $1 \leq m < n \leq 2m$  that at least  $\lfloor \frac{n}{2} \rfloor$  Bishops of each color are needed.



**Figure 3.1.** Here we have  $\gamma(B_{4,7}) = 6$  and  $\gamma(B_{4,8}) = 8$ .

**Theorem 2** For all  $1 \leq m < n \leq 2m$ , we have  $\gamma(B_{m,n}) = 2\lfloor \frac{n}{2} \rfloor$

**Proof:** For any  $1 \leq m < n \leq 2m$ , we can dominate  $B_{m,n}$  with  $2\lfloor \frac{n}{2} \rfloor$  Bishops, in squares  $(\lfloor \frac{m}{2} \rfloor - 1, 2), (\lfloor \frac{m}{2} \rfloor, 2), (\lfloor \frac{m}{2} \rfloor - 1, 4), (\lfloor \frac{m}{2} \rfloor, 4), \dots, (\lfloor \frac{m}{2} \rfloor - 1, 2\lfloor \frac{n}{2} \rfloor), (\lfloor \frac{m}{2} \rfloor, 2\lfloor \frac{n}{2} \rfloor)$  ( Fig. 3.1 shows these dominations for  $B_{4,7}$  and  $B_{4,8}$ ). Therefore  $\gamma(B_{m,n}) \leq 2\lfloor \frac{n}{2} \rfloor$ .

To show  $\gamma(B_{m,n}) = 2\lfloor \frac{n}{2} \rfloor$ , we suppose a domination exists with fewer than  $\lfloor \frac{n}{2} \rfloor$  Bishops for some color, say white. Since  $\lfloor \frac{n}{2} \rfloor \leq m$ , every white diagonal whose size is at least  $m$  must have a Bishop in it, otherwise we cannot attack every one of their squares via antidiagonals. Let us choose  $p < m$  so

be the largest integer, such that  $D_p$  is bishopless. Then  $p$  exists otherwise all white diagonals of  $D_k$  with  $1 \leq k \leq n$  would have a Bishop, and since there are less than  $\lfloor \frac{n}{2} \rfloor$  white Bishops then this is impossible. Let us also choose  $q > n$  to be smallest integer, such that  $D_q$  is also bishopless and is of the same color as  $D_p$ . One can show that  $q$  exists for a similiar reason as  $p$ . Since there are  $\frac{q-p}{2} - 1$  white diagonals between  $D_p$  and  $D_q$ , then we must assume  $\frac{q-p}{2} - 1 \leq \lfloor \frac{n}{2} \rfloor - 1$ , and assume  $q - p \leq 2 \lfloor \frac{n}{2} \rfloor$ .

*How many Bishops are needed to dominate the squares of antidiagonals  $p$  and  $q$ ?* Diagonal  $p$  includes all squares between  $(m-p+1, 1)$  and  $(m, p)$ . Diagonal  $q$  includes all squares between  $(1, q-m+1)$  and  $(n+m-q, n)$ . We know that one end of  $D_p$  is in  $A_{m-p+1}$  and the other end is in  $A_{m+p-1}$ , and one end of  $D_q$  is in  $A_{q-m+1}$  and the other end is in  $A_{2n+m-q-1}$ . Since  $\lfloor \frac{n}{2} \rfloor \leq m$ , then  $q - p \leq 2 \lfloor \frac{n}{2} \rfloor \leq 2m$ , so  $(q - m + 1) - (p + m - 1) \leq 2$ . Thus, the antidiagonals that cover  $D_p$  are contiguous with the antidiagonals that cover  $D_q$  (there may be overlap between these two sets of antidiagonals, but there is no “gap”).

So the total number of antidiagonals can be found as the difference between these antidiagonals on each side as needed to cover both  $D_p$  and  $D_q$  is  $\frac{(2n+m-q-1)-(m-p+1)}{2} = \frac{2n-q+p}{2} = n - \frac{q-p}{2} \geq n - \lfloor \frac{n}{2} \rfloor \geq \lfloor \frac{n}{2} \rfloor$ . Since there are less than  $\lfloor \frac{n}{2} \rfloor$  white Bishops, then we have a contradiction. The proof is similar for the black squares. Therefore,  $\gamma(B_{m,n}) = 2 \lfloor \frac{n}{2} \rfloor$ .  $\square$

## 4. A Conjecture for $m \times n$ Chessboards, when $n > 2m > 2$

We will introduce this section by proving a theorem that is an upper bound for all  $2m < n$  and conjecture that this is also the lower bound. We first will give a domination that for each of the separate colors combined for all values of  $m$  and values of  $n$ . We will later go and begin a proof by complete induction on  $m$  for all  $n$ .

