

ANALYSIS OF LAGRANGE MULTIPLIER BASED
DOMAIN DECOMPOSITION

by

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ABSTRACT

The convergence of a substructuring iterative method with Lagrange multipliers known as Finite Element Tearing and Interconnecting (FETI) method is analyzed in this thesis. This method, originally proposed by Farhat and Roux, decomposes finite element discretization of an elliptic boundary value problem into Neumann problems on the subdomains, plus a coarse problem for the subdomain null space components. For linear conforming elements and preconditioning by Dirichlet problems on the subdomains, the asymptotic bound on the condition number $C(1 + \log(H/h))^\gamma$, where $\gamma = 2$ or 3 , is proved for a second order problem, h denoting the characteristic element size and H the size of subdomains. A similar method proposed by Park is shown to be equivalent to FETI with a special choice of some components and the bound $C(1 + \log(H/h))^2$ on the condition number is established. Next, the original FETI method is generalized to fourth order plate bending problems. The main idea there is to enforce continuity of the transversal displacement field at the subdomain crosspoints throughout the preconditioned conjugate gradient iterations. The resulting method is shown to have a condition number that does not increase with the number of subdomains, and again grows at most

poly-logarithmically with the number of elements per subdomain; the condition number is bounded by $C(1 + \log(H/h))^3$. These optimal properties hold for numerous plate bending elements that are used in practice including the HCT, DKT, and a class of non-locking elements for the Reissner-Mindlin plate models. The theoretical results are confirmed by numerical experiments.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed _____
Jan Mandel

DEDICATION

Dedicated to my parents.

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1. Introduction

In this thesis, the Finite Element Tearing and Interconnecting (FETI) method is analyzed. It is one of domain decomposition methods for solving large systems of linear equations arising from finite element discretizations of elliptic differential equations.

Elliptic equations often arise in modeling physical phenomena. The Laplace equation,

$$-\Delta\phi = -\operatorname{div} \nabla\phi = f \quad \text{in } \Omega, \quad (1.1)$$

models electrostatic interactions and many other potential problems. Physical quantities are often governed by systems of equations, as in the case of linear elasticity which is modeled by a system of equations for the unknown displacement vector \mathbf{u}

$$-\operatorname{div} \sigma(\mathbf{u}) = \mathbf{f}.$$

In the isotropic case, the stress tensor σ can be written as $\sigma(u) = 2\mu\varepsilon(\mathbf{u}) + \lambda \operatorname{tr}(\varepsilon(\mathbf{u}))\delta$, where λ and μ are called Lamé constants, δ is the Kronecker tensor, and $\varepsilon(\mathbf{u})$ is the strain, $\varepsilon(\mathbf{u}) = 1/2(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. An equation describing a plate bending problem can be obtained from the linear elasticity equation by a limit process that considers thickness of the plate to be infinitesimally small. In Kirchhoff-Love model, this yields a fourth order elliptic equation.

In order to set up a well-posed problem, the equations need to be complemented by boundary conditions. A boundary value problem may be

reformulated in weak (variational) form [56]. Two basic kinds of boundary conditions are recognized. Essential boundary conditions, such as a Dirichlet boundary condition $\phi = 0$ on $\partial\Omega$ for (1.1), need to be explicitly imposed in the weak form of the problem. Natural boundary conditions, such as a Neumann boundary condition $\partial\phi/\partial\mathbf{n} = 0$ on $\partial\Omega$ for (1.1), are “naturally” incorporated into the weak form.

The process of discretization may be based on a Galerkin approximation. In that case, an approximate solution is sought in a finite-dimensional subspace of the space in which the weak form is posed. Finding the Galerkin approximation then requires solving a system of linear equations for the coefficients of the solution relative to a basis of the subspace. Finite element spaces are widely used in this context. They are generated by basis functions that are usually polynomial on each element of a triangulation of the domain and the supports of basis functions have only small overlaps.

The matrix of the discrete system has several special properties. It is symmetric and positive definite. Also, due to limited overlaps of finite element basis functions, it is sparse; the number of nonzero entries in a row of the matrix is a lot smaller than total number of entries in the row. The size and the condition number of this matrix are affected by fineness of the triangulation. For quasi-uniform meshes for which the finite element discretization is characterized by the mesh size h of the triangulation, the condition number of the matrix is of the order $\frac{1}{h^2}$ for second order elliptic problems and $\frac{1}{h^4}$ for fourth order elliptic problems.

Two classes of methods for solving the linear system exist, direct and iterative. Direct methods, usually based on a variant of Gaussian elimination,

are common in everyday engineering practice. They are fairly robust; their disadvantage lies in their memory requirements and their speed. Gaussian elimination leads to fill-in and the destruction of the sparse structure of the matrix. The fill-in can be reduced by re-numbering the variables, but for problems in 3D this helps only to a certain extent. In general, the number of operations required to solve the system arising from the finite element discretization is proportional to the square of the number of unknowns or worse.

Since systems arising from discretizations of problems in engineering practice constantly push available computer resources to the limits, iterative methods are often preferable for large-scale problems. Their main disadvantage is probably their lack of robustness when too wide a class of problems is considered. On the other hand, the memory requirements of iterative methods are typically much smaller than those of the direct methods. Usually only a small multiple of the number of nonzero entries of the original matrix needs to be stored. The number of operations can be as low as a multiple of the number of unknowns, but often the computational cost is not known in advance. The speed of convergence depends on the condition number of the matrix of the system. Since the condition number deteriorates with decreasing mesh size, so does the speed of convergence of simple iterative methods. It is therefore desirable to construct methods that overcome bad conditioning of the matrix and that are, if possible, independent of other singular perturbations such as inhomogeneities or bad Poisson ratios in the linear elasticity problem or, in the case of plate bending model, the problems arising from the thickness of a plate approaching zero and the model becoming a fourth order problem. Due to the elliptic nature of the underlying problem, this sort of stability is often possible

to achieve and various algorithms have been proposed.

In this thesis, we will study one of domain decomposition methods. This class of methods has gained enormous popularity in the last decade. Following a divide-and-conquer idea, domain decomposition methods divide the original problem into a number of smaller subproblems. Sometimes such a division arises from breaking up a complicated geometry. In many cases, though, it is entirely artificial. The subproblems are easier to solve because of their smaller size and often parallelism can be exploited. This is especially important with the onset of parallel computing.

Domain decomposition methods can be seen from two different points of view. They may arise from separation of a physical domain into regions, where a problem is modeled by separate partial differential equations, with the interfaces between the subdomains being handled by various conditions, such as continuity. The opposite approach is to see domain decomposition methods purely as methods for solving large algebraic linear systems arising from the discretization of PDE's. In this context, the large system is subdivided into smaller problems, whose solutions can be used to produce a preconditioner for the large system.

Every domain decomposition algorithm involves two principal issues. It breaks up the original problem into subproblems, that are solved by some known method, and it resolves interactions of local solutions, usually by the means of an iterative method. Many domain decomposition methods have been developed, originally, for the case of two subdomains. As the complexity of problems of interest demands splitting the problem into small enough subproblems, a multi-domain case proves to be of more importance. Early works

demonstrate independence of proposed algorithms on the characteristic mesh size. However, it has been shown that to achieve independence of the number of subdomains, a coarse space needs to be introduced. Solving a small coarse space problem distributes the information about the solution globally, thus resolving global characteristics of the solution not visible by the subdomains.

In the next chapter, we give an overview of several well-know domain decomposition algorithms. In particular, we will describe abstract Schwarz methods, overlapping and substructuring methods and their relationship to the finite element tearing and interconnecting method (FETI) [27, 32], the main subject of this thesis. We also briefly discuss a domain decomposition method based on smoothed prolongation [68, 64].

The remaining chapters, except for Sections 3.4 and 4.4, are based on papers [50, 51, 53] by Mandel and Tezaur, and by Mandel, Tezaur and Farhat. In Chapter 3, the derivation of the original FETI method is shown. FETI tears the computational domain into non-overlapping subdomains and enforces intersubdomain continuity via Lagrange multipliers applied at the subdomain interfaces. The Lagrange multipliers are used as the unknowns and FETI formulation is obtained by solving the saddle point problem for the Lagrangian. Consistent treatment of subdomain singularities leads to a small coarse problem which is solved in each iteration of the preconditioned conjugate gradient method iteration. In Chapter 3, we also present a generalization of the original FETI algorithm. We describe its particular application to a plate bending problem.

Chapter 4 is concerned with analyzing the original and generalized FETI methods. We show that the condition number of the preconditioned

FETI method is bounded independently of the number of subdomains and poly-logarithmically in terms of subdomain size. That is, the condition number is bounded by $C(1 + \log \frac{H}{h})^3$, where H is the characteristic subdomain size and h the mesh size. We present a complete analysis for decompositions with crosspoints in 2D and edges and crosspoints in 3D for second order elliptic problems. Furthermore, we demonstrate that the choice of the Lagrange multipliers given by Park [59, 58] assures that the condition number estimate can be improved to $C(1 + \log \frac{H}{h})^2$. Finally, we also show that the generalized FETI converges poly-logarithmically for a biharmonic problem.

In order to illustrate the potential of the generalized FETI method, Chapter 5 summarizes some computational results. We consider the plate bending problem on a unit square and demonstrate that the condition number is almost independent of the number of subdomains and the size of the problem, as predicted by the theoretical analysis.

2. Overview of Related Domain Decomposition Methods

2.1 Finite Element Approximation, Model Problem and PCG

In this section, we are going to summarize some concepts and algorithms used throughout this chapter and the rest of the thesis. We will consider the following model problem. Let Ω be a bounded Lipschitz domain in \mathbb{R}^2 or \mathbb{R}^3 . We will study the second-order elliptic equation with Dirichlet boundary condition

$$\begin{aligned} \mathcal{A}u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where

$$\mathcal{A}u = - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right).$$

with the matrix $[a_{ij}]$ symmetric, uniformly positive definite and bounded on Ω . The corresponding bilinear form is then given by

$$a(v, w) = \sum_{i,j=1}^d \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}$$

and is defined for all v and w in the Sobolev space $H^1(\Omega)$ (the space of generalized functions with square integrable first derivatives). Let us denote the $L_2(\Omega)$ inner product

$$(v, w) = \int_{\Omega} vw.$$

By the divergence theorem (integration by parts), the problem (2.1) can be written in weak (variational) form: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \tag{2.2}$$

for all v in $V = H_0^1(\Omega)$, where H_0^1 is the completion of smooth functions with support in Ω with respect to the norm in $H^1(\Omega)$. Since the bilinear form $a(u, v)$ is symmetric, continuous and elliptic (coercive) [56], the problem has a unique solution by the Lax-Milgram theorem.

Let us now use the standard Galerkin approximation. We look for an approximate solution of (2.2) in a finite-dimensional subspace $V_h(\Omega)$ of the space V . The Galerkin approximation is the solution of the following problem: Find $u_h \in V_h(\Omega)$ such that

$$a(u^h, v^h) = (f, v^h), \tag{2.3}$$

for all $v^h \in V_h(\Omega)$. This problem leads to a system of linear equation when a basis for V_h is chosen. Let $\{\varphi\}_{i=1}^N$ be a basis for $V_h(\Omega)$. Assuming

$$u^h = \sum_{i=1}^N u_i \varphi_i,$$

(2.3) becomes

$$\sum_{i=1}^N a(\varphi_i, \varphi_j) u_i = (f, \varphi_j),$$

$j = 1, \dots, N$.

Matrix $K = [a(\varphi_i, \varphi_j)]$ is called the stiffness or Gramm matrix and the right hand side vector $[(f, \varphi_j)]$ is called the load vector. The vector of unknowns, $[u_i]$, is also called the vector of degrees of freedom.

For a wide class of approximation spaces V_h , u^h is a good approximation of u . In the case of finite element methods, the construction of suitable

spaces V_h relies on triangulation, which splits the domain Ω into small disjoint regions of simple geometric shape, such as triangles or quadrangles in \mathbb{R}^2 or tetrahedrons or hexahedrons in \mathbb{R}^3 ; h refers to the characteristic size of these regions. Under certain assumptions which prevent the triangulation from degenerating (e.g, the size of the angles in triangles is bounded from below so as they do not become too sharp), the finer the triangulation, the closer a finite element Galerkin solution to the exact solution, [14].

Functions in a finite element space V_h usually arise from a polynomial interpolation on the elements of the triangulation. Every polynomial defined on a given region is uniquely determined by its values and perhaps also the values of its derivatives at some nodal points, usually the vertices of the region. Each function in V_h is thus determined by a set of values at nodal points, so called degrees of freedom. Simple examples of finite element spaces include spaces formed by continuous functions linear on triangle regions in \mathbb{R}^2 or tetrahedrons in \mathbb{R}^3 or bilinear functions on quadrangles. These spaces are being referred to as P_1 and Q_1 , respectively, and their entries are uniquely determined by their function values at the vertices of the triangulation. Another type of finite elements we will encounter later in this thesis is HCT elements, which are C^1 continuous and determined by the function values and the values of the first derivatives at the vertices of triangular regions in \mathbb{R}^2 .

The standard basis for a finite element space is the one in which each of the basis functions has exactly one degree of freedom equal to 1 and the rest 0. The unknowns in the linear system arising from the discretization are directly the degrees of freedom of the Galerkin approximation (hence the name). The overlaps of the supports of the basis functions are small, which causes

the stiffness matrix to be sparse. Due to the properties of the bilinear form, the stiffness matrix is also symmetric and positive definite. Preconditioned conjugate gradient (PCG) method is therefore often considered for solving the system. A detailed description and analysis of PCG (and other methods) can be found, e.g., in Golub and Van Loan, [38]. Here we give only a brief summary.

The conjugate gradient method is based on the observation that the solution of $Ax = b$ is the only minimum of

$$Q(x) = \frac{1}{2}x^T Ax - x^T b.$$

A sequence of vectors which converges to the solution of the system can be obtained by setting

$$x_{k+1} = x_k + \alpha_k p_k,$$

where p_k is a suitably chosen direction and vector $x_k + \alpha_k p_k$ minimizes Q on the line $x_k + \alpha p_k$. In PCG, vectors p_k are obtained by A -orthogonalization of residua, which leads to the following algorithm:

Algorithm 1 (Conjugate Gradients) Given x_0 , set

$$r_0 = b - Ax_0.$$

For $k = 1, \dots$ do:

$$\begin{aligned} \beta_k &= r_{k-1}^T r_{k-1} \\ p_k &= r_{k-1} + \frac{\beta_k}{\beta_{k-1}} p_{k-1} \quad (p_1 = r_0) \\ \alpha_k &= \frac{\beta_k}{p_k^T A p_k} \\ x_k &= x_{k-1} + \alpha_k p_k \\ r_k &= r_{k-1} - \alpha_k A p_k \end{aligned}$$

Among iteration methods the conjugate gradient method is rather exceptional. Assuming roundoff errors are not present, it gives the exact solution of the system after a finite number of iteration, which equals at most the number of unknowns. For large systems, however, performing so many iterations would be impractical. Fortunately, much smaller number of iterations is often needed to get a good enough approximate solution.

This is not true if the condition number of A is large. The performance then can be improved by introducing a preconditioner $M = E^T E$, where E is some nonsingular matrix. We point out that M defined in this way is symmetric and positive definite. Preconditioner M must have two essential properties:

- matrix $E^{-T} A E^{-1}$ is better conditioned than A ,
- system $Md = e$ can be easily solved.

E^{-T} denotes the transpose of the inverse of E .

The preconditioned conjugate gradient (PCG) method is the conjugate gradient method applied to the system

$$E^{-T} A E^{-1} (Ex) = E^{-T} b.$$

This leads to the following algorithm.

Algorithm 2 (Preconditioned Conjugate Gradients) Given x_0 , set

$$r_0 = (b - Ax_0).$$

For $i = 1, \dots$ do:

$$\begin{aligned} z_{k-1} &= M^{-1} r_{k-1} \\ \beta_k &= r_{k-1}^T z_{k-1} \end{aligned}$$

$$\begin{aligned}
p_k &= z_{k-1} + \frac{\beta_k}{\beta_{k-1}} p_{k-1} & (p_1 = z_0) \\
\alpha_k &= \frac{\beta_k}{p_k^T A p_k} \\
x_k &= x_{k-1} + \alpha_k p_k \\
r_k &= r_{k-1} - \alpha_k A p_k
\end{aligned}$$

We point out that matrix M^{-1} is not formed explicitly and, instead, a linear system with matrix M is solved in each iteration of PCG method. Standard preconditioners used in practice include diagonal scaling and incomplete factorizations (e.g., Cholesky) [38].

2.2 Abstract Schwarz Methods

Abstract Schwarz methods take their name after Herman Schwarz, a German mathematician of the past century, who used an alternating method to construct harmonic functions on regions with non-smooth boundaries. They provide an abstract framework which allows analysis of many different domain decomposition methods. Here we briefly describe the general abstract algorithm and refer to literature for details on analysis and practical/non-practical choices of particular methods. This presentation is based on [6] and [23].

Let V be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. We also consider the inner product $a(\cdot, \cdot)$ implied by the bilinear form of the variational formulation of our problem: Find $u \in V$ so that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V.$$

Let $V_i, i = 0, \dots, n$ be closed subspaces of V that form a decomposition of V , i.e.

$$V = V_0 + V_1 + V_2 + \dots + V_n.$$

In addition to the inner product implied by the the bilinear form a , we consider the inner products $a_i(\cdot, \cdot)$ implied by some bilinear forms a_i defined on $V_i \times V_i$, $i = 0, \dots, n$. The Schwarz additive method can be then written as follows.

Algorithm 3 (Additive Schwarz) Given an initial approximation u^0 , start with $k = 0$ and do:

1. Find $w_i \in V_i$ such that

$$a_i(w_i, v_i) = \langle f, v_i \rangle - a(u^k, v_i), \quad \forall v_i \in V_i.$$

2. Define the next iterate as

$$u^{k+1} = u^k + \sum_{i=0}^n w_i.$$

3. Set $k = k + 1$ and go to 1.

The Schwarz multiplicative method is then defined by the following algorithm.

Algorithm 4 (Multiplicative Schwarz) Given an initial approximation u^0 , starting with $k = 0$ do:

1. For $i = 0, \dots, n$,

(i) find $w_i \in V_i$ so that

$$a_i(w_i, v_i) = \langle f, v_i \rangle - a(u^{k+i/(n+1)}, v_i), \quad \forall v_i \in V_i,$$

(ii) set

$$u^{k+(i+1)/(n+1)} = u^{k+i/(n+1)} + w_i.$$

2. Set $k = k + 1$ and go to step 1.

The algorithms above can be combined to form hybrid methods. They are usually used as preconditioners in the conjugate gradient method. Various

choices of the subspaces V_i yield a wide variety of different algorithms. Since the multiplicative method above is not symmetric, it is usually symmetrized by considering the sequence of subspaces in forward and reverse order; that is $V_0, V_1, \dots, V_{n-1}, V_n, V_n, V_{n-1}, \dots, V_1, V_0$.

Each sub-step of an additive and a multiplicative Schwarz method can be interpreted as an approximate projection of the error. Denoting P_i the operator corresponding to the i -th sub-step of an additive method, that is the operator that satisfies

$$a_i(P_i w, v) = a(w, v) \quad \forall v \in V_i,$$

one step of the additive method satisfies

$$e^{k+1} = (I - \sum_{i=0}^n P_i) e^k,$$

where $e^k = u - u^k$ is the error of the k -th approximation. Similarly, the reduction of the error of one step of the multiplicative method is governed by the equation

$$e^{k+1} = (I - P_n) \dots (I - P_1)(I - P_0) e^k.$$

We refer to [25] and references included there for abstract analysis. Here we only summarize the main results. They are due to a number of authors: Dryja and Widlund [25, 24]; Nepomnyaschikh [55]; Bramble, Pasciak, Wang, Xu [9, 70]; Lions [44]; Bjørstad and Mandel [6], and others.

Let C_0 be the constant such that for all $u \in V$, there exists a representation $u = \sum_{i=0}^n u_i$, $u_i \in V_i$, and

$$\sum_{i=0}^n a_i(u_i, u_i) \leq C_0 a(u, u).$$

Let $B = [b_{ij}]$ be the matrix of strengthened Cauchy-Schwarz coefficients,

$$|a(v_i, v_j)| \leq b_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2} \quad \forall v_i \in V_i, \forall v_j \in V_j, i, j = 1, \dots, n.$$

Furthermore, let ω be the constant such that

$$a(u_i, u_i) \leq \omega a_i(u_i, u_i) \quad \forall u_i \in V_i, i = 0, \dots, n.$$

Then the smallest eigenvalue of the operator $\sum P_i$ of the abstract additive method used a preconditioner in the conjugate gradients method is bounded by $1/C_0$ and the largest by $\omega(\rho(B) + 1)$. The bound on the condition number of the symmetric multiplicative method used as a preconditioner in the conjugate gradients method is $C_0(1 + 2\hat{\omega}\rho(B))/(2 - \hat{\omega})$, where $\hat{\omega} = \max(1, \omega)$.

The space V_0 is usually regarded as a coarse space. It has been shown that without the presence of the coarse space, the condition number of an abstract Schwarz method grows with the number of subspaces. The coarse space ensures a mechanism of global exchange of information otherwise lacking in decompositions involving a large number of subdomains.

2.3 Overlapping Domain Decomposition

The first known domain decomposition method is due to Schwarz [63] and is known as the Schwarz alternating method. It divides the domain Ω , on which the problem is defined, into two overlapping regions Ω_1 and Ω_2 and performs a multiplicative Schwarz type algorithm.

Let us consider the model problem (2.1)

$$\begin{aligned} \mathcal{A}u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Let Ω_1 and Ω_2 be overlapping subdomains of Ω , $\Omega_1 \cup \Omega_2 = \Omega$. Furthermore, let $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$ and $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$.

Algorithm 5 (Alternating Schwarz) Given an initial guess u_2^0 for the values on Ω_2 , for $k = 1, 2, \dots$ do:

1. Solve

$$\begin{aligned} \mathcal{A}u_1^k &= f && \text{in } \Omega_1, \\ u_1^k &= 0 && \text{on } \partial\Omega_1 \setminus \Gamma_1, \\ u_1^k &= u_2^{k-1}|_{\Gamma_1} && \text{on } \Gamma_1, \end{aligned}$$

2. Solve

$$\begin{aligned} \mathcal{A}u_2^k &= f && \text{in } \Omega_2, \\ u_2^k &= 0 && \text{on } \partial\Omega_2 \setminus \Gamma_2, \\ u_2^k &= u_1^k|_{\Gamma_2} && \text{on } \Gamma_2. \end{aligned}$$

To be precise, $u_1^k|_{\Gamma_1}$ and $u_2^k|_{\Gamma_2}$ stand in the algorithm above for the trace of u_1^k on Γ_1 and u_2^k on Γ_2 , respectively, rather than the simple restrictions.

Approximate solutions of the model system can be assembled as

$$\begin{aligned} u^k &= u_2^k && \text{in } \Omega_2, \\ u^k &= u_1^k && \text{in } \Omega_1 \setminus \Omega_2. \end{aligned}$$

The variational formulation of the algorithm above is due to Lions [44]. It is the multiplicative abstract Schwarz algorithm using the subspaces V_1 and V_2 of $V = H_0^1(\Omega)$ formed by functions that vanish outside of Ω_1 and Ω_2 , respectively. The bilinear forms a_1 and a_2 are simple restrictions of a .

Choosing some other bilinear forms a_1 and a_2 can model the situation when the problems in step 1. and 2. of the algorithm above were to be solved only approximately. An additive version of the algorithm has been described by Dryja [20] and Matsokin and Nepomnyaschikh [54].