**Theorem 3** *For  $n > 2m > 2$ , we have  $\gamma(B_{m,n}) \leq 2\lfloor \frac{m+n}{3} \rfloor$*

**Proof:** The following sets describe a domination by placement of Bishops. We first note that any square  $(i, j)$  can be attacked one of three ways. These are to directly have a Bishop on the square, or to have a Bishop attack it via antidiagonal  $A_{i+j-1}$  attack it or to have a Bishop in diagonal  $D_{m+j-i}$ . The following two sets describe domination locations for Bishops, where we these sets are not necessarily unique. The first set describes the behavior observed for the white colored squares, and the second set describes the pattern for the black squares in a possible domination.

$$\left\{ \begin{array}{cccc}
(2, 2), & & & \\
(1, 5), & (1, 7), & \dots, & (1, 2m + 1), \\
(m, 3m + 2), & (m, 3m + 4), & \dots, & (m, 5m - 2), \\
\\
(1, 6m - 1), & (1, 6m + 1), & \dots, & (1, 8m - 5), \\
(m, 9m - 4), & (m, 9m - 2), & \dots, & (m, 11m - 8), \\
\\
(1, 12m - 7), & (1, 12m - 5), & \dots, & (1, 14m - 11), \\
(m, 15m - 10), & (m, 15m - 8), & \dots, & (m, 17m - 14), \\
\\
(1, 18m - 13), & (1, 18m - 11), & \dots, & (1, 20m - 17), \\
(m, 21m - 16), & (m, 21m - 14), & \dots, & (m, 23m - 20), \\
\\
\vdots & \vdots & \vdots & \vdots \\
(1, 6(m-1)k + 5), & (1, 6(m-1)k + 3), & \dots, & (1, 6(m-1)k + 2m + 1), \\
(m, 6(m-1)k + 3m + 2), & (m, 6(m-1)k + 3m + 4), & \dots, & (m, 6(m-1)k + 5m - 2),
\end{array} \right\}.$$

**Figure 4.1.** *Positions of White Bishops in my Domination Pattern.*

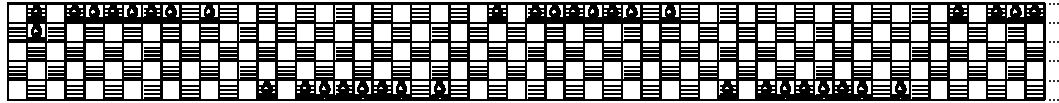
For this first subset, have shown groupings of subsets of the bishops in row 1 and row  $m$ . We can note that the white Bishops start off in a nonconforming manner attacking diagonal  $D_m$  and antidiagonal  $A_2$ , and have a maintained pattern for the Bishops at every  $2m - 4$  columns. The ending columns must then be handled with placement of Bishops off the pattern for sizes of chessboards that are not in full subsets.

This second set describes the configuration Bishops for the black squares, it can be easily noted that we do not start the set with a square that is not on the border.

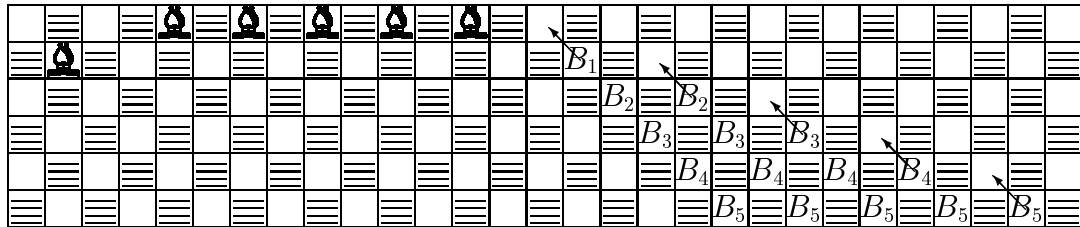
$$\left\{ \begin{array}{cccc}
 (1, 2), & (1, 4), & \dots, & (1, 2m - 2), \\
 (m, 3m - 1), & (1, 3m + 1), & \dots, & (m, 5m - 5), \\
 \\
 (1, 6m - 4), & (1, 6m - 2), & \dots, & (1, 8m - 8), \\
 (m, 9m - 7), & (1, 9m - 5), & \dots, & (m, 11m - 11), \\
 \\
 (1, 12m - 10), & (1, 12m - 8), & \dots, & (1, 14m - 14), \\
 (m, 15m - 13), & (m, 15m - 11), & \dots, & (m, 17m - 17), \\
 \\
 (1, 18m - 16), & (1, 18m - 14), & \dots, & (1, 20m - 20), \\
 (m, 21m - 19), & (m, 21m - 17), & \dots, & (m, 23m - 23), \\
 \\
 \vdots & \vdots & \vdots & \vdots \\
 (1, 6(m-1)k + 2), & (1, 6(m-1)k + 4), & \dots, & (1, 6(m-1)k + 2m - 2), \\
 (m, 6(m-1)k + 3m - 1), & (m, 6(m-1)k + 3m + 1), & \dots, & (m, 6(m-1)k + 5m - 5),
 \end{array} \right\}$$

**Figure 4.2.** *Positions of Black Bishops in my Domination Pattern.*

This next picture actually displays an example of the combined white and black bishop domination pattern for the white squares  $B_{5,n}$ . This domination is typical for all chessboards, and one can note that for every square it either has a Bishop on it or is attacked by a Bishop. We basically have groups of  $m - 1$  bishops on squares of the top and bottom rows.

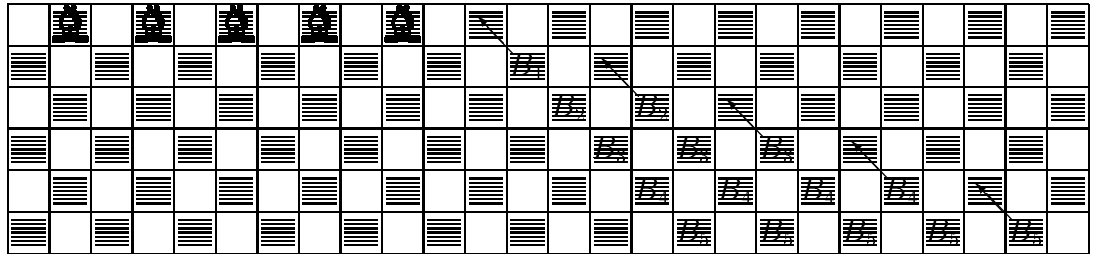


We now show that for Chessboards with sizes of  $n$  not equal to the sizes given in our pattern, I now present a configuration of Bishops that concurs with our domination number and we call it a remainder theorem to represent the remainder of the Chessboard. This pattern is represented by the following figures.



**Figure 4.4.** *Remainder Configuration for White Bishops.*

The remainder for the set of Bishops that lie on the black squares start at a smaller column index than the Bishops that lie on the white squares.



**Figure 4.5.** *Remainder Configuration for Black Bishops.*

For any chessboard, a correlation must be made when  $m$  and  $n$  do not lie on the bounds of the formula noting that there are  $\frac{2n+2m-4}{2}$  white squares and  $\frac{2n+2m-4}{2}$  black squares on the boundary. The number of Bishops can be found by noting that each Bishop independently attacks 3 squares on the boundary, thus equating to  $\left\lceil \frac{n+m-2}{3} \right\rceil = \left\lfloor \frac{n+m}{3} \right\rfloor$  Bishops.

## 5. The Conjecture is True when $n = 2m + \ell$ for $\ell \leq 5$

For the cases of  $n = 2m + 1, 2m + 2, 2m + 3, 2m + 4$  and  $2m + 5$ , I have shown the conjecture true. The proof is similar to the one found in Chapter 3.

**Theorem 4** For all  $m$ ,  $\gamma(B_{m,2m+1}) = \gamma(B_{m,2m+2}) = 2m$ .

**Proof:** Suppose that we can use  $m - 1$  Bishops to dominate the squares for a color, say white. Let  $n$  be the total number of columns in our chessboard, and so  $n = 2m + 1$  or  $n = 2m + 2$ . Let us first define  $\ell = n - 2m$ , so  $\ell = 1$  or  $\ell = 2$ . As in Theorem 2, let us choose diagonals  $p$  and  $q$  to represent bishopless diagonals with the properties that  $p$  is the largest integer less than  $m$  and  $q$  is the smallest integer greater than  $2m + \ell$ . So  $p$  exists otherwise all white diagonals of  $D_k$  with  $1 \leq k \leq 2m + \ell$  would have a Bishop, and since there are less than  $m$  white Bishops then this is impossible. One can show that  $q$  exists for a similiar reason as does  $p$ .

Since there are  $m$  white diagonals between  $D_p$  and  $D_q$ , then we will assume  $\frac{q-p}{2} - 1 \leq m - 1$  thus  $q - p \leq 2m$ . We know that one end of  $D_p$  is in  $A_{m-p+1}$  and the other end is in  $A_{p+m-1}$ , and one end of  $D_q$  is in  $A_{q-m+1}$  and the

other end is in  $A_{3m+2l-q-1}$ . Since  $q-p \leq 2m$ , we have  $(q-m+1) - (p+m-1) \leq 2$ . Thus, the antidiagonals that cover  $D_p$  are contiguous with the antidiagonals that covers the squares cover  $D_q$  (there may be overlap between these two sets of antidiagonals, but there is no “gap”). Therefore the total number of antidiagonals that pass through the squares of  $D_p$  and  $D_q$  is found as the difference between these antidiagonals on each side, as given  $\frac{(2n+m-q-1)-(m-p+1)}{2} = n - \frac{q-p}{2} \geq 2m+l-m = m+l \geq m$ . Since we assumed there are only  $m-1$  white Bishops, then we have a contradiction. The proof is similar for the black squares. Therefore for  $\ell = 1, 2$ , we have  $\gamma(B_{m,2m+\ell}) = 2m$ .  $\square$

**Theorem 5** *For all  $m$ , we have  $\gamma(B_{m,2m+3}) = \gamma(B_{m,2m+4}) = \gamma(B_{m,2m+5}) = 2(m+1)$*

**Proof:** We always have at least  $m+2$  total diagonals with  $m$  squares, where there are  $\lfloor \frac{m}{2} \rfloor + 2$  white diagonals with  $m$  squares and  $\lfloor \frac{m}{2} \rfloor + 2$  black diagonals with  $m$  squares, and thus there cannot be two of these diagonals of each color bishopless. Now we consider three cases for Bishops of a color, and let  $\ell = 3, 4, 5$ .

**Case 1: No diagonal containing  $m$  squares is bishopless:**

For any  $n$ , we can dominate  $B_{m,n}$  with  $2 \lfloor \frac{n+m}{3} \rfloor$  Bishops, (Figure 4.1 and Figure 4.2 show these dominations for general chessboards  $B_{n,m}$ ). Therefore  $\gamma(B_{m,n}) \leq 2 \lfloor \frac{2m+\ell+m}{3} \rfloor = 2 \lfloor \frac{3m+\ell}{3} \rfloor = 2 \lfloor m + \lfloor \frac{\ell}{3} \rfloor \rfloor \leq 2 \lfloor m + \lfloor \frac{5}{3} \rfloor \rfloor = 2(m+1)$ .