In the discrete, multidomain case, overlapping methods often start with a non-overlapping partition of Ω into subdomains (substructures) Ω_i , $i = 1, \dots, N_s$, the diameter of the subdomains being of order H . Then, each subdomain Ω_i is extended to a larger region Ω'_i . It is assumed that neither Ω_i nor Ω'_i cuts across any of the elements. The overlaps of subdomains Ω'_i , $i = 1, \dots, N_s$ are said to be generous, if the distance between boundaries of Ω_i and Ω'_i is greater than some fixed fraction of H . The subspaces V_i , $i = 1, \dots, n$ of the abstract Schwarz method are then chosen as restrictions of the finite element space $V_h(\Omega)$ to the subregions Ω'_i . That is,

$$V_i = V_h \cap H_0^1(\Omega'_i), \quad i = 1, \dots, N_s.$$

Approximate or exact solvers then can be chosen on the subspaces and additive, multiplicative or hybrid two-level Schwarz methods can be used. As far as parallelization is concerned, additive versions clearly gain an edge, because they can be easily parallelized. Multiplicative versions always involve sequential steps. Some parallelization is possible when mutually non-overlapping subdomains Ω'_i are grouped together. This process is often called coloring of subdomains.

Without a coarse space, the condition number of such methods is of the order $1/H^2$ [24], where H is the diameter of the subdomains. Dryja and

Widlund in [24] propose to use the space of continuous piecewise linear functions on the coarse mesh defined by the subdomain Ω_i as the coarse space V_0 . They prove that, in the case of generous overlap, the condition number relevant for the conjugate gradient iteration is uniformly bounded. Mandel shows in [47] that a similar method by Cowsar [16], which uses discrete harmonic extensions determined by piecewise constant values on interfaces of subdomains, leads to a poly-logarithmic bound on the condition number.

In their later paper [21], Dryja and Widlund mention, that their numerical experiments indicate that the convergence rate is often satisfactory even for small overlaps. Running time is often smallest when the overlaps are at minimum. The number of conjugate gradient iterations is higher in such a case, because the condition number deteriorates, but this is being compensated for by the fact that the local problems are smaller and therefore cheaper to solve. Also, they are better conditioned and, if they are solved by iteration solvers, the rate of convergence is faster.

2.4 Substructuring Methods

2.4.1 Poincaré-Steklov Operators and Schur Complement

Substructuring methods borrow their name from structural engineering. It was in the context of structural engineering that several substructuring algorithms have been pioneered. We shall concentrate on substructuring methods on interfaces of subdomains.

Let us consider the model problem (2.1) and let Ω be divided into two non-overlapping subdomains Ω_1 and Ω_2 (Figure 2.4.1). Our goal is to solve the model problem only on the subdomains. Let $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ be the subdomain

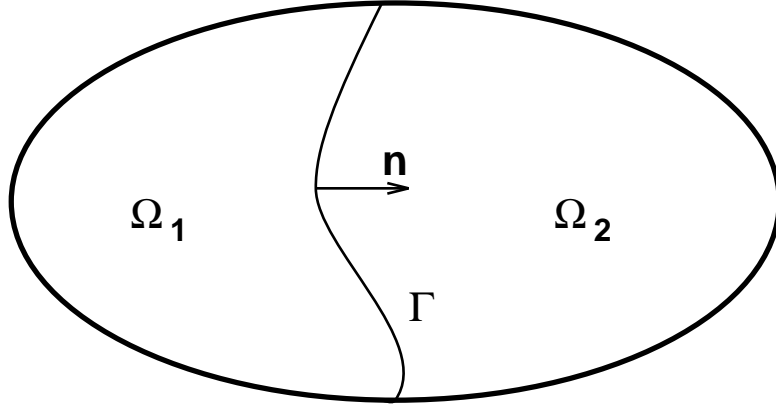


Figure 2.1. Model problem

interface. It is well known that the solution $u \in H_0^1(\Omega)$ exists for all $f \in L_2(\Omega)$. The solution satisfies continuity conditions on the subdomain interface: continuity of fluxes of the solution and continuity of the solution. Let us introduce a few notations. For $k = 1, 2$, we define the space $H_D^1(\Omega_k)$ as the subspace of $H^1(\Omega_k)$ such that the functions in $H_D^1(\Omega_k)$ vanish on $\partial\Omega_k \cap \partial\Omega$. Let $H_D^{-1}(\Omega_k)$ be the dual of $H_D^1(\Omega_k)$. Let γ_k be the trace operator mapping functions in $H_D^1(\Omega_k)$ to their traces on Γ . Let $H_{00}^{1/2}(\Gamma)$ be the fractional order Sobolev space on Γ consisting of traces of functions in $H_D^1(\Omega_k)$ and let $(H_{00}^{1/2}(\Gamma))'$ denote its dual.

Using the continuity of fluxes, we will split the problem into two subproblems, $k = 1, 2$:

$$\begin{aligned} \mathcal{A}u_k &= f_k & \text{in} & \quad \Omega_k \\ \mathbf{n} \cdot ([a_{ij}] \nabla u_k) &= (-1)^{k+1} g^* & \text{on} & \quad \Gamma \\ u_k &= 0 & \text{on} & \quad \partial\Omega \cap \partial\Omega_k. \end{aligned}$$

The vector \mathbf{n} is the normal vector to Γ oriented, for example, from Ω_1 to Ω_2 . In variational terms, these are the problems of finding $u_k \in H_D^1(\Omega_k)$, $k = 1, 2$

such that

$$a_k(u_k, v_k) = \int_{\Gamma} g^* v_k + \int_{\Omega_k} f_k v_k, \quad \forall v_k \in H_D^1(\Omega_k), \quad (2.4)$$

where

$$a_k(v, w) = \sum_{i,j=1}^d \int_{\Omega_k} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}.$$

At this point it would be possible to design an iterative method working on the subproblems using the continuity conditions. Usually, however, we first reduce the problem to a problem on the subdomain interface using Poincaré-Steklov operators.

First, we will look for the unknown Neumann data g^* on Γ . We define the Poincaré-Steklov operators $Q_k : (H_{00}^{1/2}(\Gamma))' \rightarrow H_{00}^{1/2}(\Gamma)$, $k = 1, 2$ by

$$Q_k g^* = \gamma_k u_k, \quad (2.5)$$

where, for $g^* \in (H_{00}^{1/2}(\Gamma))'$, u_k is the solution of (2.4) with $f_k = 0$. Such u_k is the harmonic function satisfying the Neumann condition given by g^* . In other words, the Poincaré-Steklov operator maps the Neumann boundary condition into the corresponding Dirichlet boundary condition

$$Q_k : \frac{\partial u_k}{\partial \mathbf{n}} \rightarrow \gamma_k u_k.$$

Furthermore, we define $R_k : H_D^{-1}(\Omega_k) \rightarrow H_{00}^{1/2}(\Gamma)$, $k = 1, 2$ by the equation

$$R_k f_k = \gamma_k u_k,$$

where for $f_k \in L_2(\Omega_k)$, u_k is the solution of (2.4) with $g^* = 0$. In terms of the Poincaré-Steklov operators, the problem is to find the solution g^* such that

$$(Q_1 + Q_2)g^* = R_2 f_2 - R_1 f_1. \quad (2.6)$$

That is to find the Neumann data g^* on Γ such that the traces of the solutions $u_k, k = 1, 2$ of (2.4) coincide on Γ .

The second possibility is to enforce continuity of the solution on the boundary a priori. This yields a dual formulation to (2.6). Consider the Dirichlet problems, $k = 1, 2$,

$$\begin{aligned} \mathcal{A}u_k &= f_k & \text{in} & \quad \Omega_k \\ u_k &= g & \text{on} & \quad \Gamma \\ u_k &= 0 & \text{on} & \quad \partial\Omega \cap \partial\Omega_k. \end{aligned}$$

We will look for the unknown Dirichlet data g on Γ such that the fluxes are continuous for the solutions $u_k, k = 1, 2$ of the problems above. Using the Poincaré-Steklov operators, this problem can be written as

$$(Q_1^{-1} + Q_2^{-1})g = Q_1^{-1}R_1f_1 + Q_2^{-1}R_2f_2. \quad (2.7)$$

The equations above and iterative methods for solving them have been studied, for example, by Agoshkov [2] and in a mixed-method setting by Glowinski and Wheller [36]. The paper by Bakhvalov and Knyazev [4] is concerned with highly discontinuous coefficients between the subdomains.

As the case of two subdomains is hardly of practical importance, we conclude this section by introducing a multi-domain discrete analog of (2.7). An analog of (2.6) is described in the next chapter.

Let us recall now the concept of Schur complement.

Definition 6 (Schur complement) Let A be an $n \times n$ matrix and $\alpha, \beta \subset \{1, \dots, n\}$ be index sets. We denote $A(\alpha, \beta)$ the submatrix that lies in the rows of A indexed by α and the columns indexed by β , and $A(\alpha', \beta')$ the submatrix given by deleting the rows indexed by α and the columns given by β .

Let $A(\alpha, \alpha)$ be nonsingular. Then the matrix

$$A(\alpha', \alpha') - A(\alpha', \alpha)A(\alpha, \alpha)^{-1}A(\alpha, \alpha')$$

is called the Schur complement of $A(\alpha, \alpha)$ in A .

We note that the Schur complement is the matrix that arises when eliminating the unknowns $x(\alpha)$ from the equation $Ax = b$. Then,

$$\begin{aligned} b(\alpha') - A(\alpha', \alpha)A(\alpha, \alpha)^{-1}b(\alpha) = \\ (A(\alpha', \alpha') - A(\alpha', \alpha)A(\alpha, \alpha)^{-1}A(\alpha, \alpha'))x(\alpha'). \end{aligned}$$

We also observe that the Schur complement S of $A(\alpha, \alpha)$ in A has the property

$$\langle Sy, y \rangle = \inf_{x \in \mathbb{R}^n, x(\alpha')=y} \langle Ax, x \rangle. \quad (2.8)$$

Let Ω be a domain in \mathbb{R}^2 or \mathbb{R}^3 decomposed into N_s non-overlapping subdomains $\Omega_1, \Omega_2, \dots, \Omega_{N_s}$. Let u_i be the vector of degrees of freedom for subdomain Ω_i corresponding to a conforming finite element discretization of the second order elliptic problem (2.1) defined on Ω , such that each subdomain is a union of some of the elements. Let u_i, K_i , and f_i , be the vector of degrees of freedom, the local stiffness matrix, and the load vectors, respectively, associated with the subdomain Ω_i .

Let L_i denote the zero-one assembly matrix mapping the subdomain degrees of freedom u_i into global degrees of freedom u , that is $u_i = L_i^T u$. The stiffness matrix then is

$$K = \sum_{i=1}^{N_s} L_i K_i L_i^T$$

and the load vector

$$f = \sum_{i=1}^{N_s} L_i f_i.$$

The problem to be solved,

$$Ku = f,$$

can be reduced to an interface problem by splitting the degrees of freedom into interface and interior degrees of freedom. Let us assume that the interior degrees of freedom are listed first. The subdomain stiffness matrices K_i and the restriction matrices L_i can be split accordingly:

$$u_i = \begin{bmatrix} \dot{u}_i \\ \bar{u}_i \end{bmatrix}, \quad K_i = \begin{bmatrix} \dot{K}_i & \tilde{K}_i^T \\ \tilde{K}_i & \bar{K}_i \end{bmatrix}, \quad L_i = [\dot{L}_i, \bar{L}_i].$$

By eliminating the interior degrees of freedom, we obtain the Schur complements of \dot{K}_i in K_i

$$S_i = \bar{K}_i - \tilde{K}_i \dot{K}_i^{-1} \tilde{K}_i^T. \quad (2.9)$$

The problem then reduces to the interface problem

$$S\bar{u} = \bar{f}$$

for the interface unknowns \bar{u} with the global Schur complement

$$S = \sum_{i=1}^{N_s} \bar{L}_i S_i \bar{L}_i^T$$

and the interface right hand side

$$\bar{f} = \sum_{i=1}^{N_s} \bar{L}_i (\bar{f}_i - \tilde{K}_i \dot{K}_i^{-1} \dot{f}_i).$$

In most cases, explicit computation of S_i would be too expensive, but for gradient-descent methods, only evaluation of the action of S_i is necessary. This evaluation can be performed efficiently by factorizing \dot{K}_i and it corresponds to solving a Dirichlet problem on every subdomain. The subdomain problems can be solved in parallel.

Let us demonstrate that the action of S_i can be evaluated by solving a Dirichlet problem on the subdomain Ω_i . Consider the discretized problem on the subdomain

$$\begin{bmatrix} \dot{K}_i & \tilde{K}_i^T \\ \tilde{K}_i & \bar{K}_i \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \bar{u}_i \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{f}_i \end{bmatrix},$$

where the Dirichlet data \bar{u}_i is known. From the first equation, we find $\dot{u}_i = \dot{K}_i^{-1} \tilde{K}_i^T \bar{u}_i$. This is the solution of the Dirichlet problem given by the data \bar{u}_i . Then, we substitute into the second equation and obtain

$$(\bar{K}_i - \tilde{K}_i \dot{K}_i^{-1} \tilde{K}_i^T) \bar{u}_i = S_i \bar{u}_i = \bar{f}_i.$$

Many preconditioners for the reduced problem are based on computations of the action of S_i^{-1} . This can be evaluated by solving a Neumann problem on the subdomain, for unknown \bar{u}_i and \dot{u}_i and given \bar{g}_i ,

$$\begin{bmatrix} \dot{K}_i & \tilde{K}_i^T \\ \tilde{K}_i & \bar{K}_i \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \bar{u}_i \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{g}_i \end{bmatrix},$$

where \bar{g}_i corresponds to the Neumann data. Then, $\bar{u}_i = S_i^{-1} \bar{g}_i$. Indeed, from the first equation, we find $\dot{u}_i = -\dot{K}_i^{-1} \tilde{K}_i^T \bar{u}_i$. Substituting into the second equation, we find

$$(\bar{K}_i - \tilde{K}_i \dot{K}_i^{-1} \tilde{K}_i^T) \bar{u}_i = S_i \bar{u}_i = \bar{g}_i.$$

We note that the Schur complement S_i is the discrete analog of the inverse of the Poincaré-Steklov operator $Q_i^{-1} : u|_\Gamma \rightarrow \frac{\partial u}{\partial \nu}$, and the inverse of the Schur complement is the analog of $Q_i : \frac{\partial u}{\partial \nu} \rightarrow u|_\Gamma$ (2.5).

One of the advantages of the reduced problem is that it improves the condition number of the original problem. For our model problem, assuming

the Poincaré-Friedrichs inequality $\|u_i^h\|_{1,2,\Omega_i}^2 \leq c_p |u_i^h|_{1,2,\Omega_i}^2$ holds, the subdomain stiffness matrix K_i satisfies

$$c \|u_i^h\|_{1,2,\Omega_i}^2 \leq u_i^T K_i u_i \leq C \|u_i^h\|_{1,2,\Omega_i}^2,$$

where u_i^h is the finite element function corresponding to the vector of degrees of freedom u_i . Then, the subdomain Schur complement S_i satisfies (cf. Lemma 26 and Lemma 33)

$$c \|\bar{u}_i^h\|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq \bar{u}_i^T S_i \bar{u}_i \leq C \|\bar{u}_i^h\|_{\frac{1}{2},2,\partial\Omega_i}^2.$$

This implies that, for a second order problem, the condition number of the reduced problem is of order $1/h$ for triangulations of characteristic mesh size h , while the condition number of the original problem is of the order $1/h^2$.

2.4.2 Various Preconditioners for the Reduced Problem

Instead of solving the reduced problem directly, iterative methods such as the preconditioned conjugate gradient method can be applied in which only matrix vector products are required. Since each iteration is quite expensive (it requires solving a Dirichlet problem on each subdomain), efficient preconditioning is important to keep the number of iterations small.

Diagonal or block diagonal preconditioning of the reduced problem requires knowing the diagonal of S . Instead of computing it directly, Chan [13] proposes a “boundary probing” technique to construct an approximate diagonal by evaluating actions of S on carefully selected vectors.

For geometrically simple subdomain in 2D, Bramble, Pasciak and Schatz in [10] propose a method for preconditioning the original problem by

first splitting the functions in the finite element space into discrete harmonics on each subdomain and functions which vanish on the boundary of each subdomain. This is equivalent to eliminating the interior degrees of freedom as in the reduced system. Then the space of discrete harmonic functions is divided into functions linear on interfaces and functions whose values are zero at subdomain crosspoints. The preconditioner is then based on a bilinear form defined on the space splitting. The condition number of the method is shown to be bounded by $C(1 + \log H/h)^2$. A similar idea is exploited in [3] for the p-version of finite elements and a bound $C(1 + \log p)^2$ is obtained.

A different algorithm [8] by Bramble, Pasciak and Schatz is described for the case of two subdomains Ω_1 and Ω_2 separated by the interface Γ . They propose to precondition the original problem in the following way: First split the local solution ϕ on Ω_2 into the component ϕ_P that satisfies the non-homogeneous equation with the zero boundary condition on $\partial\Omega_2$ and a discrete harmonic function ϕ_H that satisfies the corresponding homogeneous equation. A Dirichlet problem is solved on Ω_2 to obtain ϕ_P . This is followed by a solution of a mixed Neumann-Dirichlet problem on Ω_1 using the Neumann data obtained by solving the problem on Ω_2 . Finally another Dirichlet problem is solved to obtain ϕ_H , a discrete harmonic function, that satisfies the corresponding homogeneous equation. This is sometimes called Neumann-Dirichlet decomposition. This algorithm has also been described in [24] and it is in fact preconditioning the reduced system by either S_1^{-1} or S_2^{-1} . The inverses of the Schur complements need not be explicitly computed; the action corresponds to solving a Neumann (or mixed Neumann-Dirichlet) problem on one subdomain and then extending the solution to the other subdomain by solving a Dirichlet

problem with the Dirichlet data found from the Neumann problem. The idea of the Neumann-Dirichlet preconditioner can be extended to multiple subdomain case when there is a red-black ordering of the subdomains.

Several methods solving an equation with the global Schur complement are proposed by Mandel [47]. The methods are set in the space of all discrete harmonic functions on the union of subdomain interfaces $\Gamma = \cup_i \partial\Omega_i - \partial\Omega$. The methods are hybrid Schwarz methods using the coarse space V_0 in a multiplicative fashion and other spaces are treated additively. A method of this kind can also be interpreted as a two level variational multigrid. The methods use V_i that are associated with subdomains or globs. A glob is a vertex, an edge that does not contain its endpoints, or a face of a subdomain interface. Piecewise linear functions defined on the subdomain triangulation or a piecewise constant space based on glob-wise averaging of subdomain values is used as the coarse space. Poly-logarithmic bounds are obtained.

Many other choices of the coarse space are discussed by Dryja, Smith and Widlund [23]. Their exhaustive investigation comprises vertex based coarse spaces, based on piecewise linear functions on substructures used as elements, wire basket algorithms, that use averages on substructures, and face based algorithms.

2.4.3 The Neumann-Neumann and Balancing Domain Decomposition Methods

The balancing domain decomposition (BDD) is based on the so called Neumann-Neumann preconditioner that preconditions the reduced problem by a weighted sum of inverses of the local Schur complements. It is called

Neumann-Neumann because it corresponds to solving Neumann problems only on subdomains (as opposed to the Dirichlet-Neumann preconditioner, that uses both Dirichlet and Neumann problems). Since this is a preconditioner in a way dual to the method that is the subject of this thesis, we describe it in a little more detail. To describe the preconditioner, weight matrices $D_i, i = 1, \dots, N_s$ satisfying the decomposition of unity

$$I = \sum_{i=1}^{N_s} \bar{L}_i D_i \bar{L}_i^T$$

are used. A simple choice for D_i is a diagonal matrix with the diagonal elements being the reciprocals of the number of subdomains the degree of freedom is associated with.

The Neumann-Neumann method used as a preconditioner of the problem is as follows:

Given the residual r , distribute it to subdomains $r_i = D_i^T \bar{L}_i^T r$, solve the local problems $S_i u_i = r_i$, and average the results $M^{-1} r = \sum_{i=1}^{N_s} \bar{L}_i D_i u_i$ [7, 43].

The drawback of the Neumann-Neumann preconditioner is that it lacks a mechanism of distributing the error globally. This has been resolved by adding a coarse space to the problem. The resulting method is called the balancing domain decomposition. Let Z_i be the matrix with linearly independent columns that generate the kernel of K_i , $\text{Im } Z_i = \text{Ker } K_i$. If K_i is regular, Z_i is a void matrix. The balancing preconditioner is as follows:

1. Balance the original residual r by solving the auxiliary problems

$$Z_i^T D_i^T \bar{L}_i^T (r - S \sum_{j=1}^{N_s} \bar{L}_j D_j Z_j v_j) = 0, \quad i = 1, \dots, N_s$$

2. Distribute the balanced residual to the subdomains and find a solution of

the local problems

$$S_i u_i = D_i^T \bar{L}_i^T (r - S \sum_{j=1}^{N_s} \bar{L}_j D_j Z_j v_j), \quad i = 1, \dots, N_s$$

3. Balance by solving the auxiliary problems

$$Z_i^T D_i^T \bar{L}_i^T (r - S \sum_{j=1}^{N_s} \bar{L}_j D_j (u_j + Z_j w_j)) = 0, \quad i = 1, \dots, N_s$$

4. Average the result on the interface

$$M^{-1} r = \sum_{i=1}^{N_s} \bar{L}_i D_i (u_i + Z_i w_i)$$

The BDD is proven to be independent of number of subdomains, and its condition number is independent of jumps of coefficients between subdomains with appropriate weight matrices D_i . It can be written also as an abstract additive Schwarz algorithm [24, 41].

2.4.4 The Neumann-Neumann Domain Decomposition Algorithm for Plates and Shells

Let us introduce the Kirchhoff-Love model of plate bending following [41]. We consider a plate occupying a domain in \mathbb{R}^2 , which is clamped on the part of boundary $\partial\Omega_{00}$ and simply supported on $\partial\Omega_0 \setminus \partial\Omega_{00}$. The Kirchhoff-Love model plate model characterizes the vertical displacement $u \in V$ of the plate as the solution of the variational problem

$$a(u, v) = F(v), \quad \forall v \in V, \quad (2.10)$$

where the bilinear form on the right hand side is symmetric, continuous and coercive

$$a(u, v) = \int_{\Omega} \varepsilon(\theta(u)) : K : \varepsilon(\theta(v)),$$

the functional on the right hand side is given by

$$F(v) = \int_{\Omega} f v + \int_{\partial\Omega - \partial\Omega_{00}} m_g \frac{\partial v}{\partial \mathbf{n}} + \int_{\partial\Omega - \partial\Omega_0} g v.$$

The space of kinematically admissible fields is

$$V = \{v \in H^2(\Omega), v = 0 \text{ on } \partial\Omega_0, v = \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega_{00}\}.$$

In the definitions above f is the density of vertical forces, m_g the density of flexion moments applied on the part of the boundary where the plate is free to rotate, and g is the density of vertical boundary loading. The symbol "·" denotes a tensor product, $\varepsilon(\theta) = \frac{1}{2}(\nabla\theta + (\nabla\theta)^T)$ is the curvature tensor, $\theta(u) = \nabla u$ represents the in-plane notation of the plate and K is the plate flexural stiffness. K is symmetric, elliptic and continuous in the sense that

$$\begin{aligned} \varepsilon(\theta(u)) : K : \varepsilon(\theta(u)) &\geq ct^3 |\varepsilon(\theta(u))|^2 \\ \varepsilon(\theta(u)) : K : \varepsilon(\theta(v)) &\leq Ct^3 |\varepsilon(\theta(u))| |\varepsilon(\theta(v))|, \end{aligned}$$

where t is the plate thickness. For a simple case of an isotropic plate made of an homogeneous material with Young modulus E and Poisson coefficient ν , it is given by

$$\varepsilon(\theta(u)) : K : \varepsilon(\theta(u)) = \frac{Et^3}{12(1-\nu^2)} ((1-\nu)\nabla^2 u : \nabla^2 v + \nu\Delta u\Delta v).$$

Since V is a subspace of $H^2(\Omega)$, the appropriate finite element spaces are C^1 continuous. Examples of such elements include the discrete Kirchhoff triangle (DKT) and HCT elements.

The Neumann-Neumann and BDD preconditioner do not perform well for the plate bending problem [41]. The condition number estimate of BDD is

based on the estimate [45]

$$\kappa \leq \sup \left\{ \frac{\sum_{j=1}^{N_s} \|\bar{L}_j^T \sum_{i=1}^{N_s} \bar{L}_i D_i u_i\|_{S_j}^2}{\sum_{i=1}^{N_s} \|u_i\|_{S_i}^2} : u_i - \text{Ker } S_i, S_i u_i - \text{Im } Z_i \right\} \quad (2.11)$$

At crosspoints of subdomains, the vectors in the estimate are constructed from contributions several subdomains different from those that share an edge leading to the crosspoint. This leads to a discontinuity that is inappropriate for a fourth order problem. The BDD method for plates [42] avoids this problem arising at subdomain crosspoints by enhancing the coarse space of the balancing domain decomposition algorithm. The coarse space is again $\text{Im } Z_i$, where

$$Z_i = [x_{i1}, \dots, x_{in_i}, y_{i1}, \dots, y_{im_i}].$$

$\{x_{i1}, \dots, x_{in_i}\}$ is a basis for $\text{Ker } S_i$ and for each crosspoint $j = 1, \dots, m_i$ of the subdomain Ω_i , y_{ij} is the solution of the problem $S_i y_{ij} = e_{ij}$, with e_{ij} the vector corresponding to the unit normal load at the crosspoint j . With this choice, the normal displacement component of the vectors u_i coming out of the coarse space problem is zero since $S_i u_i - y_{ij}$ implies that $u_i - S_i y_{ij} = e_{ij}$. Then, the supremum (2.11) is taken over functions with zero at endpoints of the edges, which makes it possible to prove a poly-logarithmic bound. For another approach, imposing zeros at crosspoint directly, see [41].

2.4.5 Lagrange Multipliers and Poincaré-Steklov Operators

Following [19], we show how Lagrange multiplier approach to enforcing solution continuity is related to interface formulations using Poincaré-Steklov operators defined in Section 2.4.1. The discrete multi-domain case will be treated in the next chapter.

We consider the problem from Section 2.4.1 and the notation introduced there. We reformulate our problem as a constrained minimization problem: Find the solution $(u_1, u_2) \in H_D^1(\Omega_1) \times H_D^1(\Omega_2)$ that minimizes

$$\frac{1}{2} \sum_{k=1}^2 a_k(u_k, u_k) - \int_{\Omega_k} f v_k$$

subject to the condition $\gamma_1 u_1 = \gamma_2 u_2$. Then, for each $(u_1, u_2, g^*) \in H_D^1(\Omega_1) \times H_D^1(\Omega_2) \times (H_{00}^{1/2}(\Gamma))'$, we define the Lagrangian

$$\Lambda(u_1, u_2, \lambda^*) = \frac{1}{2} \sum_{k=1}^2 a_k(u_k, u_k) - \int_{\Omega_k} f u_k - \int_{\Gamma} \lambda^*(u_1 - u_2).$$

The Lagrange multiplier λ^* is used to enforce the continuity of the solution on the boundary. The rest of the Lagrangian is the usual quadratic functional implied by the weak form of the problem. The critical points (u_1, u_2, λ^*) of the saddle point problem of Λ now must satisfy the variational equality

$$\sum_{i=1}^2 a_k(u_k, v_k) - \int_{\Gamma} (\lambda^*(v_1 - v_2) + \mu^*(u_1 - u_2)) d\Gamma = \sum_{i=1}^2 \int_{\Omega_i} f v_k,$$

for all $(v_1, v_2, \mu^*) \in H_D^1(\Omega_1) \times H_D^1(\Omega_2) \times (H_{00}^{1/2}(\Gamma))'$. This problem has a unique solution [19] which is the solution of the problem

$$(Q_1 + Q_2)\lambda^* = R_2 f_2 - R_1 f_1$$

This is the equation (2.6). It shows that solving the Lagrange multiplier formulation is equivalent to finding Neumann interface data on Γ (cf. [18]).

The paper [19] uses the Lagrange formulation to introduce finite element spaces of Lagrangians of small dimension per interface for regular meshes. This can reduce the size of the problem substantially, but it is restricted to regular meshes. The space of Lagrangians can be chosen as the restriction of the finite elements space on the subdomains to the interface. This is similar to the approach taken by FETI as explained in the next chapter.

2.5 A Two Level Method Based on Smoothed Prolongation

The two-level method described in [68, 66] develops a simple abstract framework based on the concept of smoothed tentative prolongator introduced in [65]. The tentative prolongator is derived from a system of nonoverlapping subdomains. As opposed to the previously described methods, the union of all subdomains Ω_i , $i = 1, \dots, N_s$ does not cover whole Ω . Instead, there is a layer one element wide between each two subdomains.

Our algorithm can be written as a variational two-level multigrid with a special choice of components. We will first describe components of the method and abstract assumptions that ensure coarse space size independent convergence. Let us consider the system of linear algebraic equations

$$Ku = f,$$

where K is an $n \times n$ symmetric positive definite stiffness matrix arising from a discretization of a second order elliptic PDE, for example (2.1).

Interpolation from the coarse space to the fine space is represented by an operator MP , composed from a tentative prolongator P and prolongator smoother M . Let us denote

$$K_M = M^2 K, \quad M' = I - \frac{\omega}{\rho} K_M,$$

where $\omega \in (0, 2)$ is a given constant and $\bar{\rho}(K_M)$ is an estimated upper bound for $\rho(K_M)$, the spectral radius of K_M . The following assumption specifies requirements on M , P and $\bar{\rho}(K_M)$. are needed for proving coarse space size independent convergence.

Assumption 7 There exist positive constants C_1, C_2 independent of n and N_s and a positive constant $C_D(N_s, n)$ such that

(1) For every $u \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^{N_s}$ such that

$$\|u - Pv\| \leq C_1 C_D(N_s, n) \rho^{-1/2}(K) \|u\|_K. \quad (2.12)$$

(2) The prolongator smoother M , is symmetric, commutes with K , $\rho(M) < 1$, and

$$\rho(K_M) \leq \bar{\rho}(K_M) \leq C_2^2 C_D^{-2}(N_s, n) \rho(K). \quad (2.13)$$

We also need to define a pre-smoother \mathcal{S}_M and post-smoother $\mathcal{S}_{M'}$;

$$\mathcal{S}_M(u, f) = Mu + Nf,$$

and analogously for M' . The matrix N is chosen so the smoother is consistent with the system $Ku = f$. The smoothers \mathcal{S}_M and $\mathcal{S}_{M'}$ are relaxation operators used to smooth the approximate solution corrected by an interpolated coarse level error. The algorithm can now be written as follows.

Algorithm 8 Given an initial guess u , repeat until convergence:

- (1) $u \leftarrow \mathcal{S}_M(u, f)$,
- (2) solve $(P^T M K M P)w = P^T M(Ku - f)$,
- (3) $u \leftarrow u - MPw$,
- (4) $u \leftarrow \mathcal{S}_{M'}(u, f)$.

Post process $u \leftarrow \mathcal{S}_M(u, f)$.

The tentative prolongator P may be defined as $P = D^{1/2} \tilde{P}$, where $D = \text{diag}(K)$ and

$$\begin{aligned} \tilde{P}_{ij} &= 1, & \text{if the node corresponding to } u_i \text{ belongs to subdomain } \Omega_j, \\ &= 0, & \text{otherwise.} \end{aligned}$$

The most practical choice of M is a polynomial in K . The choice described below is a polynomial in K that attempts to minimize $\rho(K_M)$ given an upper bound on $\rho(K)$ and a chosen degree of the polynomial. For certain degrees, this polynomial can be found explicitly by the following recursive algorithm:

Algorithm 9 Let $\hat{\rho}$ be the estimate of $\rho(K)$ satisfying $\rho(K) \leq \hat{\rho} \leq C_\rho \rho(K)$, with a given constant C_ρ . Set $\hat{\rho}_i = \frac{\hat{\rho}}{9^i}$, $A_0 = K$ and for $i = 1, 2, \dots$ do:

$$(1) \quad A_i = (I - \frac{4}{3}\hat{\rho}_{i-1}^{-1}A_{i-1})^2 A_{i-1},$$

$$(2) \quad M_i = \prod_{j=0}^{i-1} (I - \frac{4}{3}\hat{\rho}_j^{-1}A_j)$$

Notice that $\deg(M_i) \leq \frac{3^i - 1}{2}$. For a $2D$ problem, we choose the prolongator smoother $M = M_k$, where

$$\deg(M_{k+1}) \geq qN_{es}^{1/2} \geq \deg(M_k), \quad (2.14)$$

where $q \in (0, 1]$ is a given parameter and N_{es} the average average number of degrees of freedom per subdomain, and M_k are the polynomials constructed by the algorithm above.

With the choices of the tentative prolongator and prolongator smoother described above, the convergence of our method can be shown to be independent of the meshsize, N_{es} , inhomogeneities between subdomains, and boundary conditions [68]. It thus overcomes one of the main disadvantages of standard two-level multigrid methods, suffering from the dependence of the rate of convergence on the size of the coarse space. On the other hand, its computational complexity is lower in comparison with domain decomposition methods using direct local solvers. This is true even for simple direct solvers as internal solvers in our method (which are preferred, as iterative solvers tend to have negative impact on the robustness of the method).

Application of the method to solid problems is treated in [64] and similar results are obtained. The method is provably robust with respect to jumps in coefficients [67].

3. Finite Element Tearing and Interconnecting (Derivation and Extensions)

3.1 Original FETI

The FETI (Finite Element Tearing and Interconnecting) method is a domain decomposition algorithm derived from a hybrid variational principle and designed for the iterative solution of systems of equations arising from the finite element discretization of self-adjoint elliptic partial differential equations. It was developed in [28, 27, 33], and also discussed in detail in monograph [34].

In this method, a given spatial domain is “torn” into *non-overlapping* subdomains where an incomplete solution of the primary field is first evaluated using a direct solver. Next, intersubdomain field continuity is enforced via Lagrange multipliers applied at the subdomain interfaces. This “gluing” phase generates a smaller size symmetric *dual* problem where the unknowns are the Lagrange multipliers, and which is best solved by a preconditioned conjugate gradient (PCG) algorithm. This idea is related to the fictitious domain method where the Lagrange multipliers enforce boundary conditions as in Dinh *et al.* [17].

In contrast with other related domain decomposition methods using Lagrange multipliers as unknowns [36, 62], the FETI method distinguishes itself with the treatment of the null spaces of the subdomain stiffness matrices (rigid

body modes) associated with the so-called floating subdomains, i.e., subdomains without a sufficient number of essential boundary conditions to prevent the local stiffness matrix from being singular. Resolving the rigid body modes leads to a small “coarse” problem that is solved in each PCG iteration. This is an added complication, but also a blessing. Farhat, Mandel, and Roux [32] have shown numerically, and proved for the FETI method without preconditioning, that the auxiliary problem plays the role of a coarse problem, namely, it causes the condition number to be bounded independently of the number of subdomains. In [26], Farhat, Chen, and Mandel extended to time-dependent problems, which lack the naturally occurring coarse problem.

The FETI method is in a sense dual to the Neumann-Neumann method with a coarse problem, developed by Mandel under the name Balancing Domain Decomposition [45] (BDD), which we described in Chapter 2 and which is based on an earlier method of de Roeck and Le Tallec [61]. A modified method was analyzed by Dryja and Widlund [25]. It should be noted that while the underlying ideas of FETI and BDD are in a way dual, FETI is not the BDD method applied to the dual problem.

3.1.1 Problem Setting and Assumptions

Let us first introduce some notation used throughout this chapter and the rest of this thesis. For $u, v \in \mathbb{R}^n$, the inner product $\langle u, v \rangle = u^T v$ serves also as duality pairing. The ℓ^2 norm is denoted $\|u\| = \langle u, u \rangle^{1/2}$. For a symmetric positive semidefinite matrix A , $\|u\|_A = \langle Au, u \rangle^{1/2}$ the induced seminorm. This is a norm if A is positive definite. The superscript $+$ denotes pseudoinverse, defined as follows.

Definition 10 (Pseudoinverse) Let A be a linear operator. The pseudoinverse A^+ is any linear operator such that if $a \in \text{Im } A$ then $AA^+a = a$.

In general, a pseudoinverse is not unique. Our algorithms will be invariant to a specific choice of the pseudoinverse. If A is a symmetric operator on a finite dimensional space, A^+ can be chosen to be also symmetric. Considering the spectral decomposition of A ,

$$A = \sum_{\sigma} \sigma v_{\sigma} v_{\sigma}^T, \quad Av_{\sigma} = \sigma v_{\sigma}, \quad v_{\sigma}^T v_{\sigma} = 1, \quad (3.1)$$

a pseudoinverse A^+ can be chosen as

$$A^+ = \sum_{\sigma \neq 0} \frac{1}{\sigma} v_{\sigma} v_{\sigma}^T.$$

For A positive semidefinite, we denote

$$A^{\alpha} = \sum_{\sigma > 0} \sigma^{\alpha} v_{\sigma} v_{\sigma}^T.$$

In particular, with this notation we have, $A = A^{1/2} A^{1/2}$ and $A^+ = A^{-1/2} A^{-1/2}$.

We point out that $\text{Ker } A^{\alpha} = \text{Ker } A$ for any real α .

We refer to Section 2.2 and the model problem (2.1) for explanation of some of the terminology used next.

Let Ω be a domain in \mathbb{R}^2 or \mathbb{R}^3 decomposed into N_s non-overlapping subdomains $\Omega_1, \Omega_2, \dots, \Omega_{N_s}$. Let u_i be the vector of degrees of freedom for subdomain Ω_i corresponding to a conforming finite element discretization of an elliptic problem (e.g. linear elasticity, plate bending) defined on Ω , such that each subdomain is a union of some of the elements. Let u_i, K_i , and f_i , be the vector of degrees of freedom, the local stiffness matrix, and the load vectors,

respectively, associated with the subdomain Ω_i . We will use the block notation

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_s} \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N_s} \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & K_{N_s} \end{bmatrix}.$$

Depending on the imposed boundary conditions and the location of the subdomain, the local stiffness matrix K_i is positive definite or positive semidefinite. A subdomain without sufficient essential boundary conditions to prevent the subdomain stiffness matrix K_i from being singular is called a *floating subdomain*. Let Z_i be the matrix with linearly independent columns that generate the kernel of K_i , $\text{Im } Z_i = \text{Ker } K_i$. If K_i is regular, Z_i is a void matrix. Denote

$$Z = \begin{bmatrix} Z_1 & 0 & \dots & 0 \\ 0 & Z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Z_{N_s} \end{bmatrix};$$

thus,

$$\text{Im } Z = \text{Ker } K, \quad \text{Ker } Z = \{0\}.$$

A single mesh point $x \in \Omega$ has several degrees of freedom associated with it if it lies on the interface (intersection of the boundaries) of two or more subdomains, see Figure 3.1. Let B be a given matrix such that $Bu = 0$ expresses the condition that for each mesh node shared by two or more subdomains the values of the degrees of freedom associated with that node coincide. We denote by W the space of all vectors of degrees of freedom, and

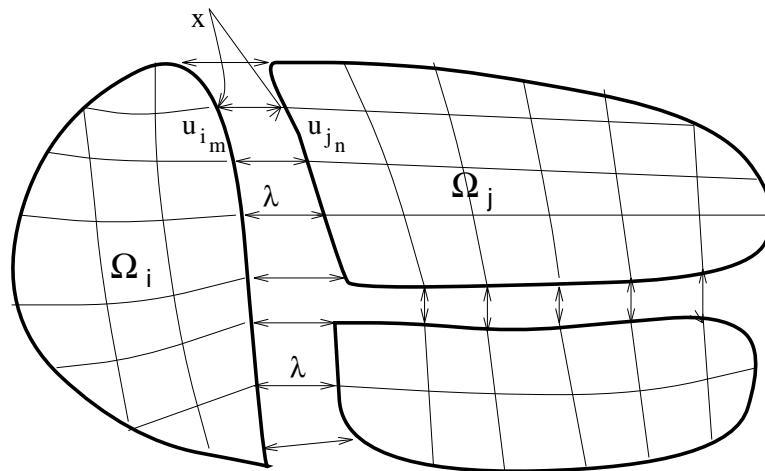


Figure 3.1. Model domain: the mesh node x shared by subdomains Ω_i and Ω_j and the degrees of freedom u_{i_m} and u_{j_n} associated with the mesh node x .

by Λ the space of the vectors of values of the continuity constraint; thus,

$$K : W \rightarrow W, \quad B : W \rightarrow \Lambda.$$

The problem to be solved is the minimization of the energy of the system subject to intersubdomain continuity conditions

$$\mathcal{E}(u) = \frac{1}{2}u^T K u - f^T u \rightarrow \min \quad \text{subject to } Bu = 0, \quad u \in W. \quad (3.2)$$

We assume that the global structure is not floating, that is, the solution of (3.2) is unique. From (3.2), this is equivalent to the assumption that

$$\text{Ker } K \cap \text{Ker } B = \{0\}. \quad (3.3)$$

The following notation will be needed to write the FETI algorithm concisely:

$$\begin{aligned} G &= BZ, \\ F &= BK^+ B^T, \\ d &= BK^+ f, \\ e &= Z^T f, \\ P &= I - G(G^T G)^{-1} G^T. \end{aligned} \quad (3.4)$$

Note that P is an ℓ^2 orthogonal projection on $(\text{Im } G)^\perp = \text{Ker } G^T$. Lemma 12 justifies the expression for P by showing that $G^T G$ is invertible.

3.1.2 Derivation of the Dual Problem

The method, as originally derived by Farhat and Roux [27], introduces Lagrange multipliers to enforce the continuity of the solution. Solving the

problem (3.2) leads to the system of equations

$$\begin{aligned} Ku + B^T \lambda &= f \\ Bu &= 0 \end{aligned} \tag{3.5}$$

A solution u of the first equation in (3.5) exists if and only if

$$f - B^T \lambda \in \text{Im } K. \tag{3.6}$$

It must then have the form

$$u = K^+(f - B^T \lambda) + Z\alpha. \tag{3.7}$$

where α is to be determined. Substituting u from (3.7) into the second equation of (3.5) yields

$$BK^+(f - B^T \lambda) + BZ\alpha = 0. \tag{3.8}$$

Now (3.8) multiplied by P together with (3.6) show that λ satisfies the system of equations

$$\begin{aligned} P(F\lambda - d) &= 0 \\ G^T \lambda &= e, \end{aligned} \tag{3.9}$$

where we have used notation (3.4).

The first of the equations (3.9) is solved by a projected preconditioned conjugate gradient method using an initial approximation λ_0 that satisfies the second equation. The conjugate gradient algorithm requires evaluating only the actions of PF . Most of the computational work in $F = BK^+B^T$ is concentrated in the action of K^+ . Since K^+ is subdomain block diagonal, its action can be computed in parallel and involves solving subdomain problems only. Application of P requires solving a small coarse problem. For a scalar problem, the size of this problem is less than the number of subdomains N_s . For a

linear elasticity problem, the size of the problem is the number of rigid body modes, which again does not exceed a small multiple of N_s . A preconditioned projected CG for the equation $PF\lambda = d$ using a symmetric preconditioner D can be written as follows.

Algorithm 11 (FETI) Given an initial $\bar{\lambda}_0$, compute the initial estimate

$$\lambda_0 = G(G^T G)^{-1}e + P\bar{\lambda}_0$$

and the initial residual

$$r_0 = P(F\lambda_0 - d).$$

Repeat for $k = 1, 2, \dots$ until convergence:

$$\begin{aligned} z_{k-1} &= Dr_{k-1} \\ y_{k-1} &= Pz_{k-1} \\ \xi_k &= r_{k-1}^T y_{k-1} \\ p_k &= y_{k-1} + \frac{\xi_k}{\xi_{k-1}} p_{k-1} \quad (p_1 = y_0) \\ \nu_k &= \frac{\xi_k}{p_k^T P F p_k} \\ \lambda_k &= \lambda_{k-1} + \nu_k p_k \\ r_k &= r_{k-1} - \nu_k P F p_k \end{aligned}$$

The choice of the preconditioner is discussed in Section 3.3.

Lagrange multipliers λ in FETI can be seen as *interface forces and moments* in the physical system. From (3.7) and the definition of F in (3.4), the residual $P(F\lambda - d) = -Bu$ has the interpretation of *jumps of the values of degrees of freedom* between subdomains. The condition $f - B^T \lambda - \text{Ker } K$

means that the action of the loads and intersubdomain forces and moments does not excite rigid body motions.

3.1.3 Saddle Point Formulation

Let us explain in more detail how enforcing the continuity using Lagrange multipliers yields the system (3.9). This approach is presented by Farhat, Mandel and Roux [32], and it will be later used to derive a generalized FETI method.

For the Lagrangian of the minimization problem (3.2),

$$\mathcal{L}(u, \lambda) = \frac{1}{2}u^T K u - f^T u + \lambda^T B u, \quad u \in W, \lambda \in \Lambda,$$

we solve the dual problem: find λ^* such that

$$\mathcal{C}(\lambda^*) = \max_{\lambda \in \Lambda} \mathcal{C}(\lambda) \equiv \max_{\lambda \in \Lambda} \inf_{u \in W} \mathcal{L}(u, \lambda). \quad (3.10)$$

By a direct computation,

$$\mathcal{C}(\lambda) = \begin{cases} -\infty & \text{if } f - B^T \lambda \notin \text{Ker } K, \\ -\frac{1}{2} \langle K^+(f - B^T \lambda), f - B^T \lambda \rangle & \text{otherwise.} \end{cases} \quad (3.11)$$

The dual problem (3.10) is thus equivalent to maximizing $\mathcal{C}(\lambda)$ on the admissible set

$$\mathcal{A} = \{\lambda \in \Lambda \mid \mathcal{C}(\lambda) > -\infty\}.$$

The space of admissible increments is

$$\{\lambda_1 - \lambda_2 \mid \lambda_1 \in \mathcal{A}, \lambda_2 \in \mathcal{A}\} = \{\mu \in \Lambda \mid B^T \mu - \text{Ker } K\} = \text{Ker } G^T. \quad (3.12)$$

At the maximum of $\mathcal{C}(\lambda)$, $\lambda \in \mathcal{A}$, the derivative of \mathcal{C} , $DC(\lambda; \mu)$, is zero in all directions in $\mu \in \text{Ker } G^T$:

$$DC(\lambda; \mu) = 0 \quad \forall \mu \in \text{Ker } G^T.$$

By a straightforward computation, this becomes

$$\langle -BK^+B^T\lambda + BK^+f, \mu \rangle = 0, \quad \forall \mu \in \text{Ker } G^T. \quad (3.13)$$

In order to express (3.13) as a linear equation in the space $\text{Ker } G^T$, we use the l_2 orthogonal projection defined by (3.4). Then for $\mu \in \text{Ker } G^T$,

$$\begin{aligned} \langle -BK^+B^T\lambda + BK^+f, \mu \rangle &= \langle -BK^+B^T\lambda + BK^+f, P\mu \rangle \\ &= \langle P(-BK^+B^T\lambda + BK^+f), \mu \rangle \end{aligned}$$

Therefore, the dual problem (3.10) is equivalent to the linear equation in $\text{Ker } G^T$ for the unknown μ ,

$$\mu \in \text{Ker } G^T, \quad P(-BK^+B^T(\mu + \lambda_0) + BK^+f) = 0, \quad (3.14)$$

where λ_0 is an arbitrary starting feasible solution, that is, $\lambda_0 \in \mathcal{A}$. Denoting $\lambda = \lambda_0 + \mu$, we obtain the system (3.9).