To show  $\gamma(B_{m,n}) = 2(m+1)$ , suppose a domination exists with at

most  $m$  Bishops for some color, say white. Let us choose  $p < m$  to be the largest integer, such that  $D_p$  is bishopless. Then  $p$  exists otherwise all white diagonals of  $D_k$  with  $1 \leq k \leq 2m + \ell$  would have a Bishop, and since there are at most  $m$  white Bishops then this is impossible. Let us also choose  $q > n$  to be smallest integer, such that  $D_q$  is also bishopless and is of the same color as  $D_p$ . One can show that  $q$  exists for a similiar reason as  $p$ . Since there are  $\frac{q-p}{2} - 1$  white diagonals between  $D_p$  and  $D_q$ , then we must assume  $\frac{q-p}{2} - 1 \leq m$ , hence  $q - p \leq 2(m + 1)$ .

*How many Bishops are needed to dominate the squares of antidiagonals  $p$  and  $q$ ?* Diagonal  $p$  includes all squares between  $(m - p + 1, 1)$  and  $(m, p)$ . Diagonal  $q$  includes all squares between  $(1, q - m + 1)$  and  $(n + m - q, n)$ . We know that one end of  $D_p$  is in  $A_{m-p+1}$  and the other end is in  $A_{m+p-1}$ , and one end of  $D_q$  is in  $A_{q-m+1}$  and the other end is in  $A_{2(2m+\ell)+m-q-1}$ . Either  $q - p \leq 2(m + 1)$  implies that  $(q - m + 1) - (p + m - 1) \leq 2$ , or  $q - p \leq 2(m + 1)$  implies that  $(q - m + 1) - (p + m - 1) \leq 4$ .

If  $q - p \leq 2(m + 1)$  implies that  $(q - m + 1) - (p + m - 1) \leq 2$ , and the antidiagonals that cover  $D_i$  are contiguous with the antidiagonals that cover  $D_j$  (there may be overlap between these two sets of antidiagonals, but there is no “gap”), where  $(q - p) - 2m + 2 \leq 2m - 2m + 2 = 2$ . So the total number of antidiagonals can be found as the difference between these antidiagonals on each side as needed to cover both  $D_p$  and  $D_q$  is  $\frac{(2(2m+\ell)+m-q-1)-(m-p+1)}{2} = \frac{2(2m+\ell)-2-(q-p)}{2} = 2m + \ell - 1 - \frac{q-p}{2} \leq 2m + \ell - 1 - \frac{2(m+1)}{2} = 2m + \ell - 1 - (m + 1) = m + \ell - 2 \geq m + 1$  Since  $\ell \geq 3$  and there are at most  $m$  white Bishops, then

we have a contradiction. The proof is similar for the black squares. Therefore,  $\gamma(B_{m,n}) = 2\lfloor \frac{m+n}{3} \rfloor$ .  $\square$

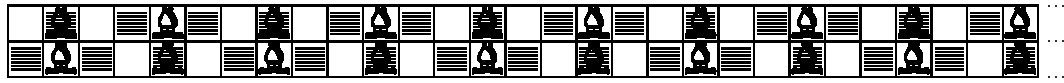
Otherwise  $q - p \leq 2(m + 1)$  implies that  $(q - m + 1) - (p + m - 1) \leq 4$  and the antidiagonals have a “gap”, where  $(q - p) - 2m + 2 \leq (2m + 2) - 2m + 2 = 4$ . So the total number of antidiagonals can be found as the difference between these antidiagonals on each side as needed to cover both  $D_p$  and  $D_q$  is  $\frac{(2(2m+\ell)+m-q-1)-(m-p+1)}{2} = \frac{2(2m+\ell)-2-(q-p)+2}{2} = 2m + \ell - \frac{q-p}{2} \leq 2m + \ell - \frac{2(m+1)}{2} = 2m + \ell - (m + 1) = m + \ell - 1 \geq m + 1$  Since  $\ell \geq 3$  and there are at most  $m$  white Bishops, then we have a contradiction. The proof is similar for the black squares. Therefore,  $\gamma(B_{m,n}) = 2(m + 1)$ .  $\square$

**Case 2: One diagonal with  $m$  squares is bishopless:** Let us label the bishopless diagonal  $D_k$ , where  $m \leq k \leq n$ . Let  $D_k$  be some color, say white. We know that  $k \neq m$ , or we have no way to attack square  $(1, 1)$ . Similarly, we cannot have  $k = n$ , but for square  $(n, m)$ .  $D_k$  has squares that range between  $(1, k - m + 1)$  to  $(m, k + 1)$  for some  $k$ . A Bishop can only attack one square of each  $A_{k-m+1}$  to  $A_{k+m+1}$ . So the number of Bishops needed are  $k - m + 1 + n + m - (k + m + 1) - n - m - 2 = 2m + \ell + m - 2 = m + \ell - 2 \geq m + 1$ . Thus, for all  $m > 2$  we are requiring more than  $m$  Bishops.  $\square$

**Case 3: More than one bishopless diagonal between  $D_p$  and  $D_q$ :** Since there are at least  $m + 1$  squares that require Bishops from at least two bishopless diagonals of size  $m$ , then my domination matches this number and uses only  $m + 1$  Bishops.  $\square$

## 6. The Conjecture is True for all $2 \times n$ and $3 \times n$ Chessboards

We can note that the Bishops in this case are also independently dominating upto three squares, and quite easily verify the conjecture.