3.1.4 Algebraic Properties of the Original FETI Method

Lemma 12 $(G^T G)^{-1}$ exists.

Proof. Let $G\alpha = BZ\alpha = 0$. Then $Z\alpha \in \text{Ker } B$, and $Z\alpha \in \text{Ker } K$ by the definition of Z . It follows from (3.3) that $Z\alpha = 0$, hence $\alpha = 0$, since Z was assumed to be of full rank. See also [34, Theorem 5.4]. \square

Theorem 13 The solution λ of (3.9) is unique up to addition of a vector from $\text{Ker } B^T$. Any solution λ of (3.9) yields the same solution u of the minimization problem (3.2), using (3.7) with $\alpha = -(G^T G)^{-1}G^T(d - F\lambda)$.

Proof. The relation between λ and α follows by a direct computation. To prove uniqueness of λ , it is sufficient to show that

$$\text{Ker } PF \cap \text{Ker } G^T = \text{Ker } F \cap \text{Ker } G^T = \text{Ker } B^T. \quad (3.15)$$

First, $B^T \lambda = 0$ implies $F \lambda = 0$ and $G^T \lambda = 0$, so

$$\text{Ker } B^T \subset \text{Ker } F \cap \text{Ker } G^T \subset \text{Ker } PF \cap \text{Ker } G^T.$$

Conversely, assume $G^T \lambda = 0$ and $PF \lambda = 0$. Then, since $G^T = Z^T B^T$, the first equation implies $B^T \lambda - \text{Im } Z = \text{Ker } K$. Thus $B^T \lambda \in \text{Im } K$. From the definition of P , $G^T \lambda = 0$ is equivalent to $P \lambda = \lambda$. From $PF \lambda = 0$, we obtain

$$0 = \lambda^T PF \lambda = \lambda^T F \lambda = (B^T \lambda)^T K^+ B^T \lambda.$$

Since K^+ is positive semidefinite, this implies that $B^T \lambda \in \text{Ker } K^+ = \text{Ker } K$. Together, $B^T \lambda \in \text{Ker } K \cap \text{Im } K = \{0\}$. \square

Non-uniqueness of the multipliers λ corresponds to redundant inter-subdomain continuity constraints, which occur naturally at crosspoints of more than two subdomains.

Define the space of the Lagrange multipliers as the factorspace

$$\tilde{\Lambda} = \Lambda / \text{Ker } B^T.$$

Since for any $\lambda \in \text{Ker } B^T$, $P \lambda$, $F \lambda$, and $G^T \lambda$ are the zero vectors, the operators P , F , and G^T induce operators on $\tilde{\Lambda}$, which will be denoted by the same symbols. To avoid confusion, all null spaces will refer to the space Λ , not the factorspace $\tilde{\Lambda}$. For example, the null space of the induced operator G^T on $\tilde{\Lambda}$ will be denoted by $\text{Ker } G^T / \text{Ker } B^T$.

It is easy to see that F is symmetric and positive semi-definite and hence so is PF . The next lemma shows that the operator PF restricted to $\text{Ker } G^T / \text{Ker } B^T$, which coincides with the restriction of PF , is positive definite.

Lemma 14 The operator PF is symmetric and positive definite on the factorspace $\text{Ker } G^T / \text{Ker } B^T$.

Proof. For $u, v \in \text{Ker } G^T$, we have from the definition of P ,

$$\langle PFPu, v \rangle = \langle PFu, v \rangle = \langle Fu, v \rangle,$$

which proves that PF is symmetric positive semidefinite on $\text{Ker } G^T$. To prove that PF is nonsingular on $\text{Ker } G^T / \text{Ker } B^T$, let $PFu = 0$ and $G^T u = 0$. Since $F = BKB^T$ and $G^T = Z^T B^T$ by definition, it follows that $B^T u \in \text{Ker } K$ and $B^T u \in \text{Ker } K$, hence $B^T u = 0$. \square

The original FETI method is therefore the method of preconditioned conjugate gradients in the factorspace $\tilde{\Lambda}$ for the operator equation, equivalent to (3.9),

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + \text{Ker } G^T, \quad (3.16)$$

where λ_0 is an initial approximation to the conjugate gradient method chosen so that $G^T \lambda_0 = e$, and all search directions are in the space $\text{Ker } G^T \subset \Lambda$.

3.2 Generalized FETI

There have been several extensions to the original method. Extension to time-dependent problems was done in [26]. Generalization to plate-bending problems is discussed in [51, 53] and, in detail, in this thesis.

For the original method for plate bending problems, the condition number was observed to grow fast with the number of elements per subdomain [32]. This is caused by the fact that plate bending is a fourth order problem, while the FETI domain decomposition method “tears” the approximate solution at subdomain crosspoints, which is suitable only for second order problems.

The limitation of method can be cured to extend the FETI methodology to obtain a non-overlapping domain decomposition method for plate bending problems. This new method has the properties one usually looks for in iterative substructuring methods: the condition number can be bounded independently of the number of subdomains, and it grows only poly-logarithmically with the number of elements per subdomain. The computational cost per iteration is proportional to the solution of a boundary value problem in each subdomain, plus the solution of a sparse *coarse problem* with only few variables per subdomain. Such methods are commonly referred to as *scalable* and *quasi-optimal*, though, of course, for very large number of subdomains, the solution of the coarse problem would dominate.

The key idea of our method is to enforce the continuity of the approximate solution at the subdomain crosspoints throughout the iterations by adding the corresponding Lagrange multipliers to the coarse problem. A similar idea was employed in the Balancing Domain Decomposition (BDD) method for plates [42], where approximate continuity at crosspoints is enforced by adding new basis functions to the original coarse space [46, 49] in order to keep the energy of the approximate solution minimal with respect to displacements that are solutions for point loads at the subdomain crosspoints (cf. Section 2.4.4). The distinguishing features of both the present method and the method from [42] is that they are *non-overlapping* and work for standard finite elements used in everyday engineering practice.

For other domain decomposition methods for the biharmonic equation and plate bending see, for example, [12, 71]. Extensions to shells, implementation issues, and further computational can be found in [29, 31].

3.2.1 Duality Derivation of the Generalized FETI

In this section, we present a derivation and formulation [52] of the generalized FETI, based on the concept of coarse optimality of a dual objective function.

Preserving the condition $G^T \lambda_k = e$ throughout the iterations of the Algorithm 11 can be interpreted as enforcing that every λ_k be optimal with respect to all possible increments of the form $G\alpha$:

$$G^T \lambda_k = e \iff \mathcal{C}(\lambda_k) \geq \mathcal{C}(\lambda_k - G\alpha), \quad \forall \alpha, \quad (3.17)$$

where \mathcal{C} is defined by (3.11).

The key to the generalization of the FETI method to plate bending problems is to make all λ_k optimal in more directions. Let C be some given matrix with the same number of rows as G . Each column of C will give rise to an additional variable in the coarse problem. We shall satisfy in each iteration the *coarse optimality property*

$$\mathcal{C}(\lambda) \geq \mathcal{C}(\lambda - G\alpha - C\beta), \quad \forall \alpha, \beta. \quad (3.18)$$

with $\lambda = \lambda_k$. To satisfy this property, consider an auxiliary problem: For a given $\tilde{\lambda}$ find α and β so that

$$\mathcal{C}(\lambda) \rightarrow \max, \quad \lambda = \tilde{\lambda} - G\alpha - C\beta \quad (3.19)$$

Since we only need solutions satisfying $\mathcal{C}(\lambda) > -\infty$, we consider the maximization problem (3.19) along with the constraint

$$G^T \lambda = e \quad (3.20)$$

Introducing new Lagrange multipliers μ for (3.20), we get that α , β , and μ solve the saddle point problem

$$\inf_{\alpha, \beta} \sup_{\mu} \tilde{\mathcal{L}}(\tilde{\lambda} - G\alpha - C\beta, \mu) = \sup_{\mu} \inf_{\alpha, \beta} \tilde{\mathcal{L}}(\tilde{\lambda} - G\alpha - C\beta, \mu)$$

where

$$\tilde{\mathcal{L}}(\lambda, \mu) = -\frac{1}{2}\lambda^T F \lambda + \lambda^T d + \mu^T (G^T \lambda - e)$$

From the optimality conditions

$$\frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \alpha} = 0, \quad \frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \beta} = 0, \quad \frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \mu} = 0, \quad \lambda = \tilde{\lambda} - G\alpha - C\beta,$$

we obtain that (3.19) is equivalent to the block linear system

$$M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \tilde{\lambda} - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \quad (3.21)$$

where

$$M = \begin{bmatrix} G^T F G & G^T F C & G^T G \\ C^T F G & C^T F C & C^T G \\ G^T G & G^T C & 0 \end{bmatrix} = \begin{bmatrix} G^T & 0 \\ C^T & 0 \\ 0 & G^T \end{bmatrix} \begin{bmatrix} F & I \\ I & 0 \end{bmatrix} \begin{bmatrix} G & C & 0 \\ 0 & 0 & G \end{bmatrix}. \quad (3.22)$$

The solution λ of (3.19) is unique up to the addition of a vector in $\text{Ker } F \cap \text{Ker } G^T$, and we write it as

$$\lambda = \tilde{\lambda} - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \left(\begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \tilde{\lambda} - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \right). \quad (3.23)$$

Note that since (3.19) has a solution for any $\tilde{\lambda}$, so does (3.21), hence

$$\text{Im} \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \subset \text{Im } M. \quad (3.24)$$

Further, if λ is coarse optimal, then $\lambda + \delta$ is also coarse optimal if and only if

$$\begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \delta \in \text{Ker } F \cap \text{Ker } G^T. \quad (3.25)$$

The generalized FETI algorithm is thus obtained from Algorithm 11 by prescribing the initial approximation λ_0 by

$$\lambda_0 = \bar{\lambda}_0 - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \left(\begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \bar{\lambda}_0 - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \right)$$

and by replacing the step $y_{k-1} = Pz_{k-1}$ by

$$y_{k-1} = z_{k-1} - [G, C, 0] M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} z_{k-1},$$

cf. Algorithm 20 in the next section.

3.2.2 Derivation of Generalized FETI Algorithm without Duality

The generalized FETI method can be also obtained by forcing the iterates to satisfy also a weighted residual condition [53]. That is, we require throughout the iterations that

$$C^T P(F\lambda - d) = 0, \quad G^T \lambda = e, \quad (3.26)$$

where C is some given matrix. Search directions that preserve (3.26) form the space

$$V' = \{\lambda \in \Lambda \mid G^T \lambda = 0, C^T P F \lambda = 0\}. \quad (3.27)$$

The corresponding space of residuals is

$$V = \{u \in \text{Im } B \mid G^T u = 0, C^T u = 0\}. \quad (3.28)$$

Denote the associated subspaces of the factorspace $\tilde{\Lambda}$ as

$$\tilde{V}' = \{u + \text{Ker } B^T \mid u \in V'\} = V' / \text{Ker } B^T, \quad \tilde{V} = \{u + \text{Ker } B^T \mid u \in V\}.$$

Note that V and \tilde{V} are isomorphic: since $V \subset \text{Im } B$ and $\text{Im } B \cap \text{Ker } B^T = \{0\}$, each class of \tilde{V} contains exactly one element of V .

We will need several properties of the spaces \tilde{V} and \tilde{V}' .

Lemma 15

$$\tilde{V} = P F \tilde{V}'.$$

Proof. From the definition, clearly $P F V' \subset V$, hence $P F \tilde{V}' \subset \tilde{V}$. To show that $\tilde{V} \subset P F \tilde{V}'$, let $u + \text{Ker } B^T \in \tilde{V}$. Since $P F$ is a bijection on $\text{Ker } G^T / \text{Ker } B^T$ by Lemma 14, there is $\tilde{\lambda} \in \text{Ker } G^T / \text{Ker } B^T$ such that $P F \tilde{\lambda} = u + \text{Ker } B^T$. It follows that $\tilde{\lambda} \in V'$. \square

Lemma 16 The space \tilde{V}' is the dual of \tilde{V} with the duality pairing $\langle \cdot, \cdot \rangle$.

Proof. Any $\lambda \in \tilde{V}'$ defines a linear functional λ' on \tilde{V} by $\lambda'(v) = \langle \lambda, v \rangle$. Let λ' be an arbitrary linear functional on \tilde{V} . From Lemma 15, it follows that $\lambda' P F$ is a linear functional on \tilde{V}' , and, from Riesz representation theorem, there is a unique $\lambda \in \tilde{V}'$ such that $\lambda'(P F v) = \langle \lambda, v \rangle$ for all $v \in \tilde{V}'$. From Lemma 14, $P F$ is a bijection on \tilde{V} , and it follows that the mapping between a linear functional λ' and its representation $\lambda \in \tilde{V}'$ is an isomorphism. \square

Lemma 17

$$\tilde{\Lambda} = \tilde{V}' \oplus \tilde{V}^-.$$

Proof. From Lemma 14, PF is symmetric positive definite on $\text{Ker } G^T / \text{Ker } B^T$. Hence, $(PF)^{-1}$ is also symmetric, positive definite on $\text{Ker } G^T / \text{Ker } B^T$, and, using Lemma 15, it follows that

$$\tilde{\Lambda} = \tilde{V}' \oplus (\tilde{V}')^{-(PF)^{-1}} = \tilde{V}' \oplus ((PF)\tilde{V})^{-(PF)^{-1}} = \tilde{V}' \oplus \tilde{V}^-, \quad (3.29)$$

which was to be proved. \square

We now define a projection operator Q by

$$Q : \tilde{\Lambda} \rightarrow \tilde{\Lambda}, \quad Q^2 = Q, \quad \text{Im } Q = \tilde{V}', \quad \text{Ker } Q = \tilde{V}^-, \quad (3.30)$$

and compute the matrix representation of Q . This representation defines also an operator on Λ , which will be denoted by the same symbol Q .

Lemma 18 The projector Q is given by the formula

$$Q = I - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix},$$

where M is defined by (3.22).

Proof. Let $\lambda, \bar{\lambda} \in \Lambda$, $Q(\bar{\lambda} + \text{Ker } B^T) = \lambda + \text{Ker } B^T$. Since $\lambda \in V'$, from the definition of P , there exists μ such that $PF\lambda = F\lambda + G\mu$. From the definition of V' , $G^T\lambda = 0$ and $C^T PF\lambda = 0$, hence

$$\begin{aligned} G^T F\lambda + G^T G\mu &= 0, \\ C^T F\lambda + C^T G\mu &= 0, \\ G^T \lambda &= 0. \end{aligned} \quad (3.31)$$

From the definition of Q , we have $\lambda - \bar{\lambda} - V$, so for any $u \in \text{Im } B$, $G^T u = 0$ and $C^T u = 0$ implies $\langle \lambda - \bar{\lambda}, u \rangle = 0$. Consequently, there exist α and β such that for all $u \in \text{Im } B$,

$$\langle \lambda - \bar{\lambda}, u \rangle = \langle G\alpha, u \rangle + \langle C\beta, u \rangle,$$

which implies that

$$\lambda = \bar{\lambda} + G\alpha + C\beta + \gamma, \quad \gamma \in (\text{Im } B)^\perp = \text{Ker } B^T \quad (3.32)$$

Since $\lambda \in V'$, substituting (3.32) into (3.31) gives that α, β, μ satisfy the linear system

$$\begin{aligned} G^T F(\bar{\lambda} + G\alpha + C\beta) + G^T G\mu &= 0 \\ C^T F(\bar{\lambda} + G\alpha + C\beta) + C^T G\mu &= 0 \\ G^T(\bar{\lambda} + G\alpha + C\beta) &= 0. \end{aligned} \quad (3.33)$$

On the other hand, it is easy to see from (3.31) and the definitions of Q and V that if α, β, μ satisfy (3.33), then $\lambda = \bar{\lambda} + G\alpha + C\beta \in Q\bar{\lambda} + \text{Ker } B^T$. \square

Similarly as in the case of the original FETI, we need to find an initial approximation λ_0 satisfying certain conditions, in this case (3.26).

Lemma 19 For any $\bar{\lambda}_0 \in \Lambda$, the system of equations

$$\begin{aligned} G^T F(\bar{\lambda}_0 + G\alpha + C\beta) + G^T G\mu &= G^T d \\ C^T F(\bar{\lambda}_0 + G\alpha + C\beta) + C^T G\mu &= C^T d \\ G^T(\bar{\lambda}_0 + G\alpha + C\beta) &= e \end{aligned} \quad (3.34)$$

has a solution α, β, μ , and

$$\lambda_0 = \bar{\lambda}_0 + G\alpha + C\beta \quad (3.35)$$

satisfies

$$C^T P(F\lambda_0 - d) = 0, \quad G^T \lambda_0 = e. \quad (3.36)$$

Proof. As in the preceding proof, λ_0 satisfies (3.36) if and only if there is a μ such that

$$\begin{aligned} G^T F \lambda_0 + G^T G \mu &= G^T d \\ C^T F \lambda_0 + C^T G \mu &= C^T d \\ G^T \lambda_0 &= e \end{aligned} \tag{3.37}$$

Substituting λ_0 from (3.35) into (3.37) yields the system (3.34), which can be written as

$$M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = X^T \begin{bmatrix} d \\ G(G^T G)^{-1} e \end{bmatrix}, \quad X = \begin{bmatrix} G & C & 0 \\ 0 & 0 & G \end{bmatrix}.$$

From the factorization (3.22), it follows that $\text{Ker } M = \text{Ker } X$. Using symmetry of M , we have $\text{Im } M = (\text{Ker } M)^- = (\text{Ker } X)^- = \text{Im } X^T$; hence, (3.34) has a solution. \square

The generalized FETI method is the conjugate gradients method for the operator $PF : V' \rightarrow V$, preconditioned by $QDQ^T : V \rightarrow V'$, where the $D : \Lambda \rightarrow \Lambda$ is a given operator symmetric on V . Since Q^T is also a projection and $\text{Im } Q^T = (\text{Ker } Q)^- = V$, the application of Q^T on V can be omitted, and one obtains the following algorithm similar to the algorithm of the original FETI (Algorithm 11).

Algorithm 20 (Generalized FETI) Given an initial $\bar{\lambda}_0$, compute the initial λ_0 from (3.34) and (3.35), and compute the initial residual

$$r_0 = P(F\lambda_0 - d).$$

Repeat for $k = 1, 2, \dots$ until convergence:

$$z_{k-1} = Dr_{k-1}$$

$$\begin{aligned}
y_{k-1} &= Qz_{k-1} \\
\xi_k &= r_{k-1}^T y_{k-1} \\
p_k &= y_{k-1} + \frac{\xi_k}{\xi_{k-1}} p_{k-1} \quad (p_1 = y_0) \\
\nu_k &= \frac{\xi_k}{p_k^T P F p_k} \\
\lambda_k &= \lambda_{k-1} + \nu_k p_k \\
r_k &= r_{k-1} - \nu_k P F p_k
\end{aligned}$$

The choice of the preconditioner is discussed in Section 3.3.

3.2.3 Method Selection for Plate Problems and Other Generalizations

We choose the columns of matrix C , which appears in the description of the generalized FETI method above, as vectors with a one at the position of a Lagrange multiplier that enforces the continuity of the transversal displacement at a crosspoint, and zeros everywhere else. By a crosspoint we understand an interface node adjacent to at least three subdomains or to two subdomains and the complement of Ω . For a more precise formulation, see Section 4.5.1.

A similar idea can be exploited for shell problems. The continuity of the normal displacement needs to be enforced. To avoid finding normals and the added complexity of enforcing continuity of the normal displacement, one may enforce continuity of all displacement degrees of freedom. This, however, increases the size of the coarse space. Other possibilities and computational issues are discussed in [30].

Farhat et al. in [26] studies the case of time-dependent problems where the subdomain stiffness matrices K_i are perturbed by the addition of

a multiple of the subdomain mass matrix, thus making the new local matrix positive definite. Consequently, all matrices Z_i are void and the natural coarse problem is lost in time-dependent applications. The methodology developed in [26] for reintroducing a coarsening operator in the FETI algorithm for dynamics problems is a special case of the present generalization where C is taken to be the matrix G before the perturbation, that is, $C = [B_i \tilde{Z}_i]$ where the columns of \tilde{Z}_i are the basis of the kernel of the local stiffness matrix of the subdomain Ω_i . The reason why the preconditioner works for the dynamics problems is quite different from that for the plate bending.

3.3 The Dirichlet Preconditioner

Let us decompose the space of all degrees of freedom W into the space of internal degrees of freedom and the degrees of freedom on subdomain interfaces,

$$W = \dot{W} \times \bar{W}.$$

In the corresponding block notation,

$$B = [0 \ \bar{B}], \quad \bar{B} : \bar{W} \rightarrow \Lambda,$$

since B has nonzero entries for interface degrees of freedom only. Also,

$$Z = \begin{bmatrix} \dot{Z} \\ \bar{Z} \end{bmatrix},$$

and we have

$$G = BZ = \bar{B}\bar{Z}, \quad \text{Ker } B^T = \text{Ker } \bar{B}^T.$$

Let S be the Schur complement of K obtained by elimination of degrees of freedom internal to all subdomains. Then

$$F = BK^+B^T = \bar{B}S^+\bar{B}^T \quad (3.38)$$

and $\text{Ker } S = \text{Im } \bar{Z}$. Evaluation of the matrix-vector product S^+u reduces to the solution of independent *Neumann problems* on all subdomains, cf. Section 2.4.1. Inspired by (3.38), we choose $D = \bar{B}S\bar{B}^T$, giving the preconditioner

$$QD = Q\bar{B}S\bar{B}^T. \quad (3.39)$$

This preconditioner is called the *Dirichlet preconditioner*, since evaluating the matrix-vector product Sr is equivalent to solving independent *Dirichlet problems* on all subdomains, cf. Section 2.4.1.

3.4 Interface Formulation of Park et al.

In [59, 58], K.C. Park has developed a similar substructuring method. We will show that this method can essentially be written as FETI with a special choice of the interface continuity operator B .

The method augments the Lagrangian by introducing another variable u_g , the global vector of degrees of freedom, on the interfaces between subdomains. This variable is redundant and is later eliminated, yielding an analog of FETI. We will show that the method can be written as FETI and discuss the resulting choice of B and its implications.

Let L be the subdomain assembly matrix, that is a zero-one matrix mapping the local subdomain degrees of freedom to global ones. Using this matrix, the continuity of the solution is enforced through the constraint

$$u = Lu_g.$$

The Lagrangian can be written as

$$\mathcal{L}(u, u_g, \lambda) = \frac{1}{2}u^T K u - f^T u + \lambda^T (u - L u_g).$$

Decomposing $u = Z\alpha + \beta$ into a rigid body mode component and its complement (pure deformation component), we obtain

$$\mathcal{L}(\alpha, \beta, u_g, \lambda) = \frac{1}{2}\beta^T K \beta - f^T \beta - f^T Z\alpha + \lambda^T (Z\alpha + \beta - L u_g).$$

The stationarity condition leads to the system of equations

$$\begin{bmatrix} K & I & 0 & 0 \\ I & 0 & Z & -L \\ 0 & Z^T & 0 & 0 \\ 0 & -L & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \lambda \\ \alpha \\ u_g \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ -Zf \\ 0 \end{bmatrix}$$

Eliminating the deformation component, $\beta = K^+(f - \lambda)$, the system becomes

$$\begin{bmatrix} K^+ & Z & -L \\ Z^T & 0 & 0 \\ -L^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha \\ u_g \end{bmatrix} = \begin{bmatrix} K^+ f \\ -Zf \\ 0 \end{bmatrix}$$

The first equation of the system is

$$K^+ \lambda + Z\alpha - L u_g = K^+ f. \quad (3.40)$$

We first eliminate the term $Z\alpha$ by multiplying (3.40) by the orthogonal projection P_Z onto $(\text{Im } Z)^\perp = \text{Ker } Z^T$, $P_Z = I - Z(Z^T Z)^{-1} Z^T$, and obtain the equation

$$P_Z K^+ \lambda - P_Z L u_g = P_Z K^+ f, \quad (3.41)$$

equivalent to (3.40) holding for some α . Next, using the orthogonal projection P_l onto $(\text{Im } P_Z L)^\perp = \text{Ker } (P_Z L)^T$,

$$P_l = I - P_Z L (L^T P_Z L)^{-1} L^T P_Z,$$

in a similar way,

$$P_l P_Z K^+ \lambda = P_l P_Z K^+ f.$$

This is the equation (15) in [58].

We may reverse the order of elimination of the terms, and we obtain from (3.40)

$$P_z P_L K^+ \lambda = P_z P_L K^+ f, \quad (3.42)$$

where the orthogonal projections P_L and P_z are given by

$$\begin{aligned} P_L &= I - L(L^T L)^{-1} L^T \\ P_z &= I - P_L Z (Z^T P_L Z)^{-1} Z^T P_L. \end{aligned}$$

Since $\text{Im } P_z \subset \text{Im } P_L = \text{Ker } L^T$, we have $\nu = P_L \nu$ for $\nu \in \text{Im } (P_z)$. Thus,

$$P_L K^+ \nu = P_L K^+ P_L \nu.$$

Now choosing

$$B = P_L, \quad (3.43)$$

we may write

$$P_z = I - P_L Z (Z^T P_L Z)^{-1} Z^T P_L = I - G(G^T G)^{-1} G^T = P,$$

where we have used the notation (3.4). This shows that the method with the equation (3.42) can be written as FETI with the special choice of B (3.43).

Let us illustrate this choice. If a mesh node is on the interface of two subdomains, then, omitting all other degrees of freedom and Lagrange multipliers, the corresponding block of the assembly matrix L is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the corresponding block of the projection P_L is

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Similarly for a three node and four node interfaces, we get

$$\frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and

$$\frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix},$$

respectively. The choice of $B = P_L$ has some interesting implications on the analysis, which will be discussed in Section 4.4.

4. Convergence Analysis

In this chapter, we prove that for a second order problem the condition number of the preconditioned FETI method is bounded independently of the number of subdomains and poly-logarithmically in terms of subdomain size, similarly as it is in the case for other optimal non-overlapping domain decomposition methods [11, 23, 25, 49, 48]. We prove that the choice of the interface continuity operator as given by Park et al. assures that the condition number is bounded by $C(1 + \log \frac{H}{h})^2$. Finally we also show that the generalized FETI converges poly-logarithmically for the plate bending problem.

Analysis of domain decomposition methods typically demonstrates spectral equivalence of the quadratic form that defines the problem in a variational setting and the quadratic form that defines the preconditioner, often by way of P.L. Lions lemma [6, 23, 24, 44]. Since the preconditioner in the FETI method is quite complicated and is not defined in terms of a quadratic form, we proceed differently. We find a bound on the norm of the product of the system operator and the preconditioner, so as to bound the maximal eigenvalue, and a bound on the inverse, to bound the minimal eigenvalue. Related analyses were previously done for methods without crosspoints between the subdomains, or done formally in functional spaces, cf., for example, Glowinski and Wheeler [37] and Bakhvalov and Knyazev [4]. We present a complete analysis in terms of upper and lower bound on the preconditioned operator for decompositions with crosspoints in 2D and edges and crosspoints in 3D for

second order elliptic problems. We show that the condition number is bounded by $C(1 + \log \frac{H}{h})^\gamma$, where $\gamma = 2, 3$, h is the characteristic mesh size, and H the diameter of the subdomains.

4.1 Preliminaries

In this section, we present some results that will be used in analysis of FETI methods. The results are mostly concerned with estimates for finite element functions used to discretize our model problems.

We will assume that a domain Ω is divided into a set of nonoverlapping subdomains $\Omega_i, i = 1, \dots, N_s, \bar{\Omega} = \bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_{N_s}$. The subdomains are assumed to be shape regular of diameter H according to the definition below. We will formulate all results for one of the subdomains Ω_i and assume that constants in the estimates do not depend on the index of the subdomain.

Definition 21 A subdomain $\Omega_i \subset \mathbb{R}^d$ is said to be shape-regular of diameter $O(H)$ if it can be generated from a reference domain (square or cube) $\hat{\Omega}$ of unit diameter by a mapping \mathcal{F}_i such that $\Omega_i = \mathcal{F}_i(\hat{\Omega})$. The mapping is assumed to satisfy

$$\|\partial\mathcal{F}_i\| \leq c_s H, \quad \|\partial\mathcal{F}_i^{-1}\| \leq c_s H^{-1} \quad (4.1)$$

where $\partial\mathcal{F}_i$ is the Jacobian of the mapping, $\|\cdot\|$ is the Euclidean \mathbb{R}^d matrix norm, and c_s is a positive constant.

We will be using the Sobolev spaces $W_{k,p}$. For $p = 2$, the spaces are Hilbert spaces and we denote them $H^k \equiv W_{k,2}$. Definitions of Sobolev spaces can be found, for example, in [1]. For $k \neq 1/2, 1$, we use $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ to denote the standard Sobolev norms and seminorms of functions in $W_{k,p}$. Following [22], we define the scaled Sobolev norm for a scalar function

$u \in H^1(\Omega_i)$

$$\|u\|_{1,2,\Omega_i}^2 = |u|_{1,2,\Omega_i}^2 + \frac{1}{H^2} \|u\|_{0,2,\Omega_i}^2,$$

where the Sobolev seminorm is defined by

$$|u|_{1,2,\Omega_i}^2 = \int_{\Omega_i} \|\nabla u(x)\|^2 dx.$$

We define the scaled Sobolev norm for a scalar function $u \in H^{1/2}(\partial\Omega_i)$

$$\|u\|_{1/2,2,\partial\Omega_i}^2 = |u|_{1/2,2,\partial\Omega_i}^2 + \frac{1}{H} \|u\|_{0,2,\partial\Omega_i}^2,$$

where the Sobolev seminorm is defined by

$$|u|_{1/2,2,\partial\Omega_i}^2 = \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|u(x) - u(y)|^2}{\|x - y\|^d} dx dy.$$

Here $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d . We note that the space $H^{1/2}(\partial\Omega_i)$ is the space of traces of functions in the space $H^1(\Omega_i)$.

If $u = (u_1, u_2)$ is a vector function, then we define, for example,

$$\|u\|_{1,2,\Omega_i}^2 = \|u_1\|_{1,2,\Omega_i}^2 + \|u_2\|_{1,2,\Omega_i}^2.$$

Other norms and seminorms of vector functions are defined analogously.

We present two variants of the Poincaré-Friedrichs inequality. We prove the first for the sake of completeness and to demonstrate the technique that is used to prove its variants for finite element functions. We note that $P_k, k = 0, 1, \dots$ denotes the space of polynomials of degree at most k .

Lemma 22 Let \mathcal{G} be a continuous linear functional on $H^{1/2}(\partial\Omega_i)$ such that for all $u \in P_0$, $\mathcal{G}(u) = 0$ implies $u = 0$. Then there exists a constant c independent of H such that for all $u \in H^{1/2}(\partial\Omega_i), u \in \text{Ker } \mathcal{G}$

$$\|u\|_{\frac{1}{2},2,\partial\Omega_i} \leq c |u|_{\frac{1}{2},2,\partial\Omega_i}.$$

Proof. We prove the theorem for a reference domain of diameter 1 by contradiction. The result in the scaled norms then follows.

If the inequality is not true, there exists a sequence $\{u_n\} \subset \text{Ker } \mathcal{G}$ such that

$$\|u_n\|_{\frac{1}{2},2,\partial\hat{\Omega}} = 1 \quad \text{and} \quad |u_n|_{\frac{1}{2},2,\partial\hat{\Omega}} \rightarrow 0.$$

Due to the compact imbedding of $H^{1/2}(\partial\hat{\Omega})$ in $L_2(\partial\hat{\Omega})$, there exists a subsequence of $\{u_n\}$ that converges in $L_2(\partial\hat{\Omega})$. Since $|u_n|_{\frac{1}{2},2,\partial\hat{\Omega}} \rightarrow 0$, the subsequence $\{u_{n_k}\}$ is Cauchy in $H^{1/2}(\partial\hat{\Omega})$. Therefore it converges in $H^{1/2}(\partial\hat{\Omega})$ to some $u \in H^{1/2}(\partial\hat{\Omega})$. The continuity of the norm and the seminorm on $H^{1/2}(\partial\hat{\Omega})$ implies

$$\|u\|_{\frac{1}{2},2,\partial\hat{\Omega}} = 1 \quad \text{and} \quad |u|_{\frac{1}{2},2,\partial\hat{\Omega}} = 0. \quad (4.2)$$

Therefore, by the second equation in (4.2), $u = k$ almost everywhere, where k is some real number. Let us assume without loss of generality that $u = k$ everywhere; that is $u \in P_0$. Since \mathcal{G} is continuous $\mathcal{G}(u_{n_k}) \rightarrow \mathcal{G}(u) = 0$. This yields, by the assumption, $u = 0$ which contradicts the first equation in (4.2).

□

The proof of this Poincaré-Friedrichs inequality in scaled norms is similar to the proof of the previous lemma.

Lemma 23 Let \mathcal{G} be a continuous linear functional on $H^1(\Omega_i)$ such that for all $u \in P_0$, $\mathcal{G}(u) = 0$ implies $u = 0$. Then there exists a constant c independent of H such that for all $u \in H^1(\Omega_i)$, $u \in \text{Ker } \mathcal{G}$

$$\|u\|_{1,2,\Omega_i} \leq c|u|_{1,2,\Omega_i}.$$

4.1.1 Estimates for P1/Q1 Elements

Let $V_h^{P1}(\Omega_i)$ be a conforming finite element space of P1 or Q1 elements [14] satisfying the usual regularity and inverse properties and possibly some essential boundary conditions. That is, for example in \mathbb{R}^2 , we assume that

$$\Omega_i = \cup_{\mathcal{K} \in \mathcal{T}_{h,i}} \mathcal{K},$$

where each element \mathcal{K} of the triangulation $\mathcal{T}_{h,i}$ is a triangle or a rectangle. Furthermore, for all $\mathcal{K} \in \mathcal{T}_{h,i}$

$$c_1 d(\mathcal{K}) \leq h \leq c_2 \rho(\mathcal{K}), \tag{4.3}$$

where $d(\mathcal{K})$ is the diameter of \mathcal{K} , and $\rho(\mathcal{K})$ the diameter of the circle inscribed in \mathcal{K} . Then h is called the characteristic mesh size. A vertex of an element $\mathcal{K} \in \mathcal{T}_{h,i}$ will be referred to as a mesh point, a nodal point, or just a vertex. If essential boundary conditions are prescribed on $\Sigma \subset \partial\Omega_i$, we assume that $\mu(\Sigma) \geq c_c \mu(\partial\Omega_i)$, where $\mu(\cdot)$ denotes the measure. We note that functions in $V_h^{P1}(\Omega_i)$ are continuous.

As in the previous chapter, the corresponding space of vectors of degrees of freedom is denoted W_i . Let $I_{P1} : W_i \rightarrow V_h^{P1}(\Omega_i)$ denote the linear one-to-one transformation that maps a vector of degrees of freedom to the corresponding finite element function. This transformation is often called finite element interpolation.

We decompose a vector of degrees of freedom $U \in W_i$ into internal and boundary degrees of freedom assuming the boundary degrees of freedom are listed last. That is, in block notation, $U = [\dot{U}^T, \bar{U}^T]^T$, where \bar{U} is the vector of boundary degrees of freedom. We denote \bar{W}_i the space of boundary degrees

of freedom and define the matrix T_i so that $\bar{U} = T_i U$. Then, $I_{P_1} U$ defines a function on $\partial\Omega_i$ which depends on \bar{U} only. By abuse of notation, we will write $I_{P_1} \bar{U} = I_{P_1} [0^T, \bar{U}^T]^T$ on $\partial\Omega_i$.

We summarize some well known results and inequalities in a form suitable for our purposes. The next lemma summarizes the fact that the $H^{1/2}$ norm of a zero extension of a "piece" of a function can be bounded by the norm of the function.

Lemma 24 Let $\Sigma \subset \partial\Omega_i$ be a vertex, edge, or face (if $d = 3$) of subdomain Ω_i . A face is understood not to contain adjacent edges, and an edge does not contain its endpoints. For for all $z \in V_h^{P_1}(\partial\Omega_i)$, define $w \in V_h^{P_1}(\partial\Omega_i)$ by $w(x) = z(x)$ on all nodes of triangulation $x \in \Sigma$, $w(x) = 0$ on all other nodes of $\partial\Omega_i$. Then

$$\|w\|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq C(1 + \log \frac{H}{h})^\beta (|z|_{\frac{1}{2}, 2, \partial\Omega_i}^2 + \frac{1}{H} \|z\|_{0, 2, \partial\Omega_i}^2),$$

where

$\beta = 1$ if $d = 2$ and Σ is a vertex, or $d = 3$ and Σ is an edge or a vertex;

$\beta = 2$ if $d = 2$ and Σ is an edge, or $d = 3$ and Σ is a face.

Proof. The inequality for $d = 2$ was proved in [49, 48]. The case when $d = 3$ follows from Lemmas 4.1 and 4.2 in [11] if Σ is an edge or a vertex, and Lemma 4.3 in [11] if Σ is a face. Cf. also [23]. \square

The following lemma can be proved by using Lemmas 4.1 and 4.2 in [11] and estimates the $H^{1/2}$ seminorm of a spike function.

Lemma 25 There exists a constant c independent of h and H such that for all $u \in V_h^{P_1}(\Omega_i)$, $u(x) = 0$ for all mesh points $x \in \partial\Omega_i$, $x \neq x_0$,

$$|u|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq c \|u\|_{0, \infty, \partial\Omega_i}^2$$

The following lemma shows the equivalence of the discrete seminorm defined by the Schur complement and the $H^{1/2}$ seminorm. In the next section we will apply this result to the stiffness matrix K_i , which is why we use the same symbol here. This result is standard [10, 69] and it is proved here for the sake of completeness.

Lemma 26 Let K_i be a symmetric matrix that satisfies

$$c|I_{P1}U|_{1,2,\Omega_i}^2 \leq \langle K_i U, U \rangle \leq C|I_{P1}U|_{1,2,\Omega_i}^2 \quad \forall U \in W_i.$$

Let $\bar{U} = T_i U$ and the matrix K_i be decomposed so that

$$K_i U = \begin{bmatrix} \dot{K} & \tilde{K}^T \\ \tilde{K} & \bar{K} \end{bmatrix} \begin{bmatrix} \dot{U} \\ \bar{U} \end{bmatrix}.$$

Let S_i be the Schur complement of \dot{K} in K_i . Then, there exist constants c_0 and C_0 independent of h and H such that

$$c_0|I_{P1}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq \langle S_i \bar{U}, \bar{U} \rangle \leq C_0|I_{P1}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}^2 \quad \forall \bar{U} \in \bar{W}_i.$$

Proof. From the definition of the Schur complement (Definition 6) and the property (2.8), it follows that

$$\langle S_i \bar{U}, \bar{U} \rangle = \langle \bar{K} \bar{U}, \bar{U} \rangle - \langle \dot{K}^{-1} \tilde{K}^T \bar{U}, \tilde{K}^T \bar{U} \rangle = \inf_{U=[\dot{U}^T, \bar{U}^T]^T} \langle K_i U, U \rangle$$

Hence, from the assumption by invariance of the $|\cdot|_{1,2,\Omega_i}$ seminorm to adding a constant and by the discrete extension theorem [22] for the scaled norms, we obtain

$$\langle S_i \bar{U}, \bar{U} \rangle \leq C \inf_{U=[\dot{U}^T, \bar{U}^T]^T} |I_{P1}U + k|_{1,2,\Omega_i} \leq C c_e \|I_{P1}\bar{U} + k\|_{\frac{1}{2},2,\partial\Omega_i},$$

where $k \in \mathbb{R}$. Choosing $k = -\frac{\int_{\partial\Omega_i} I_{P1} \bar{U} dx}{\int_{\partial\Omega_i} dx}$, yields the equivalence of the scaled norm to the seminorm by the Poincaré-Friedrichs inequality (Lemma 22). On the other hand, by the trace theorem, invariance of the seminorms to adding a constant, and the Poincaré-Friedrichs inequality (Lemma 23), we have

$$\frac{1}{c_t c_p} c |I_{P1} \bar{U}|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq c \inf_{U=[\bar{U}^T, \bar{U}^T]^T} |I_{P1} U|_{1, 2, \Omega_i} \leq \langle S_i \bar{U}, \bar{U} \rangle,$$

where c_t and c_p are constants arising from the trace and Poincaré-Friedrichs inequality respectively. \square

The following theorem shows a variant of the Poincaré-Friedrichs inequality. We note that the dimension of \bar{W}_i implicitly depends on h and for a uniform mesh, the lemma follows from Lemma 22.

Lemma 27 Let S_i be the Schur complement from Lemma 26. There exists a constant C independent of H and h such that for all $\bar{U} \in \bar{W}_i$, $\bar{U} - \text{Ker } S_i$

$$\|I_{P1} \bar{U}\|_{\frac{1}{2}, 2, \partial\Omega_i} \leq C |I_{P1} \bar{U}|_{\frac{1}{2}, 2, \partial\Omega_i}.$$

Proof. If there is an essential boundary condition imposed on $\partial\Omega_i$, then $\text{Ker } S_i = \{0\}$ and the statement follows from the Poincaré-Friedrichs inequality Lemma 22. Otherwise, $\text{Ker } S_i$ is spanned by the vector of ones. Then the condition $\bar{U} - \text{Ker } S_i$ implies $\sum_{j=1}^N \bar{U}_j = 0$ for $\bar{U} = [\bar{U}_1, \dots, \bar{U}_N]^T$. Following along the lines of the proof of Lemma 22, we obtain a sequence of vectors $\bar{U}^n \in \bar{W}_i \equiv \mathbb{R}^{N(n)}$ such that

$$(I_{P1} \bar{U}^n) \circ \mathcal{F}_i \rightarrow u \quad \text{in} \quad L_2(\partial\hat{\Omega}),$$

and

$$\|u\|_{\frac{1}{2}, 2, \partial\hat{\Omega}} = 1 \quad \text{and} \quad |u|_{\frac{1}{2}, 2, \partial\hat{\Omega}} = 0 \quad (4.4)$$

Therefore, $u = k$ almost everywhere, where $k \in \mathbb{R}$. Thus, $(I_{P1}\bar{U}^n) \circ \mathcal{F}_i \rightarrow k$ in $L_2(\partial\hat{\Omega})$. Then, from the quasi-uniformity of the triangulation, we obtain

$$\|(I_{P1}\bar{U}^n) \circ \mathcal{F}_i - k\|_{0,2,\partial\hat{\Omega}}^2 = c \frac{1}{N(n)} \|\bar{U}^n - k\|^2 \geq \frac{c}{N(n)^2} \left(\sum_{j=1}^{N(n)} (\bar{U}_j^n - k) \right)^2$$

Since $\bar{U}^n - \text{Ker } S_i$, it holds that $\sum_{j=1}^{N(n)} \bar{U}_j^n = 0$. Thus,

$$ck^2 \leq \|(I_{P1}\bar{U}^n) \circ \mathcal{F}_i - k\|_{0,2,\partial\hat{\Omega}}^2 \rightarrow 0.$$

This is possible only if $k = 0$ which is a contradiction with (4.4). \square

4.1.2 Estimates for HCT Elements

Let $V_h^{HCT}(\Omega_i)$ be the finite element space of HCT elements satisfying the usual regularity and inverse properties and possibly some essential boundary conditions. That is, we assume that

$$\Omega_i = \cup_{K \in \mathcal{T}_{h,i}} K,$$

where each element K of the triangulation $\mathcal{T}_{h,i}$ is a triangle. Furthermore, condition (4.3) is satisfied and h denotes the characteristic mesh size. We note that functions in $V_h^{HCT}(\Omega_i)$ are C_1 continuous and the space $V_h^{HCT}(\Omega_i)$ can be written as

$$\begin{aligned} V_h^{HCT}(\Omega_i) = \\ \{v \in C^1(\Omega_i) : \forall K \in \mathcal{T}_{h,i}, v|_{\mathcal{K}_j} \in P_3(\mathcal{K}_j) \text{ for all subtriangles } \mathcal{K}_j \text{ of } K, \\ \frac{\partial v}{\partial n}|_{a_k a_j} \in P_1(a_k a_j) \text{ for all sides } (a_k, a_j) \text{ of the triangle } K\} \end{aligned}$$

On each triangle, a function v in $V_h^{HCT}(\Omega_i)$ is determined by the values $v(a_i)$ and the values of its derivatives $\frac{\partial v}{\partial x_j}(a_i)$ at the vertices of the triangle. The

internal vertex of subtriangles is not arbitrary, but it is determined to guarantee the unisolvence of the element. This is as much detail as we will need in our considerations. We refer to [14] for more details about the HCT element.

The corresponding space of vectors of degrees of freedom is denoted W_i . Let $I_{P1} : W_i \rightarrow V_h^{HCT}(\Omega_i)$ denote the linear one-to-one transformation that maps a vector of degrees of freedom to the corresponding finite element function. As in the previous section, we decompose a vector of degrees of freedom U into internal and boundary degrees of freedom assuming the boundary degrees of freedom are listed last. That is, in block notation, $U = [\dot{U}^T, \bar{U}^T]^T$, where \bar{U} is the vector of boundary degrees of freedom. We denote \bar{W}_i the space of boundary degrees of freedom and define the matrix T_i so that $\bar{U} = T_i U$. Then, $I_{HCT}U$ defines a function on $\partial\Omega_i$ which depends on \bar{U} only. By abuse of notation, we will write $I_{HCT}\bar{U} = I_{HCT}[0^T, \bar{U}^T]^T$ on $\partial\Omega_i$.

We summarize here some well known results and inequalities in a form suitable for our purposes. The following lemma estimates the norm of a “spike” function and is proved in [42].

Lemma 28 Let x be a vertex of a subdomain Ω_i . For $u \in V_h^{HCT}(\Omega_i)$ such that $u(x) = 0$, define $z \in V_h^{HCT}(\Omega_i)$ by $z(x) = u(x) = 0$, $\nabla z(x) = \nabla u(x)$, and $z(y) = 0$, $\nabla z(y) = 0$ at all other nodes y of $\partial\Omega_i$. Then

$$\|\nabla z\|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq C \left(1 + \log \frac{H}{h}\right) \left(\|\nabla u\|_{\frac{1}{2}, 2, \partial\Omega_i}^2 + \frac{1}{H} \|\nabla u\|_{0, 2, \partial\Omega_i}^2\right).$$

The following estimate of the trace norm of the extension by zero is proved as in [10, Lemma 3.5].