**Figure 6.1.** *Dominations for all  $2 \times n$  Chessboards.*

**Theorem 6** *When  $m = 2$  and for all  $n$ , we have  $\gamma(B_{2,n}) = 2 \lfloor \frac{n+2}{3} \rfloor$*

**Proof:** Let  $P_n$  be defined as the set of disjoint paths of length  $n$ . Then  $B_{2,n} = P_n \cup P_n$  is disjoint union of all the paths of length  $n$ . It is well known that  $\gamma(P_n) = \lfloor \frac{n}{3} \rfloor$ , thus  $\gamma(B_{2,n}) = \gamma(P_n \cup P_n) = \gamma(P_n) + \gamma(P_n) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{3} \rfloor = 2 \lfloor \frac{n}{3} \rfloor$ .  
□

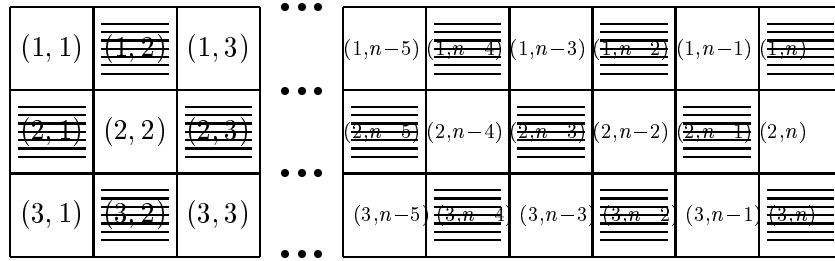
**Theorem 7** **For all  $n$ , we have  $\gamma(B_{3,n}) = 2 \lfloor \frac{n+3}{3} \rfloor = 2 \lfloor \frac{n}{3} \rfloor + 2$**

**Proof:** This will be done via induction on  $n$ , and conjecture that this holds true for all possible  $m$ . There are a total of four possible configurations of the last six columns that the dominations fall into. These six columns repeat in

pattern and I begin with a basis of  $n = 4, 5, 6, 7, 8$  and  $9$ .

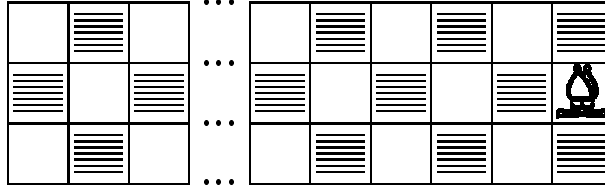
Applying Theorem 2 for  $n = 4, 5$ , we can state that the number of Bishops required are 2 for a color. Also  $n = 6$ , we can apply Theorem 2, and state that the number of Bishops required are 3 for a color. When we apply our formula, we have  $\lfloor \frac{4}{3} + 1 \rfloor = 2$ ,  $\lfloor \frac{5}{3} + 1 \rfloor = 2$  and  $\lfloor \frac{6}{3} + 1 \rfloor = 3$ . So the formula matches these results. Applying Theorem 4 for  $n = 7, 8$ , we can state that the number of Bishop required are 3 for a color. When we apply our formula, we have  $\lfloor \frac{7}{3} + 1 \rfloor = 3$  and  $\lfloor \frac{8}{3} + 1 \rfloor = 3$ , and these formula also matches these results. Applying Theorem 5 for  $n = 9$ , we can state that the number of Bishops required are 4 and applying our formula we have the number of Bishops required is  $\lfloor \frac{9}{3} + 1 \rfloor = 4$  and is also the same result. So these values form a basis for induction.

Now we can assume that for columns  $n > 9$  that  $\gamma(B_{3,k}) \leq 2 \lfloor \frac{k}{3} + 1 \rfloor$  holds true for  $4 \leq k \leq n$  as the upper bound for the number of Bishops required to dominate  $B_{3,k}$ . I must show through two cases of the last six columns where  $(1, n)$  is either a white or a black square that  $\gamma(B_{3,k}) \geq 2 \lfloor \frac{k}{3} + 1 \rfloor$ .



**Figure 6.2.** For all  $B_{3,n}$  where  $(1, n)$  is black.

**Case where square  $(1, n)$  is black:** Let  $D$  be the set of Bishop squares that dominates  $B_{3,n}$ . Square  $(2, n)$  must be dominated, and so a Bishop can either be at square  $(2, n)$  or we need to have a Bishop at either of the squares  $(1, n - 1)$  or  $(3, n - 1)$ .