Lemma 29 Let $u \in V_h^{HCT}(\Omega_i)$. Then there exists a constant C such that if

$\text{supp } u \cap \partial\Omega_i$ is contained in a segment σ of $\partial\Omega_i$ of length τ , then

$$|u|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq |u|_{\frac{1}{2},2,\sigma}^2 + C \left(1 + \log \frac{\tau}{h}\right) \|u\|_{0,\infty,\sigma}^2.$$

The following estimate is a modification of the previous lemma.

Lemma 30 Let $u \in V_h^{HCT}(\Omega_i)$ and $\text{supp } u \cap \partial\Omega_i$ be contained in a segment of $\partial\Omega_i$ and x_1 and y_1 be its endpoints. Let x_2 and y_2 be the nodal points next to x_1 and y_1 respectively in the segment. Let (x_2, y_2) denote the segment between x_2 and y_2 . Let the length of this segment be $\tau \geq h$. Assume that the function u satisfies the condition $\|u\|_{0,\infty,(x_1,y_1)} \leq c\|u\|_{0,\infty,(x_2,y_2)}$. Then

$$|u|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq |u|_{\frac{1}{2},2,(x_2,y_2)}^2 + C \left(1 + \log \frac{\tau}{h}\right) \|u\|_{0,\infty,(x_2,y_2)}^2.$$

Proof. By the previous lemma, the inequality holds with (x_2, y_2) replaced by the segment (x_1, y_1) . By the definition of the $H^{1/2}$ seminorm, we have

$$|u|_{\frac{1}{2},2,(x_1,y_1)}^2 = |u|_{\frac{1}{2},2,(x_2,y_2)}^2 + \int_{(x_1,x_2)} \int_{(x_1,y_2)} \frac{|u(x) - u(y)|^2}{\|x - y\|^2} dx dy + \int_{(y_1,y_2)} \int_{(x_1,y_2)} \frac{|u(x) - u(y)|^2}{\|x - y\|^2} dx dy.$$

Using the fact that

$$|u(x) - u(y)| \leq \min \left\{ \frac{c_1 c}{h} \|u\|_{0,\infty,(x_2,y_2)} \|x - y\|, 2c \|u\|_{0,\infty,(x_2,y_2)} \right\},$$

the last two terms are bounded by $c_2 \|u\|_{0,\infty,(x_2,y_2)}$. \square

We will also need a straightforward extension of the discrete Sobolev inequality of Dryja [22] to piecewise polynomial functions of order $p > 1$ [57].

Lemma 31 For every $p \geq 1$ there exists a constant $C = C(p)$ such that for every u continuous on $\sigma \subset \partial\Omega_i$ of length τ such that $u \in P_p$ on the side of every element,

$$\|u\|_{0,\infty,\sigma}^2 \leq C \left(1 + \log \frac{\tau}{h}\right) \left(|u|_{\frac{1}{2},2,\sigma}^2 + \frac{1}{\tau} \|u\|_{0,2,\sigma}^2 \right).$$

The following lemma establishes a useful inequality between discrete and Sobolev norms.

Lemma 32 Let $u \in V_h^{HCT}(\Omega_i)$ and $u(x_0) = 0$ for some $x_0 \in \partial\Omega_i$. Let $U = I_{HCT}^{-1}u$ and $\bar{U} = T_i U$. Then, for all such \bar{U} ,

$$h\|\bar{U}\|^2 \leq c(1 + H^2)\|\nabla u\|_{0,2,\partial\Omega_i}^2,$$

where c is independent of h and H .

Proof. Let E be an element edge on $\partial\Omega_i$ and x_1 and x_2 be its endpoints. Each component of ∇u is a polynomial of degree at most two on E . Since all norms in a finite dimensional space are equivalent, it holds that

$$\begin{aligned} h(\|\nabla u(x_1)\|^2 + \|\nabla u(x_2)\|^2) &\leq c_1 \min_{f \in P_2 \times P_2, f(x_1) = \nabla u(x_1), f(x_2) = \nabla u(x_2)} \|f\|_{0,2,E}^2 \\ &\leq c_1 \|\nabla u\|_{0,2,E}^2 \end{aligned}$$

Summing over all edges of the boundary, we obtain

$$h\|\bar{U}\|^2 \leq \frac{c_1}{2} \|\nabla u\|_{0,2,\partial\Omega_i}^2 + h\|U_0\|^2,$$

where U_0 denotes the displacement degrees of freedom of \bar{U} . We show that $h\|U_0\|^2 < c_2 H^2 \|\nabla u\|_{0,2,\partial\Omega_i}^2$. Since $u(x_0) = 0$, we can write for any mesh point x on the boundary

$$u(x) = \int_{(x_0,x)} \nabla u(y) \cdot \tau(y) dy,$$

where $(x_0, x) \subset \partial\Omega_i$ is the part of the boundary between x_0 and x , and τ is the vector tangential to the boundary. Squaring, using the Cauchy inequality, and considering that the length of (x_0, x) is bounded by $c_2 H$, we obtain

$$u(x)^2 \leq c_2 H \|\nabla u\|_{0,2,\partial\Omega_i}^2.$$

The statement of the lemma then follows by summing over all mesh points on the boundary. \square

The following lemma shows the equivalence of the discrete seminorm defined by the Schur complement and the $H^{1/2}$ seminorm of the gradient of the corresponding finite element function and is an analog of Lemma 26 of the previous section.

Lemma 33 Let K_i be a matrix that satisfies

$$c|I_{HCT}U|_{2,2,\Omega_i}^2 \leq \langle K_i U, U \rangle \leq C|I_{HCT}U|_{2,2,\Omega_i}^2 \quad \forall U \in W_i.$$

Let $\bar{U} = T_i U$ and let the matrix K_i be decomposed so that

$$K_i U = \begin{bmatrix} \dot{K} & \tilde{K}^T \\ \tilde{K} & \bar{K} \end{bmatrix} \begin{bmatrix} \dot{U} \\ \bar{U} \end{bmatrix}.$$

Let S_i be the Schur complement of \dot{K} in K_i . Then, there exist constants c_0 and C_0 independent of h and H such that

$$c_0|\nabla I_{HCT}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq \langle S_i \bar{U}, \bar{U} \rangle \leq C_0|\nabla I_{HCT}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}^2 \quad \forall \bar{U} \in \bar{W}_i.$$

Proof. From the definition of the Schur complement it follows that

$$\langle S_i \bar{U}, \bar{U} \rangle = \langle \bar{K} \bar{U}, \bar{U} \rangle - \langle \dot{K}^{-1} \tilde{K}^T \bar{U}, \tilde{K}^T \bar{U} \rangle = \inf_{U=[\dot{U}^T, \bar{U}^T]^T} \langle K_i U, U \rangle$$

Hence, by the discrete extension theorem [41], we obtain

$$\langle S_i \bar{U}, \bar{U} \rangle \leq C \inf_{U=[\dot{U}^T, \bar{U}^T]^T} |I_{P1}U|_{2,2,\Omega_i} \leq C c_e |\nabla I_{P1}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}.$$

On the other hand, the trace theorem and invariance of the $|\cdot|_{\frac{1}{2},2,\Omega_i}$ seminorm to adding a linear function, and the Poincaré-Friedrichs theorem,

$$|\nabla I_{P1}\bar{U}|_{\frac{1}{2},2,\partial\Omega_i}^2 \leq c_t c_p \inf_{U=[\dot{U}^T, \bar{U}^T]^T} |I_{P1}U|_{2,2,\Omega_i}.$$

□

The following theorem shows another variant of the Poincaré-Friedrichs inequality. We again emphasize that the dimension of \bar{W}_i implicitly depends on h and is an analog of Lemma 27 of the previous section.

Lemma 34 Let S_i be the Schur complement from Lemma 33. There exists a constant C independent of H and h such that for all $\bar{U} \in \bar{W}_i$, $\bar{U} - \text{Ker } S_i$

$$\|\nabla I_{HCT} \bar{U}\|_{\frac{1}{2}, 2, \partial\Omega_i} \leq C \|\bar{U}\|_{\frac{1}{2}, 2, \partial\Omega_i}.$$

Proof. Let us prove the case when $\text{Ker } S$ is nontrivial. In the other case, the statement follows from the standard Poincaré-Friedrichs inequality. Following along the lines of the proof of Lemma 22, we obtain a sequence of vectors $\bar{U}^n \in \bar{W}_i \equiv \mathbb{R}^{N(n)}$ such that

$$(\nabla I_{HCT} \bar{U}^n) \circ \mathcal{F}_i \rightarrow u \quad \text{in} \quad L_2(\partial\hat{\Omega}) \times L_2(\partial\hat{\Omega}),$$

and

$$\|u\|_{\frac{1}{2}, 2, \partial\hat{\Omega}} = 1 \quad \text{and} \quad |u|_{\frac{1}{2}, 2, \partial\hat{\Omega}} = 0 \quad (4.5)$$

Therefore, $u = (k_1, k_2) \in \mathbb{R} \times \mathbb{R}$ almost everywhere. Thus, $(\nabla I_{HCT} \bar{U}^n) \circ \mathcal{F}_i \rightarrow (k_1, k_2)$ in $L_2(\partial\hat{\Omega}) \times L_2(\partial\hat{\Omega})$. We decompose the vector \bar{U}^n into two vectors corresponding to the partial derivative with respect to x , \bar{U}_1^n , and the partial derivative with respect to y , \bar{U}_2^n . Then, by the proof of Lemma 32, we obtain

$$\begin{aligned} \|(\nabla I_{HCT} \bar{U}^n) \circ \mathcal{F}_i - (k_1, k_2)\|_{0, 2, \partial\hat{\Omega}}^2 &\geq c \frac{1}{N(n)} (\|\bar{U}_1^n - k_1\|^2 + \|\bar{U}_2^n - k_2\|^2) \\ &\geq \frac{c}{N(n)^2} \left(\sum_{i=1}^{N(n)} (\bar{U}_{1,i}^n - k_1) \right)^2 + \frac{c}{N(n)^2} \left(\sum_{i=1}^{N(n)} (\bar{U}_{2,i}^n - k_2) \right)^2. \end{aligned}$$

Since $\bar{U}^n - \text{Ker } S_i$, it holds that $\sum_{i=1}^{N(n)} \bar{U}_{1,i}^n = 0$ and $\sum_{i=1}^{N(n)} \bar{U}_{2,i}^n = 0$. Thus,

$$c(k_1^2 + k_2^2) \leq \|(\nabla I_{HCT} \bar{U}^n) \circ \mathcal{F}_i - (k_1, k_2)\|_{0, 2, \partial\hat{\Omega}}^2 \rightarrow 0.$$

This is possible only if $(k_1, k_2) = (0, 0)$ which is a contradiction with (4.5). \square

4.2 Abstract Analysis Framework

This section develops a framework for estimating the condition number of our method. We estimate the minimum and maximum eigenvalues of the matrix of the FETI formulation and the preconditioner.

The following simple lemma will be the basis of our estimates. It will allow to reduce estimates of norms to estimates of boundedness and coercivity. The proof follows a standard argument and it is presented for completeness only.

Lemma 35 Let X be a Banach space, X' the dual of X , and $A : X \rightarrow X'$ a linear operator that is onto and satisfies the conditions

$$\langle y, Ax \rangle = \langle x, Ay \rangle, \quad \forall x, y \in X \quad (4.6)$$

$$c_A \|x\|_X^2 \leq \langle x, Ax \rangle \leq C_A \|x\|_X^2, \quad \forall x \in X \quad (4.7)$$

with constants $C_A, c_A > 0$. Then

$$\|A\|_{X \rightarrow X'} \leq C_A, \quad \|A^{-1}\|_{X' \rightarrow X} \leq \frac{1}{c_A}.$$

Proof. From (4.6),

$$\|A\|_{X \rightarrow X'} = \sup_{x \in X} \frac{\|Ax\|_{X'}}{\|x\|_X} = \sup_{x, \tilde{x} \in X} \frac{\langle x, A\tilde{x} \rangle}{\|x\|_X \|\tilde{x}\|_X} = \sup_{x \in X} \frac{\langle x, Ax \rangle}{\|x\|_X^2} \leq C_A.$$

From (4.7) and the fact that A is onto, we obtain

$$\frac{1}{\|A^{-1}\|_{X' \rightarrow X}} = \inf_{x \in X} \frac{\|Ax\|_{X'}}{\|x\|_X} = \inf_{x \in X} \sup_{\tilde{x} \in X} \frac{\langle x, A\tilde{x} \rangle}{\|x\|_X \|\tilde{x}\|_X} \geq \inf_{x \in X} \frac{\langle x, Ax \rangle}{\|x\|_X^2} \geq c_A,$$

concluding the proof. \square

Assume that the preconditioner of a FETI method can be written in the form RDR^T , where R is a projection, and RDR^T is an isomorphism from \mathcal{H} onto \mathcal{H}' . Let \mathcal{H} and its dual \mathcal{H}' are subspaces of $\text{Ker } G^T$. Furthermore, let $\mathcal{H} = \text{Im } R^T$. The original FETI satisfies this with $R = P$ and $\mathcal{H} = \text{Ker } G^T / \text{Ker } B^T$. The generalized FETI satisfies the assumption with $R = Q$ and $\mathcal{H} = \tilde{V}$.

For the purpose of analysis, we equip the space \mathcal{H} with the norm

$$\|v\|_{\mathcal{H}} = \|\bar{B}^T v\|_S = \langle S\bar{B}^T v, \bar{B}^T v \rangle^{1/2}. \quad (4.8)$$

Since $\bar{B}^T v \perp \text{Ker } S$ for $v \in \mathcal{H}$, (4.8) indeed defines a norm rather than only seminorm. The dual space \mathcal{H}' is equipped with the dual norm

$$\|\lambda\|_{\mathcal{H}'} = \sup_{v \in \mathcal{H}} \frac{\langle \lambda, v \rangle}{\|v\|_{\mathcal{H}}}. \quad (4.9)$$

Since $\mathcal{H} \subset \tilde{\Lambda}$, it immediately follows

$$\|\lambda\|_{\mathcal{H}'} = \sup_{v \in \mathcal{H}} \frac{\langle \lambda, v \rangle}{\|\bar{B}^T v\|_S} = \sup_{w \in \tilde{W}, \bar{B}w \in \mathcal{H}} \frac{\langle \lambda, \bar{B}w \rangle}{\|\bar{B}^T \bar{B}w\|_S}. \quad (4.10)$$

The norm on \mathcal{H} was chosen so that the preconditioner RD is trivially coercive and bounded.

Lemma 36 For all $v \in \mathcal{H}$,

$$\langle v, RDv \rangle = \|v\|_{\mathcal{H}}^2.$$

Proof. Let $v \in \mathcal{H} = \text{Im } R^T$. Since R^T is a projection, we have by definition of the preconditioner D ,

$$\langle v, RDv \rangle = \langle R^T v, \bar{B}S\bar{B}^T v \rangle = \langle v, \bar{B}S\bar{B}^T v \rangle = \langle \bar{B}^T v, S\bar{B}^T v \rangle = \|v\|_{\mathcal{H}}^2,$$

which was to be shown. \square

Coercivity and boundedness of the system operator PF on \mathcal{H}' will be estimated using the following lemma.

Lemma 37 For all $\lambda \in \mathcal{H}'$,

$$\langle \lambda, F\lambda \rangle = \sup_{w \in \bar{W}, w \perp \text{Ker } S} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|w\|_S^2} = \sup_{w \in \bar{W}} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|w\|_S^2}.$$

Proof. Let $\lambda \in \mathcal{H}'$. Then

$$\begin{aligned} \langle \lambda, F\lambda \rangle &= \langle S^+ \bar{B}^T \lambda, \bar{B}^T \lambda \rangle = \langle S^{-1/2} \bar{B}^T \lambda, S^{-1/2} \bar{B}^T \lambda \rangle = \\ &= \|S^{-1/2} \bar{B}^T \lambda\|^2 = \sup_{x \in \bar{W}} \frac{\langle S^{-1/2} \bar{B}^T \lambda, x \rangle^2}{\|x\|^2} = \\ &= \sup_{\substack{x \in \bar{W}, x = x_1 + x_2 \\ x_1 \in \text{Ker } S, x_2 \perp \text{Ker } S}} \frac{\langle \bar{B}^T \lambda, S^{-1/2} x \rangle^2}{\|x_1 + x_2\|^2} = \sup_{x_2 \in \bar{W}, x_2 \perp \text{Ker } S} \frac{\langle \bar{B}^T \lambda, S^{-1/2} x_2 \rangle^2}{\|x_2\|^2} \end{aligned}$$

since $S^{-1/2} x_1 = 0$ and $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Now write any $w \in \bar{W}$ as

$$w = w_1 + w_2, \quad w_1 \in \text{Ker } S, \quad w_2 = S^{-1/2} x_2 \perp \text{Ker } S.$$

$\lambda \in \mathcal{H}'$ implies that $\langle \bar{B}^T \lambda, w_1 \rangle = 0$, hence $\langle \bar{B}^T \lambda, w_2 \rangle = \langle \bar{B}^T \lambda, w \rangle = \langle \lambda, \bar{B}w \rangle$. It follows that

$$\langle \lambda, F\lambda \rangle = \sup_{w_2 \in \bar{W}, w_2 \perp \text{Ker } S} \frac{\langle \bar{B}^T \lambda, w_2 \rangle^2}{\langle w_2, S w_2 \rangle} = \sup_{w \in \bar{W}} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|w\|_S^2},$$

which was to be proved. \square

It is well known [39] that after k iterations of the preconditioned conjugate gradient method, the energy norm or the error $\|e\| = \langle PFe, e \rangle^{1/2}$ is reduced by a factor of at least $2((\sqrt{\kappa}-1)/(\sqrt{\kappa}+1))^k$, where κ is the condition number. The condition number is given in our case by

$$\kappa = \kappa(RDPF) = \frac{\lambda_{\max}(RDPF)}{\lambda_{\min}(RDPF)}, \quad (4.11)$$

where λ_{\max} and λ_{\min} are the maximum and minimum eigenvalue, respectively.

We are now ready to prove an abstract bound on κ .

Theorem 38 Assume there exist constants C_1, C_2 such that

(i) for any $\lambda \in \mathcal{H}'$ and $w \in \bar{W}$ such that $\bar{B}w \in \mathcal{H}$, there is $\tilde{w} \in \bar{W}$ such that

$$\langle \lambda, \bar{B}\tilde{w} \rangle = \langle \lambda, \bar{B}w \rangle, \quad \text{and} \quad \|\tilde{w}\|_S^2 \leq C_1 \|\bar{B}^T \bar{B}w\|_S^2;$$

(ii) for any $\lambda \in \mathcal{H}'$ and $w \in \bar{W}$, $w \perp \text{Ker } S$, there is $\tilde{w} \in \bar{W}$ such that $\bar{B}\tilde{w} \in \mathcal{H}$,

$$\langle \lambda, \bar{B}\tilde{w} \rangle = \langle \lambda, \bar{B}w \rangle, \quad \text{and} \quad \|\bar{B}^T \bar{B}\tilde{w}\|_S^2 \leq C_2 \|w\|_S^2.$$

Then $\kappa(RDPF) \leq C_1 C_2$.

Proof. Lemma 35 applied to the operator RD together with Lemma 36 give

$$\|QD\|_{\mathcal{H} \rightarrow \mathcal{H}'}^2 \leq 1, \quad \|(RD)^{-1}\|_{\mathcal{H}' \rightarrow \mathcal{H}}^2 \leq 1. \quad (4.12)$$

From assumption (i), we have

$$\sup_{w \in \bar{W}} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|w\|_S^2} \geq \frac{1}{C_1} \sup_{w \in \bar{W}, \bar{B}w \in \mathcal{H}} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|\bar{B}^T \bar{B}w\|_S^2} \quad (4.13)$$

while assumption (ii) gives the converse bound

$$\sup_{w \in \bar{W}, w \in \text{Ker } S} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|w\|_S^2} \leq C_2 \sup_{w \in \bar{W}, \bar{B}w \in \mathcal{H}} \frac{\langle \lambda, \bar{B}w \rangle^2}{\|\bar{B}^T \bar{B}w\|_S^2}. \quad (4.14)$$

Using (4.10) and Lemma 37, we see that the inequalities (4.13) and (4.14)

imply the inequality

$$\frac{1}{C_1} \|\lambda\|_{\mathcal{H}'}^2 \leq \langle \lambda, PF\lambda \rangle \leq C_2 \|\lambda\|_{\mathcal{H}'}^2, \quad \forall \lambda \in \mathcal{H}'.$$

Applying Lemma 35 to the operator PF , we obtain

$$\|PF\|_{\mathcal{H}' \rightarrow \mathcal{H}}^2 \leq C_2, \quad \|(PF)^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}'}^2 \leq C_1. \quad (4.15)$$

From (4.12) and (4.15), we have

$$\|QDPF\|_{\mathcal{H}' \rightarrow \mathcal{H}'} \leq \|QD\|_{\mathcal{H} \rightarrow \mathcal{H}'} \|PF\|_{\mathcal{H}' \rightarrow \mathcal{H}} \leq C_2$$

and

$$\|(QDPF)^{-1}\|_{\mathcal{H}' \rightarrow \mathcal{H}'} \leq \|(QD)^{-1}\|_{\mathcal{H}' \rightarrow \mathcal{H}} \|(PF)^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}'} \leq C_1.$$

The result follows. \square

4.3 Analysis of the Original FETI

In this section, we use the abstract analysis framework from the previous section to prove a bound on the condition number of FETI for a second elliptic problem, utilizing results from Section 4.1.

4.3.1 Assumptions

We need more specific assumptions in order to be able to prove a bound on the condition number κ . Let us recall the model problem (2.1). Assume we are solving the boundary value problem

$$\mathcal{A}u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where \mathcal{A} is a second order elliptic operator

$$\mathcal{A}v = - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(\alpha(x) \frac{\partial v(x)}{\partial x_j} \right),$$

with $\alpha(x)$ a measurable function such that $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1$ a.e. in Ω .

The domain Ω is assumed to be divided into non-overlapping subdomains Ω_i , $i = 1, \dots, N_s$, that are shape regular of diameter H (Definition 21). Assume that $V_h(\Omega)$ is a conforming P1 or Q1 finite element space on a triangulation of Ω , which satisfies the standard regularity and inverse assumptions (cf. Section 4.1). In particular, we recall that the degrees of freedom are values at nodes of the triangulation h denotes the characteristic element size. Each subdomain Ω_i is assumed to be a union of some of the elements, and all functions in $V_h(\Omega)$ are zero on $\partial\Omega$.

Let us define the restriction operator $R_i : \bar{W} \rightarrow \bar{W}_i$ by the equation

$$R_i w = w_i,$$

where $w = [w_1^T w_2^T \dots w_{N_s}^T]^T$, $w_i \in \bar{W}_i, i = 1, \dots, N_s$.

We will assume that the interface continuity operator \bar{B} is defined as follows. Let $w_r(x)$ and $w_s(x)$ denote the pair of degrees of freedom corresponding to the mesh node $x \in \partial\Omega_r \cap \partial\Omega_s$ and let $(\bar{B}\bar{w})_{rs}(x)$ be the entry of vector $\bar{B}\bar{w}$ that corresponds to the mesh point x , and subdomains Ω_r and Ω_s . We require each such $(\bar{B}\bar{w})_{rs}(x)$ to have form

$$(\bar{B}w)_{rs}(x) = \sigma_{rs}(w_r(x) - w_s(x)), \quad (4.16)$$

where σ_{rs} is either 1 or -1. Note that σ_{rs} does not depend on x , that is, coefficients σ_{rs} are uniform along edges (and faces, in 3D) between two subdomains. For each node x that belongs to the interface of two and more subdomains, $x \in \cap \partial\Omega_i, i = s_1, s_2, \dots, s_n$, vector $\bar{B}w$ contains $n - 1$ entries, $(\bar{B}w)_{s_k s_{k+1}}(x), k = 1, \dots, s_{n-1}$. For an example of the definition of the values of $\bar{B}w$ for $(s_1, s_2, s_3) = (1, 3, 2)$ in 2D around a crosspoint, see Fig. 4.1. We point out that \bar{B} chosen in this way has full row rank, that is, this definition implies that there are no redundant constraints in enforcing the continuity of the solution at the nodes where more than two subdomains meet.

Only the improved estimate in statement 3 of Lemma 39 will require the specific definition of this section. If redundant constraints are allowed, the estimates, with the exceptions of the improved estimates, still hold which can be shown by minor modifications of the proofs. See Section 4.4.

An additional connectivity assumption on the decomposition is needed. Let $N_{rs}, r, s = 1, \dots, N_s, r \neq s$ be the number of interface conditions

between subdomains Ω_r and Ω_s , N_{rr} the number of degrees of freedom of subdomain Ω_r and $N = \max_{i=1}^{N_s} N_{ii}$. We will assume that there exists constants c and n_0 independent of h and H such that for all $r, s = 1, \dots, N_s, r \neq s$, for which $N_{rs} > 0$, there exists a sequence of indices $\{r_i\}$, $r_0 = r, r_k = s, k < n_0$ such that

$$N_{r_{i-1}r_i} \geq cN \quad \forall i = 1, \dots, k.$$

Throughout the next section, $c, C, c_1, c_2, c_3, c_4, c_5$ and c_6 denote positive constants independent of H and h .

4.3.2 Discrete Norm Bounds

The stiffness matrices $K_i, i = 1, \dots, N_s$ satisfy the assumption of Lemma 26. Therefore, using Lemma 26 for each subdomain and summing over all subdomains, we obtain

$$c_1 \sum_{i=1}^{N_s} |I_{P1} R_i w|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq \|w\|_S^2 \leq c_2 \sum_{i=1}^{N_s} |I_{P1} R_i w|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \quad \forall w \in W, \quad (4.17)$$

where the positive constants c_1, c_2 are independent of the characteristic mesh size h and the subdomain diameter H . The constants c_1, c_2 may depend on the regularity of the shape of subdomains.

The following lemmas verify the assumptions of Theorem 38.

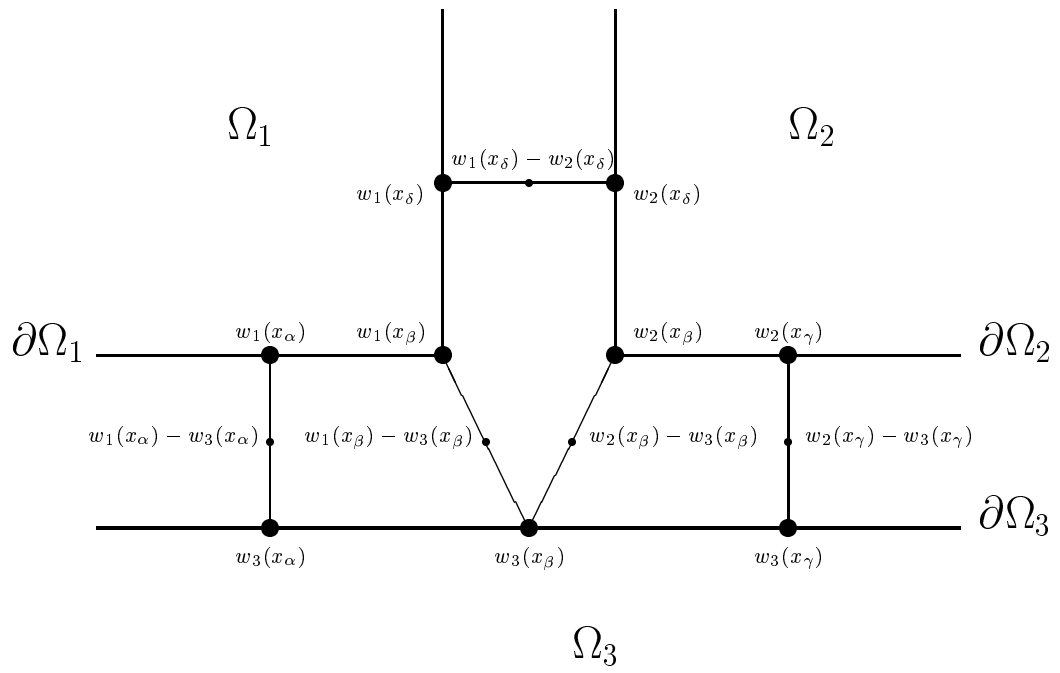
Lemma 39 For all $\lambda \in \text{Ker } G^T$ and all $w \in \bar{W}$ such that $\bar{B}^T \bar{B} w \in \text{Ker } S$, there exists $\tilde{w} \in \bar{W}$ such that

$$\langle \lambda, \bar{B} \tilde{w} \rangle = \langle \lambda, \bar{B} w \rangle \quad \text{and} \quad \|\tilde{w}\|_S^2 \leq C(1 + \log H/h)^\alpha \|\bar{B}^T \bar{B} w\|_S^2.$$

where $\alpha = 1$, and $\alpha = 0$ in the following special cases:

- (1) $\frac{1}{2} \bar{B} \bar{B}^T = I$, which means that there are no nodes shared by more than two subdomains.

Figure 4.1. Definition of \bar{B}



- (2) $d = 2$, and the matrix \bar{B} has the following property: If $\bar{w} \in \text{Im } \bar{B}^T$, x is a crosspoint (node shared by more than two subdomains), $I_{P_1}R_i\bar{w}(x) = I_{P_1}R_i\bar{w}(y)$ for all i such that $x \in \partial\Omega_i$ and all nodes y that are adjacent to x on $\partial\Omega_i$, then $I_{P_1}R_i\bar{w}(x) = 0$ for all i such that $x \in \partial\Omega_i$.
- (3) $d = 2$, \bar{B} is defined as in Section 4.3.1 and all nodes in the triangulation belong to either one, two, or an odd number of subdomains.

Proof. Let us first prove that, in the general case, we obtain $\alpha \leq 1$. Let $w \in \bar{W}$ and define $\tilde{w} = \bar{B}^T(\bar{B}\bar{B}^T)^{-1}\bar{B}w$. The triangle inequality shows that

$$\|\tilde{w}\|_S \leq \left\| \frac{1}{2}\bar{B}^T\bar{B}w \right\|_S + \left\| \frac{1}{2}\bar{B}^T \left(I - \left(\frac{1}{2}\bar{B}\bar{B}^T \right)^{-1} \right) \bar{B}w \right\|_S. \quad (4.18)$$

Denote $z = \frac{1}{2}\bar{B}^T(I - (\frac{1}{2}\bar{B}\bar{B}^T)^{-1})\bar{B}w$. From the definition of \bar{B} in (4.16), z is zero at all nodes that belong to at most two subdomains. The remaining nodes lie on crosspoints or edges (in the 3D case) of subdomains. From the definition of B , at every such node x , z is a linear combination of the entries of $\bar{B}^T\bar{B}w$ that correspond to the same node x and the coefficients of the linear combinations are bounded only in terms of the number of subdomains the node belongs to. Using Lemma 24 for the crosspoints of subdomains and the equivalence (4.17), we obtain for the 2D case that

$$\|z\|_S^2 \leq c \sum_{i=1}^{N_s} |I_{P_1}R_i z|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq C(1 + \log(H/h)) \sum_{i=1}^{N_s} \|I_{P_1}R_i \bar{B}^T \bar{B}w\|_{\frac{1}{2}, 2, \partial\Omega_i}^2. \quad (4.19)$$

The Poincaré inequality, Lemma 27, yields

$$\sum_{i=1}^{N_s} \|I_{P_1}R_i \bar{B}^T \bar{B}w\|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq \|\bar{B}^T \bar{B}w\|_S^2$$

In the 3D case, the argument for subdomain crosspoints is the same. In addition, we note that the coefficients of the linear combination do not change along a subdomain edge, so it remains to apply Lemma 24 on every edge.

Let us now turn to the special cases that give $\alpha = 0$. If $\frac{1}{2}\bar{B}\bar{B}^T = I$, we choose $\tilde{w} = \frac{1}{2}\bar{B}^T\bar{B}w = w$, which proves the special case 1.

Now we prove special case 2. From the definition of the $H^{1/2}$ norm, the equivalence of norms (4.17), and the fact that $I_{P_1}R_i\bar{B}^T\bar{B}w$ is a piecewise linear function, it follows that

$$\begin{aligned} \|\bar{B}^T\bar{B}w\|_S^2 &\geq c \sum_{i=1}^{N_s} |I_{P_1}R_i\bar{B}^T\bar{B}w|_{\frac{1}{2},2,\partial\Omega_i}^2 \\ &\geq c \sum_{\substack{x \text{ crosspoint, } x \in \partial\Omega_i \\ y \text{ adjacent to } x, y \in \partial\Omega_i}} \left(I_{P_1}R_i\bar{B}^T\bar{B}w(x) - I_{P_1}R_i\bar{B}^T\bar{B}w(y) \right)^2. \end{aligned} \quad (4.20)$$

For any crosspoint x , it follows from the assumption that for every $\bar{w} \in \text{Im } \bar{B}^T$,

$$\sum_{\substack{i, \partial\Omega_i \ni x \\ y \text{ adjacent to } x, y \in \partial\Omega_i}} (I_{P_1}R_i\bar{w}(x) - I_{P_1}R_i\bar{w}(y))^2 = 0 \Rightarrow \sum_{i, \partial\Omega_i \ni x} (I_{P_1}R_i\bar{w}(x))^2 = 0.$$

Consequently, by compactness, and since there are only finitely many different numbers of subdomains sharing a crosspoint, for all $\bar{w} \in \text{Im } \bar{B}^T$

$$\sum_{i, \partial\Omega_i \ni x} (I_{P_1}R_i\bar{w}(x))^2 \leq C \sum_{\substack{i, \partial\Omega_i \ni x \\ y \text{ adjacent to } x, y \in \partial\Omega_i}} (I_{P_1}R_i\bar{w}(x) - I_{P_1}R_i\bar{w}(y))^2.$$

By summation over all crosspoints x , using Lemma 25 and (4.20), we get

$$\|z\|_S^2 \leq C \|\bar{B}^T u\|_S^2,$$

which concludes the proof of this case.

In order to prove case 3, we verify the assumptions of case 2. We formulate the proof only for a crosspoint shared by three subdomains (Fig. 4.1). The proof is similar for a different odd number of subdomains. Let $\bar{w} \in \text{Im } \bar{B}^T$. Since $I_{P_1}R_1\bar{w}(x_\beta) - I_{P_1}R_1\bar{w}(x_\alpha) = 0$, and $I_{P_1}R_1\bar{w}(x_\beta) - I_{P_1}R_1\bar{w}(x_\delta) = 0$, we have $I_{P_1}R_1\bar{w}(x_\alpha) = I_{P_1}R_1\bar{w}(x_\delta)$. Similarly, we obtain $I_{P_1}R_2\bar{w}(x_\alpha) =$

$I_{P_1}R_2\bar{w}(x_\gamma)$, and $I_{P_1}R_3\bar{w}(x_\delta) = I_{P_1}R_3\bar{w}(x_\gamma)$. Moreover $\bar{w} \in \text{Im } \bar{B}^T$ implies $I_{P_1}R_1\bar{w}(x_\alpha) = -I_{P_1}R_2\bar{w}(x_\alpha)$, $I_{P_1}R_2\bar{w}(x_\gamma) = -I_{P_1}R_3\bar{w}(x_\gamma)$, and $I_{P_1}R_3\bar{w}(x_\delta) = -I_{P_1}R_1\bar{w}(x_\delta)$, which can be satisfied only if $I_{P_1}R_1\bar{w}(x_\alpha) = I_{P_1}R_1\bar{w}(x_\delta) = \dots = 0$. \square

Remark 40 In general, the exponent $\alpha = 1$ in Lemma 39 cannot be improved. To see that, let us consider the configuration with values of u and $\bar{B}u$ in the neighborhood of a crosspoint as in Fig. 4.2, which violate the assumptions of special case 2. Extending the values of u in Fig. 4.2 to decay as $\log^\gamma(t/H)$, $\gamma < 1/2$, where t is the distance from the crosspoint, we obtain a vector $u \in \Lambda$ such that

$$\|\bar{B}^T u\|_S \approx C, \quad \|I_{P_1}u\|_{\frac{1}{2}, 2, \partial\Omega_1 \cap \partial\Omega_2} \approx |\log h/H|^\gamma.$$

If $u = \bar{B}w$, then on $\partial\Omega_1 \cap \partial\Omega_2$, $u = w_2 - w_1$, which gives

$$\begin{aligned} |I_{P_1}u|_{\frac{1}{2}, 2, \partial\Omega_1 \cap \partial\Omega_2} &\leq |I_{P_1}w_1|_{\frac{1}{2}, 2, \partial\Omega_1 \cap \partial\Omega_2} + |I_{P_1}w_2|_{\frac{1}{2}, 2, \partial\Omega_1 \cap \partial\Omega_2} \\ &\leq |I_{P_1}w_1|_{\frac{1}{2}, 2, \partial\Omega_1} + |I_{P_1}w_2|_{\frac{1}{2}, 2, \partial\Omega_2} \\ &\leq \|w\|_S, \end{aligned}$$

and therefore $\|w\|_S \geq C(\gamma)|\log h/H|^\gamma$ for all $\gamma < 1/2$.

Lemma 41 Let $\lambda \in \text{Ker } G^T$. Then for all $w \in \bar{W}$, $w \perp \text{Ker } S$, there is a $\tilde{w} \in \bar{W}$ such that $\bar{B}^T B\tilde{w} \perp \text{Ker } S$,

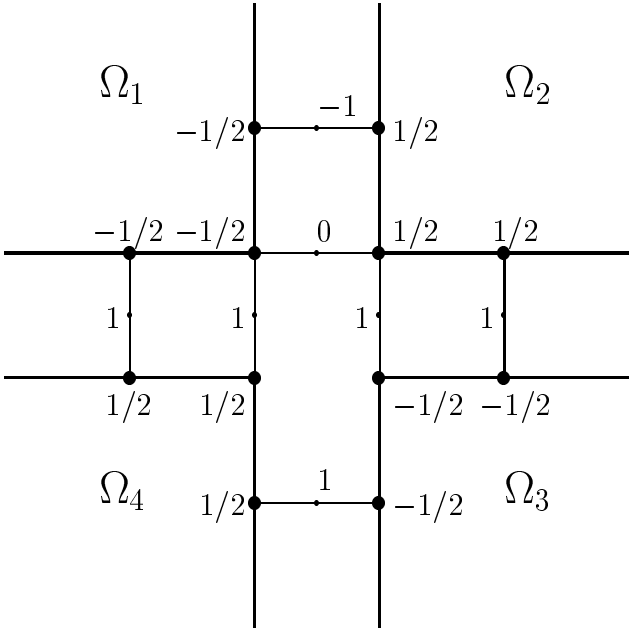
$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle, \quad \text{and} \quad \|\bar{B}^T \bar{B}\tilde{w}\|_S^2 \leq C(1 + \log H/h)^2 \|w\|_S^2$$

Proof. Let $w \in \bar{W}$ be arbitrary, and put $\bar{B}\tilde{w} = PBw$. Then

$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle. \quad (4.21)$$

Denote $Bz = \bar{B}w - \bar{B}\tilde{w}$, $z \in \text{Ker } S$. Then, since P is an orthogonal projection, $\|\bar{B}z\| \leq \|\bar{B}w\|$.

Figure 4.2. Counter-example



Now, from the definition of \bar{B} and the Poincaré inequality, Lemma 27, we obtain

$$h\|\bar{B}w\|^2 \leq 2h\|w\|^2 \leq CH\|w\|_S^2.$$

Also, since $z \in \text{Ker } S$, it is constant on each $\partial\Omega_i$, using the connectivity assumption, we have the following by Lemma 24:

$$\|\bar{B}^T \bar{B}z\|_S^2 \leq C \frac{h}{H} \|Bz\|^2 (1 + \log H/h)^2.$$

Together this yields

$$\|\bar{B}^T \bar{B}z\|_S^2 \leq C(1 + \log H/h)^2 \|w\|_S^2.$$

By the definition of \bar{B} , $R_i \bar{B}^T \bar{B}w$ on $\partial\Omega_i \cup \partial\Omega_j$ is a linear combination (with bounded coefficients) of (a bounded number of) w_k from all $\partial\Omega_k$ adjacent to $\partial\Omega_i \cup \partial\Omega_j$. From Lemma 24 and the Poincaré-Friedrichs inequality Lemma 27, we obtain

$$\|\bar{B}^T \bar{B}w\|_S \leq C(1 + \log(H/h))\|w\|_S, \quad \forall w \in \bar{W}.$$

Finally, summarizing,

$$\|\bar{B}^T \bar{B}\tilde{w}\|_S \leq \|\bar{B}^T \bar{B}w\|_S + \|\bar{B}^T \bar{B}z\|_S \leq C(1 + \log H/h)\|w\|_S.$$

From this and (4.21), the statement of the lemma follows. \square

4.3.3 Condition Number Estimate

We are now ready to prove the final result. It follows from the abstract estimate in Lemma 35 with the assumptions verified by Lemma 39 and Lemma 41.

Theorem 42 Under the assumptions of section 4.3.1, the condition number of the FETI method with the Dirichlet preconditioner satisfies

$$\kappa = \frac{\lambda_{max}(PDPF)}{\lambda_{min}(PDPF)} \leq C(1 + \log \frac{H}{h})^\gamma$$

with $\gamma = 3$, and $\gamma = 2$ in the special cases listed in Lemma 39.

4.4 Analysis of the Method by Park

We consider the same assumptions as for the original FETI for second order problems except for the choice of \bar{B} . We consider \bar{B} given by projection matrix P_L (3.43). The essential property of this matrix \bar{B} used here is that $\bar{B} = \bar{B}^T \bar{B} = \bar{B}^T$, since \bar{B} is an orthogonal projection.

We present two lemmas that verify assumptions of Theorem 38.

Lemma 43 For all $\lambda \in \text{Ker } G^T / \text{Ker } \bar{B}^T$ and all $w \in \bar{W}$ such that $\bar{B}w \in \text{Ker } G^T / \text{Ker } \bar{B}^T$, there exists $\tilde{w} \in \bar{W}$ such that

$$\langle \lambda, \bar{B}\tilde{w} \rangle = \langle \lambda, \bar{B}w \rangle, \quad \|\tilde{w}\|_S = \|\bar{B}^T \bar{B}w\|_S$$

Proof. Since $\bar{B}^T = \bar{B}$ and $\bar{B}^2 = \bar{B}$, we may choose $\tilde{w} = \bar{w}$. \square

Lemma 44 Let $\lambda \in \text{Ker } G^T / \text{Ker } B^T$. Then for all $w \in \bar{W}$, $w - \text{Ker } S$, there is a $\tilde{w} \in \bar{W}$ such that $\bar{B}^T B\tilde{w} - \text{Ker } S$ and

$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle, \quad \|\bar{B}^T \bar{B}\tilde{w}\|_S^2 \leq C(1 + \log H/h)^2 \|w\|_S^2$$

Proof. Let $w \in \bar{W}$ and $\bar{B}\tilde{w} = PBw$. Then

$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle \tag{4.22}$$

Let $z \in \text{Ker } S$, $Bz = \bar{B}w - \bar{B}\tilde{w}$. Then, since P is an orthogonal projection, $\|\bar{B}z\| \leq \|\bar{B}w\|$.

Now, from the definition of \bar{B} and the Poincaré inequality, we obtain

$$h\|\bar{B}w\|_S^2 \leq 2h\|w\|^2 \leq CH\|w\|_S^2.$$

Also, since $z \in \text{Ker } S$, it is constant on each $\partial\Omega_i$, and using the connectivity assumption, we have the following by Lemma 24

$$\|\bar{B}z\|_S^2 \leq C\frac{h}{H}\|Bz\|^2(1 + \log H/h)^2.$$

Together this yields

$$\|\bar{B}z\|_S^2 \leq C(1 + \log H/h)^2\|w\|_S^2.$$

By the definition of \bar{B} , $(\bar{B}w)_i$ on $\partial\Omega_i \cup \partial\Omega_j$ is a linear combination (with bounded coefficients) of (a bounded number of) w_k from all $\partial\Omega_k$ adjacent to $\partial\Omega_i \cup \partial\Omega_j$. From Lemma 24 and the Poincaré-Friedrichs inequality (Lemma 27),

$$\|\bar{B}w\|_S \leq C(1 + \log(H/h))\|w\|_S, \quad \forall w \in \bar{W}.$$

Finally, summarizing,

$$\|\bar{B}\tilde{w}\|_S \leq \|\bar{B}w\|_S + \|\bar{B}^T\bar{B}z\|_S \leq C(1 + \log H/h)\|w\|_S.$$

From this, (4.22), and $\bar{B}^T\bar{B} = \bar{B}$, the result follows. \square

Theorem 45 Under the assumption of Section 4.3.1, the condition number of the Park's variant of FETI method with the Dirichlet preconditioner (3.39) satisfies

$$\kappa = \frac{\lambda_{\max}(QDPF)}{\lambda_{\min}(QDPF)} \leq C \left(1 + \log \frac{H}{h}\right)^2.$$

Proof. Lemmas 43 and 44 verify the assumptions (i) and (ii) of Theorem 38 with $C_1 = 1$ and $C_2 = C(1 + \log H/h)^2$ respectively. \square

4.5 Convergence Estimates for Plate Bending

The following *approximate parametric variational principle* formulated in [42] will allow us to considerate to the plate bending problem and the biharmonic equation.

Assumption 46 ([41, 42]) We consider elements with displacement and rotation degrees of freedom at the vertices only, and assume that there exist constants $c_1 > 0$, c_2 such that if the plate thickness t satisfies $0 < t \leq h$, then for each element T , the local stiffness matrix K_T satisfies

$$c_1 K_T^{HCT} \leq K_T \leq c_2 K_T^{HCT} \quad (4.23)$$

where K_T^{HCT} is the HCT element level stiffness matrix of the biharmonic equation [15], with the rotations interpreted as derivatives of the transversal displacement in the HCT element.

That is, as the thickness of the plate goes to zero, the stiffness matrix of the element should be spectrally equivalent to that of the HCT element for the biharmonic equation. We refer to [14] and Section 4.1.2 for more details on HCT elements. Here we just summarize that HCT elements are in C^1 and they use cubic splines for values on element sides, linear interpolation for normal derivatives on the sides, and piecewise polynomial extension into the element interior. In [41], Assumption 46 is verified for the particular case of the DKT element [5]. Assumption 46 also holds for the following general class of non-locking $P1$ Reissner-Mindlin elements.

Theorem 47 ([41]) Assume that the energy functional for an element T is spectrally equivalent to

$$\int_T |\nabla \theta|^2 dx + \frac{1}{t^2 + h^2} \int_T |\theta - \nabla u|^2 dx, \quad (4.24)$$

where $h = \text{diam}(T)$, $u \in P_1(T)$ is the transversal displacement, and $\theta \in (P_1(T))^2$, is the rotation. Then (4.23) holds.

Elements with the energy functional of the form (4.24) include the DKT element as restated in [60]. It should be noted that for the related Timoshenko beam element, the thin limit is exactly the discretization by cubic splines of the biharmonic equation [35].