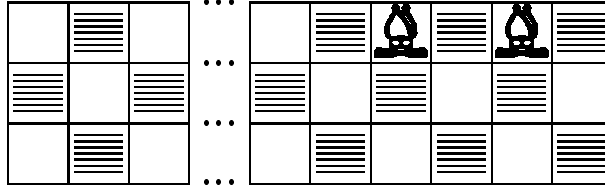


**Figure 6.3.** For all  $B_{3,n}$  where  $(1, n)$  is black,  $\gamma(B_{3,n}) \geq \gamma(B_{3,n-2}) + 1$ .

Since we need to dominate square  $(2, n)$  then either a Bishop lies on square  $(2, n)$  or there is a Bishop on either square  $(3, n - 1)$  or square  $(1, n - 1)$ . So assume square  $(2, n)$  is contained in  $D$  (see Figure 6.3), and we have used one Bishop. Thus we have fully dominated two columns from  $B_{3,n}$  and are left with  $B_{3,n-2}$  to dominate. So our result is:

$$\gamma(B_{3,n}) \geq \gamma(B_{3,n-2}) + 1 \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1 \geq \left\lfloor \frac{n}{3} \right\rfloor + 1. \quad \square$$

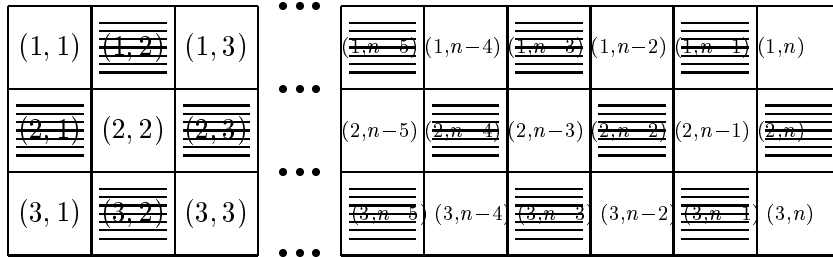
If a Bishop is not at square  $(2, n)$ , then we must have a Bishop at square  $(1, n - 1)$  and square  $(3, n - 1)$ . So without loss of generality let us assume square  $(1, n - 1)$  is in  $D$ . So now we have dominated two columns from  $B_{3,n}$ . So there must be a Bishop at either square  $(3, n - 1)$  or square  $(2, n - 3)$  or square  $(1, n - 4)$ . Having a Bishop lie on square  $(1, n - 4)$  dominates the same squares as having the Bishop lie on the other squares and more. So assume there is a second Bishop on square  $(1, n - 4)$  (see Figure 6.4).



**Figure 6.4.** For all  $B_{3,n}$  where  $(1, n)$  is black,  $\gamma(B_{3,n}) \geq \gamma(B_{3,n-6}) + 2$ .

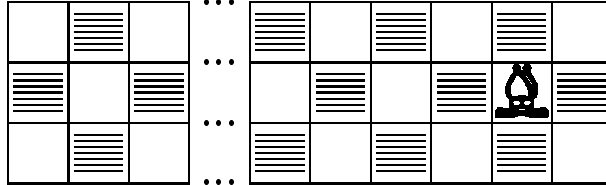
We can observe from Figure 6.4 that square  $(1, n - 5)$  is not dominated by a Bishop. It is also possible to have square  $(3, n - 5)$  not dominated by a Bishop. So either an additional Bishop lies on square  $(1, n - 5)$ , square  $(2, n - 6)$  or on square  $(3, n - 7)$ . Since this Bishop is from  $B_{3,n-6}$  and would dominate more squares, let us assume that a Bishop lies on square  $(2, n - 6)$ , which gives rise to the same domination result as given from the previous case. So now we have used two Bishops to dominate a total of six columns with the assumption of the domination of a square by a Bishop from  $B_{3,n-6}$ . So our result is:

$$\gamma(B_{3,n-6}) + 2 \geq \gamma(B_{3,n}) \geq 1 + \left\lfloor \frac{n-6}{3} \right\rfloor + 2 \geq \left\lfloor \frac{n}{3} \right\rfloor + 1. \quad \square$$



**Figure 6.5.** For all  $B_{3,n}$  where  $(1, n)$  is white.

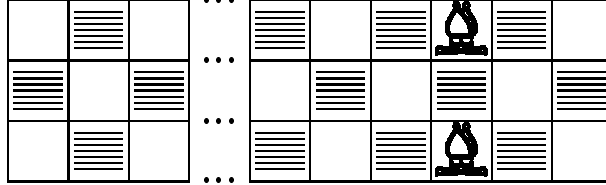
**Case where square  $(1, n)$  is white.** Let  $D$  be the set of Bishop squares that dominates  $B_{3,n}$ , as we did when square  $(1, n)$  was black. Since a Bishop from  $D$  must dominate square  $(1, n)$ , then this Bishop must lie on either square  $(1, n)$ , square  $(2, n - 1)$  or square  $(3, n - 2)$ . We can note that having square  $(2, n - 1)$  contained in  $D$  is also a domination of the same minimal size as having square  $(1, n)$  contained in  $D$ . So either square  $(2, n - 1)$  or square  $(3, n - 2)$  is contained in  $D$ .