4.5.1 Assumptions

Before proving a bound on the condition number κ of the generalized FETI method, we need to introduce some specific assumptions. We refer to the model problem (2.10) and using the spectral equivalence, that we discussed in the previous section, we consider the biharmonic boundary value problem in a variational form. Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^2(\Omega),$$

where

$$a(u, v) = \int_{\Omega} \partial_{11}u \partial_{11}v + \partial_{12}u \partial_{12}v + \partial_{22}u \partial_{22}v, \quad \forall u, v \in H_0^2(\Omega)$$

Let the domain Ω be divided into non-overlapping subdomains Ω_i , $i = 1, \dots, N_s$, that are shape regular of diameter $O(H)$ (Definition 21). Without loss of generality, we suppose that $H < 1$. Furthermore, we require that, for all $r, s = 1, \dots, N_s$, if $\partial\Omega_r \cap \partial\Omega_s$ is an edge, then its length is larger than a certain prescribed fraction of the length of $\partial\Omega_r$. We note that this condition implies the connectivity condition as that of the original FETI. It is possible to prove the result using the general condition, but, for the sake of simplicity, we prove it using the stronger assumption.

We assume that the problem is discretized using reduced HCT elements. The general case of plate bending then follows from spectral equivalence of the local element stiffness matrices following Assumption 46. Let $V_h^{HCT}(\Omega)$ denote the corresponding finite element space, and h denote the characteristic element size. Each subdomain Ω_i is assumed to be a union of some of the elements. The degrees of freedom are values of the transversal displacement and its derivatives (rotations) at the nodal points of the discretization.

We define \bar{B} as follows, cf., Fig. 4.3. For each node x on subdomain interface and each pair (r, s) such that $x \in \Omega_r \cap \partial\Omega_s$ and $\partial\Omega_r$ and $\partial\Omega_s$ share an edge (i.e., do not meet only at a crosspoint), $\bar{B}w$ includes one subvector of three elements,

$$(\bar{B}w)_{rs}(x) = \sigma_{rs}(w_r(x) - w_s(x)),$$

where $w_r(x)$ and $w_s(x)$ denote subvectors of w containing the three degrees of freedom associated with node x and subdomain Ω_r and Ω_s , respectively, and σ_{rs} is chosen to be either 1 or -1 . We point out that σ_{rs} does not depend on $x \in \partial\Omega_r \cap \partial\Omega_s$, i.e., coefficients σ_{rs} are uniform along the edge between Ω_r and Ω_s . Note that, unlike in the case described in Section 4.3.1, this definition implies redundant constraints at subdomain crosspoints.

Let us make the definition of the matrix C from Section 3.2.3 more precise. Let $m = \dim \Lambda$ and $\{k_j, j = 1, \dots, n\}$ be the complete list of indices of Lagrange multipliers corresponding to conditions (expressed through B) enforcing continuity of the solution (but not the continuity of the derivatives) at crosspoints. Then $C = [c_{ij}]$ is the $m \times n$ matrix satisfying

$$c_{ij} = 1 \quad \text{if } i = k_j$$

$$c_{ij} = 0 \quad \text{otherwise.}$$

4.5.2 Discrete Norm Bounds

Using Lemma 33 for subdomain stiffness matrices K_i for each subdomain and summing over all subdomains, we obtain

$$c_1 \sum_{i=1}^{N_s} |\nabla I_{HCT} R_i w|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \leq \|w\|_S^2 \leq c_2 \sum_{i=1}^{N_s} |\nabla I_{HCT} R_i w|_{\frac{1}{2}, 2, \partial\Omega_i}^2 \quad \forall w \in W, \quad (4.25)$$

where the positive constants c_1, c_2 are independent of the characteristic mesh size h and the subdomain diameter H . The constants c_1, c_2 may depend on the regularity of the shape of subdomains. For definitions of Sobolev spaces, see, for example [1]. Throughout this section, $c, C, c_1, c_2, c_3, c_4, c_5$ and c_6 denote positive constants independent of H and h .

The next two lemmas contain the principal technical estimates.

Lemma 48 For all $\lambda \in V'$ and all $w \in \bar{W}$ such that $\bar{B}w \in V$, there exists a $\tilde{w} \in \bar{W}$ such that

$$\langle \lambda, \bar{B}\tilde{w} \rangle = \langle \lambda, \bar{B}w \rangle \quad \text{and} \quad \|\tilde{w}\|_S^2 \leq C(1 + \log H/h)^\alpha \|\bar{B}^T \bar{B}w\|_S^2,$$

where $\alpha = 1$, and $\alpha = 0$ if $\frac{1}{2}\bar{B}\bar{B}^T = I$, which happens when there are no nodes shared by more than two subdomains.

Proof. Let us first prove that, in the general case, we obtain $\alpha \leq 1$. Let $w \in \bar{W}$ and $\bar{B}w \in V$. That is $\bar{Z}^T \bar{B}^T \bar{B}w = 0$ and $C^T \bar{B}w = 0$. We define $\tilde{w} = \bar{B}^T (\bar{B}\bar{B}^T)^+ \bar{B}w$. Then, $\bar{B}\tilde{w} = \bar{B}w$. By the triangle inequality, we may write

$$\|\tilde{w}\|_S \leq \|\frac{1}{2}\bar{B}^T \bar{B}w\|_S + \|\frac{1}{2}\bar{B}^T (I - (\frac{1}{2}\bar{B}\bar{B}^T)^+) \bar{B}w\|_S. \quad (4.26)$$

Denote $z = \frac{1}{2}\bar{B}^T (I - (\frac{1}{2}\bar{B}\bar{B}^T)^+) \bar{B}w$. From the definition of \bar{B} , z is zero at all nodes that belong to at most two subdomains. The remaining

nodes lie on subdomain crosspoints. At every such node, $I_{HCT}R_i z(x)$ is a linear combination of the entries of $\bar{B}^T \bar{B}w$ that correspond to the same node x and the coefficients of the linear combinations are bounded only in terms of the number of subdomains to which the node belongs. In addition, since $C^T \bar{B}w = 0$, the transversal displacement components of z at crosspoints are zero. Using (4.25) and Lemma 24 for the subdomain crosspoint vertices, one subdomain at a time, we obtain

$$\|z\|_S^2 \leq C(1 + \log \frac{H}{h}) \sum_{i=1}^{N_s} \left(\frac{1}{H} \|\nabla I_{HCT}R_i \bar{B}^T \bar{B}w\|_{0,2,\partial\Omega_i}^2 + |\nabla I_{HCT}R_i \bar{B}^T \bar{B}w|_{\frac{1}{2},2,\partial\Omega_i}^2 \right)$$

This together with the Poincaré inequality, Lemma 34, and (4.26) yields the result.

If $\frac{1}{2}\bar{B}\bar{B}^T = I$, we simply choose $\tilde{w} = \frac{1}{2}\bar{B}^T \bar{B}w = w$. \square

Now we derive the converse bound.

Lemma 49 For all $\lambda \in V'$ and $w \in \bar{W}, w \perp \text{Ker } S$, there is a $\tilde{w} \in \bar{W}$ such that $\bar{B}\tilde{w} \in V$,

$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle, \quad \text{and} \quad \|\bar{B}^T \bar{B}\tilde{w}\|_S^2 \leq C \left(1 + \log \frac{H}{h}\right)^2 \|w\|_S^2.$$

Proof. Let $\lambda \in V', w \in \bar{W}, w \perp \text{Ker } S$, and consider $\bar{B}\tilde{w} = P\bar{B}w + PFPC\alpha$. Since $\lambda \in V'$, it is easily verified that $\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle$. Vector α can be found from the condition $C^T \bar{B}\tilde{w} = 0$, which can be rewritten as

$$\langle PFPC\alpha, C\tilde{\alpha} \rangle = -\langle P\bar{B}w, C\tilde{\alpha} \rangle \quad \forall \tilde{\alpha}.$$

By the definition of F , we can rewrite this as

$$\langle S^+ \bar{B}^T PC\alpha, \bar{B}^T PC\tilde{\alpha} \rangle = -\langle w, \bar{B}^T PC\tilde{\alpha} \rangle \quad \forall \tilde{\alpha}.$$

Since S^+ is positive semidefinite, the equation yields α such that

$$\langle S^{+1/2} \bar{B}^T PC\alpha, S^{+1/2} \bar{B}^T PC\alpha \rangle = -\langle S^{1/2} w, S^{+1/2} \bar{B}^T PC\alpha \rangle$$

and, from the Cauchy inequality, $\|S^{+1/2} \bar{B}^T PC\alpha\| \leq \|S^{1/2} w\|$. Therefore,

$$\|S^+ \bar{B}^T PC\alpha\|_S \leq \|w\|_S. \quad (4.27)$$

We need to estimate $\|\bar{B}^T \bar{B} \tilde{w}\|_S$. Let \bar{B}_{ij} be the matrix constructed from \bar{B} by zeroing out all the rows that do not correspond to the interface conditions between Ω_i and Ω_j . Then,

$$\bar{B}^T \bar{B} \tilde{w} = \sum_{i,j=1, i < j}^{N_s} \bar{B}_{ij}^T \bar{B} \tilde{w},$$

and, by triangle inequality and (4.25),

$$\|\bar{B}^T \bar{B} \tilde{w}\|_S \leq \sum_{i,j=1, i < j}^{N_s} \left(|\nabla I_{HCT} \bar{B}_{ij}^T \bar{B} \tilde{w}|_{\frac{1}{2}, 2, \partial\Omega_i} + |\nabla I_{HCT} \bar{B}_{ij}^T \bar{B} \tilde{w}|_{\frac{1}{2}, 2, \partial\Omega_j} \right).$$

Since $C^T \bar{B} \tilde{w} = 0$, Lemma 30 can be applied to $\nabla I_{HCT} \bar{B}_{ij}^T \bar{B} \tilde{w}$, $i, j = 1, \dots, N_s$.

It follows that

$$\|\bar{B}^T \bar{B} \tilde{w}\|_S^2 \leq c_1 \sum_k \left(|\nabla I_{HCT} \bar{B} \tilde{w}|_{\frac{1}{2}, 2, \Gamma_k}^2 + \left(1 + \log \frac{H}{h}\right) \|\nabla I_{HCT} \bar{B} \tilde{w}\|_{0, \infty, \Gamma_k}^2 \right),$$

where summation is carried out over all edges Γ_k of the intersubdomain interface. The L^∞ norm, by Lemma 31, can be bounded as follows

$$\begin{aligned} & \|\nabla I_{HCT} \bar{B} \tilde{w}\|_{L^\infty(\Gamma_k)}^2 \leq \\ & c_2 \left(1 + \log \frac{H}{h}\right) \left(\frac{1}{H} \|\nabla I_{HCT} \bar{B} \tilde{w}\|_{0, 2, \Gamma_k}^2 + |\nabla I_{HCT} \bar{B} \tilde{w}|_{\frac{1}{2}, 2, \Gamma_k}^2 \right). \end{aligned}$$

Thus,

$$\begin{aligned} & \|\bar{B}^T \bar{B} \tilde{w}\|_S^2 \leq \\ & c_1 (c_2 + 1) \left(1 + \log \frac{H}{h}\right)^2 \sum_k \left(\frac{1}{H} \|\nabla I_{HCT} \bar{B} \tilde{w}\|_{0, 2, \Gamma_k}^2 + |\nabla I_{HCT} \bar{B} \tilde{w}|_{\frac{1}{2}, 2, \Gamma_k}^2 \right). \end{aligned}$$

Denote $u = w + S^+ \bar{B}^T P C \alpha$. Then, $\bar{B} \tilde{w} = P \bar{B} u$. The triangle inequality and (4.27) yield

$$\|u\|_S \leq \|w\|_S + \|S^+ \bar{B}^T P C \alpha\|_S \leq 2\|w\|_S.$$

So now we only have to prove that

$$\sum_k \left(\frac{1}{H} \|\nabla I_{HCT} P \bar{B} u\|_{0,2,\Gamma_k}^2 + |\nabla I_{HCT} P \bar{B} u|_{\frac{1}{2},2,\Gamma_k}^2 \right) \leq c_3 \|u\|_S^2.$$

We first use the triangle inequality to get

$$|\nabla I_{HCT} P \bar{B} u|_{\frac{1}{2},2,\Gamma_k} \leq |\nabla I_{HCT} \bar{B} u|_{\frac{1}{2},2,\Gamma_k} + |\nabla I_{HCT} (I - P) \bar{B} u|_{\frac{1}{2},2,\Gamma_k}.$$

From the definition of \bar{B} , each entry reduces to a linear combination with coefficients ± 1 , constant along every edge. Since $(I - P) \bar{B} u \in \text{Im } G$, $I_{HCT} (I - P) \bar{B} u$ is a restriction of a linear function on every edge, $|\nabla I_{HCT} (I - P) \bar{B} u|_{\frac{1}{2},2,\Gamma_k} = 0$.

Furthermore,

$$\sum_k |\nabla I_{HCT} \bar{B} u|_{\frac{1}{2},2,\Gamma_k}^2 \leq 2 \sum_{i=1}^{N_s} |\nabla I_{HCT} u|_{\frac{1}{2},2,\partial\Omega_i}^2. \quad (4.28)$$

To estimate the L^2 terms, we again first use the triangle inequality:

$$\|\nabla I_{HCT} P \bar{B} u\|_{0,2,\Gamma_k} \leq 2 \|\nabla I_{HCT} u\|_{0,2,\Gamma_k} + \|\nabla I_{HCT} (I - P) \bar{B} u\|_{0,2,\Gamma_k}.$$

Since $\nabla I_{HCT} (I - P) \bar{B} u$ is a linear function on every edge and P is an orthogonal projection, we have

$$\begin{aligned} \sum_k \|\nabla I_{HCT} (I - P) \bar{B} u\|_{0,2,\Gamma_k}^2 &\leq c_4 h \|(I - P) \bar{B} u\|^2 \\ &\leq c_5 h \|\bar{B} u\|^2 \leq 2c_5 h \|u\|^2. \end{aligned}$$

Furthermore, Lemma 32 shows that

$$h \|u\|^2 \leq c_6 (1 + H^2) \sum_{i=1}^{N_s} \|\nabla I_{HCT} u\|_{0,2,\partial\Omega_i}^2$$

Finally, since $u \in \text{Ker } S$, the Poincaré inequality (Lemma 34) and the equivalence of the norms (4.25) conclude the estimate. \square

4.5.3 Condition Number Estimate

We have now everything ready to prove the final result.

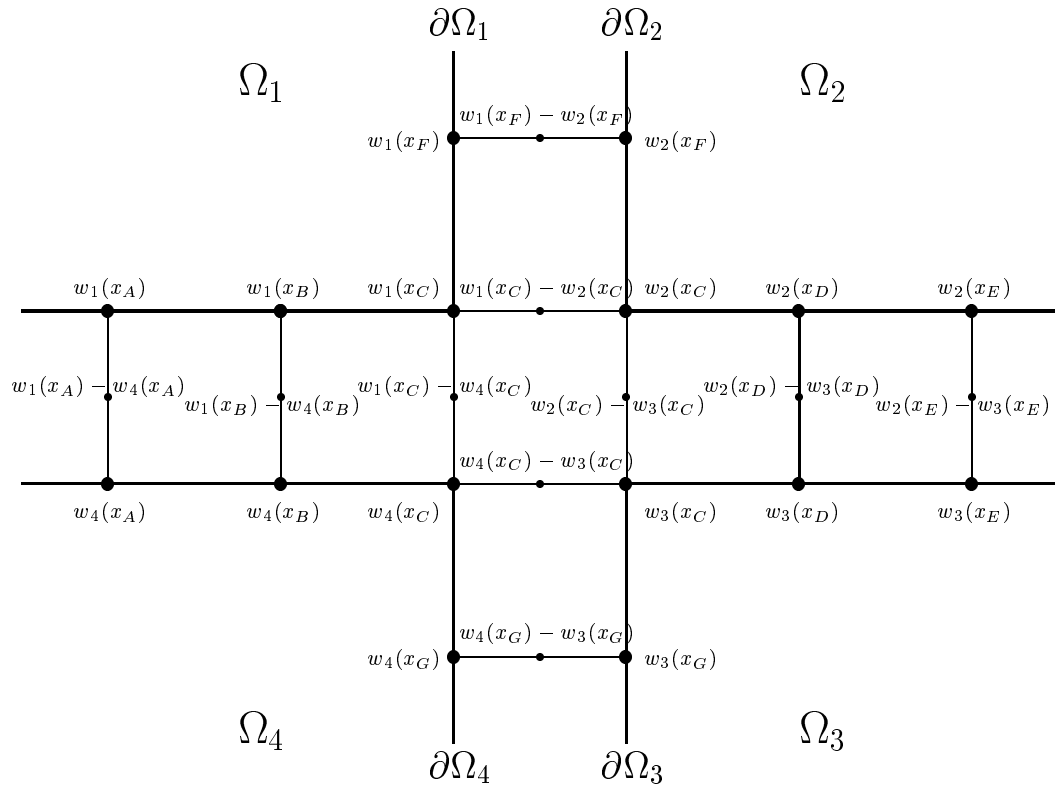
Theorem 50 Under the assumptions of Section 4.5.1, the condition number of the generalized FETI method with the Dirichlet preconditioner (3.39) satisfies

$$\kappa = \frac{\lambda_{max}(QDPF)}{\lambda_{min}(QDPF)} \leq C \left(1 + \log \frac{H}{h}\right)^\gamma$$

with $\gamma = 3$, and $\gamma = 2$ if there are no crosspoints between more than two subdomains.

Proof. Since V and V' can be replaced by the factorspaces \tilde{V} and \tilde{V}' , Lemmas 48 and 49 verify the assumptions (i) and (ii), of Theorem 38, with $C_1 = C(1 + \log H/h)^\alpha$, $\alpha = 0$ or 1 , and $C_2 = C(1 + \log H/h)^2$, respectively. \square

Figure 4.3. Definition of B for plate bending



5. Computational Results

We refer to the paper by Farhat et al. [32] for numerical results for second order problems confirming our theoretical results. The paper also deals with parallel implementation issues and performance. Here we summarize results for plate bending problems based on the paper by Mandel, Tezaur and Farhat [53], due to Farhat.

We consider the plate bending problem on a unit square. The plate is discretized by a uniform mesh of three-node triangular DKT plate elements and subjected to a uniform pressure load. The thickness of the plate is 10^{-3} , the Young modulus $E = 10^6$ and the Poisson coefficient $\nu = 0.3$. The FETI and generalized FETI methods are preconditioned by the Dirichlet preconditioner (Section 3.3) [32, 34], and the following stopping criterion is used

$$\frac{\|z_{k-1}\|}{\|f\|} \leq \varepsilon = 10^{-3} \quad (5.1)$$

where z_{k-1} is the preconditioned residual in Algorithm 20. This condition is a good estimator of the global error as shown in [34].

We construct several meshes with different mesh size h and several mesh partitions with different subdomain size H , and report the performance of the original FETI and the generalized FETI. Three series of computational experiments are reported. The condition number is denoted by κ and the number of iterations by n_{it} in the tables.

The first series of experiments shows the performance of the method

for three different numbers of subdomains ($H = 1/2, 1/4, 1/8$) and three different meshes corresponding to $h = 1/10$, $h = 1/20$, and $h = 1/40$. Results in Table 5.1) demonstrate that for a given H , the condition number of the original FETI grows fast with the mesh size h , while that of the new FETI method is much smaller and grows only weakly with h . For large number of subdomains ($N_s = 64$) the new FETI method converges about 7 times faster than the original one.

In the second series of experiments (Table 5.2), the mesh size is fixed to $h = 1/120$ (28800 elements and 86400 degrees of freedom), and H is varied between $H = 1/2$ (4 subdomains) and $H = 1/12$ (144 subdomains). In this case, the condition numbers of both FETI methods are shown to decrease with the number of subdomains. This is an expected result because when h is fixed and H is decreased, the size of the coarse problem increases for both algorithms. The generalized FETI shows only a small dependence on the number of subdomains as predicted by the theory. For large number of subdomains, the new FETI method clearly outperforms the original FETI.

Finally, the subdomain problem size is fixed to $h/H = 1/15$, and the number of subdomains as well as the size of the global problem are increased. The performance results reported in Table 5.3 show that in this case too, the new FETI method outperforms significantly the original one.

Since solution time is ultimately the most important criterion for assessing performance, Both FETI methods were also benchmarked for the same plate bending problem described above with 960000 degrees of freedom and 64 subdomains. The performance results obtained on a 64-processor IBM SP2 are summarized in Table 5.4. They show that even though the new FETI

Table 5.1. Fixed number of subdomains, series of refined meshes

	FETI		Generalized FETI	
$\frac{h}{H}$	n_{it}	κ	n_{it}	κ
2x2 subdomains, $H=\frac{1}{2}$				
$\frac{1}{10}$	18	2578	12	7.6
$\frac{1}{20}$	22	30101	15	12.6
$\frac{1}{40}$	26	409987	17	18.6
4x4 subdomains, $H=\frac{1}{4}$				
$\frac{1}{10}$	61	6795	21	11.5
$\frac{1}{20}$	86	84199	27	17
$\frac{1}{40}$	119	1038120	36	24.4
8x8 subdomains, $H=\frac{1}{8}$				
$\frac{1}{10}$	172	21707	25	13
$\frac{1}{20}$	247	275004	34	19.4
$\frac{1}{40}$	323	3920613	42	27.6

Table 5.2. Fixed global mesh 120x120, $h=\frac{1}{120}$, series of refined mesh partitions

Decomposition		FETI		Generalized FETI	
H	$\frac{h}{H}$	n_{it}	κ	n_{it}	κ
$\frac{1}{2}$	$\frac{1}{60}$	27	2079032	18	23.2
$\frac{1}{3}$	$\frac{1}{40}$	64	839240	29	22.4
$\frac{1}{4}$	$\frac{1}{30}$	104	391470	32	21
$\frac{1}{5}$	$\frac{1}{24}$	135	234504	33	19.9
$\frac{1}{6}$	$\frac{1}{20}$	164	160173	32	18.6
$\frac{1}{8}$	$\frac{1}{15}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{12}$	255	63896	29	14.9
$\frac{1}{12}$	$\frac{1}{10}$	245	46921	27	13.6

Table 5.3. Fixed local mesh ($15 \times 15, \frac{h}{H} = \frac{1}{15}$), series of refined meshes and mesh partitions

Decomposition		FETI		Generalized FETI	
H	h	n_{it}	κ	n_{it}	κ
$\frac{1}{2}$	$\frac{1}{30}$	20	11088	13	10
$\frac{1}{3}$	$\frac{1}{45}$	49	19004	21	13.4
$\frac{1}{4}$	$\frac{1}{60}$	74	29041	25	14.6
$\frac{1}{5}$	$\frac{1}{75}$	109	40120	28	15.4
$\frac{1}{6}$	$\frac{1}{90}$	145	55068	29	15.9
$\frac{1}{8}$	$\frac{1}{120}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{150}$	318	144556	32	16.9

Table 5.4. Performance results for 960000 degrees of freedom and 64 subdomains

FETI			Generalized FETI		
n_{it}	Total time	Time per iter.	n_{it}	Total time	Time per iter.
314	265 s	0.8 s	45	105 s	1.1 s

method consumes an amount of CPU time (55.5 s.) equivalent to that of 50 of its iterations to set up and preprocess the coarse problem (3.33), and even though each of its iterations is 1.3 times more expensive than an iteration of the original FETI method, the new FETI method is 2.5 times faster than the original one at solving the system of 960000 plate bending equations on a 64-processor IBM SP2.

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