**Figure 6.6.** For all  $B_{3,n}$  and  $(1, n)$  is white,  $\gamma(B_{3,n}) \geq \gamma(B_{3,n-3}) + 1$ .

If square  $(1, n)$  is contained in  $D$ , then we know we get the same domination of equal size as if square  $(2, n - 1)$  or if square  $(3, n - 2)$  were contained in  $D$ . So let us assume that square  $(1, n)$  is not in  $D$ . If square  $(2, n - 1)$  is contained in  $D$ , then we can note that this dominates all the white squares of  $B_{3,3}$  from  $B_{3,n}$ . This reduces our problem by three columns, and if we let  $k = n - 3$  we can refer to inductive hypothesis for  $B_{3,k}$ . So our our result is:

$$\gamma(B_{3,n}) \geq \gamma(B_{3,n-3}) + 1 \geq 1 + \left\lfloor \frac{n-3}{3} \right\rfloor + 1. = \left\lfloor \frac{n}{3} \right\rfloor + 1. \quad \square$$



**Figure 6.7.** Another form where  $(1, n)$  is white,  $\gamma(B_{3, n-5}) \geq \gamma(B_{3, n}) + 2$ .

In the last configuration, square  $(2, n - 1)$  is now not in  $D$  and square  $(3, n - 2)$  is contained in  $D$ . We now note that we need to dominate square  $(3, n)$ . If we had a Bishop there, we would dominate the same squares as placing a Bishop at square  $(2, n - 1)$ . We also can find that placing a Bishop at square  $(1, n - 2)$  also dominates the same squares as placing the Bishop at square  $(2, n - 1)$  and more squares. Since there are no other ways to dominate these other squares, let us assume that the second Bishop lies on square  $(1, n - 2)$  (see Figure 6.7). Now we can note that our result is  $\gamma(B_{3, n}) \geq \gamma(B_{3, n-5}) + 2$ . Having a Bishop lie on square  $(2, n - 5)$  would only dominate that square, and we would want to have the Bishop lie on either square  $(1, n - 6)$  or square  $(3, n - 6)$ . This Bishop is not part of the domination of  $B_{3, n-5}$ . So our result is:

$$\gamma(B_{3, n}) \geq \gamma(B_{3, n-5}) + 2 \geq \left\lfloor \frac{n-2}{3} \right\rfloor + 2 \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1 \geq \left\lfloor \frac{n}{3} \right\rfloor + 1. \quad \square$$

## 7. Integer Programing Results

An alternate method to model this problem is using systems of linear inequalities for specific Chessboards. I wrote a c program to generate such inequalities into specific text files that contained the system of linear inequalities describing diagonals and the squares that belong to each diagonal and antidiagonal for each specific Chessboard, which were run through LINDO. I ran these processes in hopes of finding a counterexample to the conjecture at hand. I was not able to find such a counter example, my results ran for values  $m \leq 9$  with respective values of  $n \leq 110$ .

Squares of the diagonals range from from the Northeast to the Southwest, and squares of antidiagonals range from the Southeast to the NorthWest. I for the sake of the program, my variable names were Diag# and Antid# and were numbered sequentially, and squares were row-column coordinates, starting with r#c#. The summation of a diagonal and antidiagonal subtracting a double counted square was an inequality valued as greater than or equal to one. Diagonals minus the squares of the antidiagonal were valued as equal to zero. Antidiagonals minus the squares of the diagonal were valued as equal to zero.

We now give an sample of systems of linear inequalities that describe

the solution to the white squares for a  $B_{5,16}$ . I also have displayed a sample Chessboard with the solution obtained. The solution obtained is interesting that it is similar to the upper bound domination I have given.

The Integer program that modeled the white squares is given as follows:

```

min
Diag1+Diag2+Diag3+Diag4+Diag5+Diag6+Diag7+Diag8+Diag9+Diag10
st
Antid8-r1c1-r2c2-r3c3-r4c4-r5c5=0
Diag1-r1c1=0
Diag1+Antid8-r1c1>=1
Antid1-r1c3-r2c4-r3c5-r4c6-r5c7=0
Diag2+Antid1-r1c3>=1
Antid2-r1c5-r2c6-r3c7-r4c8-r5c9=0
Diag3+Antid2-r1c5>=1
Antid3-r1c7-r2c8-r3c9-r4c10-r5c11=0
Diag4+Antid3-r1c7>=1
Antid4-r1c9-r2c10-r3c11-r4c12-r5c13=0
Diag5+Antid4-r1c9>=1
Antid5-r1c11-r2c12-r3c13-r4c14-r5c15=0
Diag6+Antid5-r1c11>=1
Antid6-r1c13-r2c14-r3c15-r4c16=0
Diag7+Antid6-r1c13>=1
Antid7-r1c15-r2c16=0
Diag8+Antid7-r1c15>=1
Diag2+Antid8-r2c2>=1
Diag3+Antid1-r2c4>=1
Diag4+Antid2-r2c6>=1
Diag5+Antid3-r2c8>=1
Diag6+Antid4-r2c10>=1
Diag7+Antid5-r2c12>=1
Diag8+Antid6-r2c14>=1
Diag9+Antid7-r2c16>=1

```

$\text{Antid9-r3c1-r4c2-r5c3}=0$   
 $\text{Diag2-r3c1-r2c2-r1c3}=0$   
 $\text{Diag2+Antid9-r3c1}\geq 1$   
 $\text{Diag3+Antid8-r3c3}\geq 1$   
 $\text{Diag4+Antid1-r3c5}\geq 1$   
 $\text{Diag5+Antid2-r3c7}\geq 1$   
 $\text{Diag6+Antid3-r3c9}\geq 1$   
 $\text{Diag7+Antid4-r3c11}\geq 1$   
 $\text{Diag8+Antid5-r3c13}\geq 1$   
 $\text{Diag9+Antid6-r3c15}\geq 1$   
 $\text{Diag3+Antid9-r4c2}\geq 1$   
 $\text{Diag4+Antid8-r4c4}\geq 1$   
 $\text{Diag5+Antid1-r4c6}\geq 1$   
 $\text{Diag6+Antid2-r4c8}\geq 1$   
 $\text{Diag7+Antid3-r4c10}\geq 1$   
 $\text{Diag8+Antid4-r4c12}\geq 1$   
 $\text{Diag9+Antid5-r4c14}\geq 1$   
 $\text{Diag10+Antid6-r4c16}\geq 1$   
 $\text{Antid10-r5c1}=0$   
 $\text{Diag3-r5c1-r4c2-r3c3-r2c4-r1c5}=0$   
 $\text{Diag3+Antid10-r5c1}\geq 1$   
 $\text{Diag4-r5c3-r4c4-r3c5-r2c6-r1c7}=0$   
 $\text{Diag4+Antid9-r5c3}\geq 1$   
 $\text{Diag5-r5c5-r4c6-r3c7-r2c8-r1c9}=0$   
 $\text{Diag5+Antid8-r5c5}\geq 1$   
 $\text{Diag6-r5c7-r4c8-r3c9-r2c10-r1c11}=0$   
 $\text{Diag6+Antid1-r5c7}\geq 1$   
 $\text{Diag7-r5c9-r4c10-r3c11-r2c12-r1c13}=0$   
 $\text{Diag7+Antid2-r5c9}\geq 1$   
 $\text{Diag8-r5c11-r4c12-r3c13-r2c14-r1c15}=0$   
 $\text{Diag8+Antid3-r5c11}\geq 1$   
 $\text{Diag9-r5c13-r4c14-r3c15-r2c16}=0$   
 $\text{Diag9+Antid4-r5c13}\geq 1$   
 $\text{Diag10-r5c15-r4c16}=0$   
 $\text{Diag10+Antid5-r5c15}\geq 1$

```
end
terse
gin
200
go
99999999
no
quit
```

The Bishop square locations produced by LINDO are given in the following set as  $\{R2C2, R5C3, R5C1, R1C9, R1C11, R1C13, R1C15\}$ .

LINDO 5.3 (October 1994)  
LINDO Systems, Chicago, IL

LDW83-531000  
University of Colorado

: : ? : LP OPTIMUM FOUND AT STEP 102  
OBJECTIVE VALUE = 5.58333349

NEW INTEGER SOLUTION OF 7.00000000  
AT BRANCH 5 PIVOT 352  
BOUND ON OPTIMUM: 6.111111  
ENUMERATION COMPLETE. BRANCHES= 5 PIVOTS= 352

LAST INTEGER SOLUTION IS THE BEST FOUND  
RE-INSTALLING BEST SOLUTION...  
99999999

:  
OBJECTIVE FUNCTION VALUE

1) 7.000000

VARIABLE	VALUE	REDUCED COST
DIAG2	1.000000	1.000000
DIAG3	1.000000	1.000000
DIAG4	1.000000	1.000000
DIAG5	1.000000	1.000000
DIAG6	1.000000	1.000000
DIAG7	1.000000	1.000000
DIAG8	1.000000	1.000000
ANTID8	1.000000	0.000000

R2C2	1.000000	0.000000
ANTID4	1.000000	0.000000
R1C9	1.000000	0.000000
ANTID5	1.000000	0.000000
R1C11	1.000000	0.000000
ANTID6	1.000000	0.000000
R1C13	1.000000	0.000000
ANTID7	1.000000	0.000000
R1C15	1.000000	0.000000
ANTID9	1.000000	0.000000
R5C3	1.000000	0.000000
ANTID10	1.000000	0.000000
R5C1	1.000000	0.000000

ROW SLACK OR SURPLUS DUAL PRICES

NO. ITERATIONS= 371  
 BRANCHES= 5 DETERM.= 1.000E 0  
 :

Again, we see from this example alone that there is a tend for the placement of the Bishops toward the top and bottom rows.

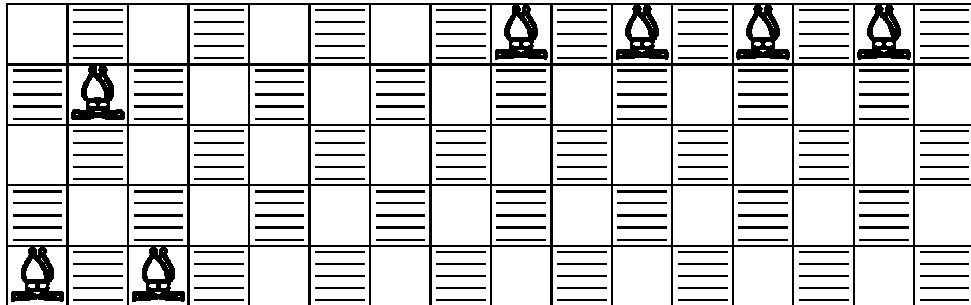


Figure 7.1. LINDO results for white Bishops  $\gamma(B_{5,16})$ .

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