

# Solving Symmetric Indefinite Systems with Symmetric Positive Definite Preconditioners

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- Brief intro: iterative methods, SPD and non-SPD preconditioning
- An ideal SPD preconditioner for a symmetric indefinite linear system
- **Absolute value preconditioning.** Definition
- Absolute value preconditioners for linear systems with strictly (block) diagonally dominant matrices
- MG absolute value preconditioner for a model problem.  
Numerical examples
- Conclusions

# Symmetric indefinite linear systems

We consider a nonsingular linear system

$$Ax = b, \quad A = A^* \in \mathbb{R}^{n \times n}.$$

## Several origins of the problem

- Mixed finite element discretizations of PDEs in fluid and solid mechanics, acoustics
- Inner steps of interior point methods in linear and nonlinear optimization
- Solution of the correction equation in the Jacobi-Davidson method for a symmetric eigenvalue problem

## General setting

- Large problem size, ill-conditioned
- Sparse matrices or matrix-free environment
- Direct methods are inapplicable. Iterate!
- Use of preconditioners to improve the convergence

# How to solve a symmetric indefinite linear system?

What iterative method shall we use to approximate the solution of

$$Ax = b, \quad A = A^*$$

Let  $T$  be a *preconditioner*. Consider the *preconditioned* linear system

$$TAx = Tb.$$

- $T$  is *symmetric indefinite* (e.g.,  $T \approx A^{-1}$ ) or *nonsymmetric*.  
 $\Rightarrow TA$  is generally *nonsymmetric* in any inner product.

The symmetry is lost.

Possible solution techniques: **GMRES**, **BiCG**, **BiCGstab**, **QMR**, etc.

- $T$  is *symmetric positive definite (SPD)*.  
 $\Rightarrow TA$  is *symmetric* in the  $T^{-1}$ -based inner product;  $(x, y)_{T^{-1}} = (x, T^{-1}y)$ .

The symmetry is preserved.

It is possible to use an **optimal short-term recurrent scheme**, e.g., **PMINRES**.

# The Preconditioned Minimal Residual method

Let  $T$  be an SPD preconditioner.

**The Preconditioned Minimal Residual method**, at step  $i$ , constructs an approximation  $x^{(i)}$  to the solution of system  $Ax = b$  of the form

$$x^{(i)} \in x^{(0)} + \mathcal{K}_i(TA, Tr^{(0)}),$$

such that the residual vector  $r^{(i)} = b - Ax^{(i)}$  satisfies the optimality condition

$$\|r^{(i)}\|_T = \min_{u \in \mathcal{K}_i(TA, Tr^{(0)})} \|r^{(0)} - u\|_T,$$

where  $\mathcal{K}_i(TA, Tr^{(0)}) = \text{span} \{Tr^{(0)}, (TA)Tr^{(0)}, \dots, (TA)^{i-1}Tr^{(0)}\}$  is the  $i$ -dimensional preconditioned Krylov subspace,  $x^{(0)}$  is the initial guess.

**Stable implementation:** The PMINRES algorithm (Paige, Saunders, 1975).

**Question:** How do we define the SPD preconditioner  $T$ ?

# Matrix absolute value

Consider the eigenvalue decomposition of a symmetric matrix  $A$ ,

$$A = V\Lambda V^*,$$

where  $V$  is an orthogonal matrix of eigenvectors and  $\Lambda = \text{diag}\{\lambda_j\}$  is a diagonal matrix of eigenvalues of  $A$ .

- The **matrix absolute value** of  $A$  is defined as

$$|A| = V|\Lambda|V^*, \quad |\Lambda| = \text{diag}\{|\lambda_j|\}.$$

- The **matrix sign** of  $A$  is defined as

$$\text{sign}(A) = V\text{sign}(\Lambda)V^*, \quad \text{sign}(\Lambda) = \text{diag}\{\text{sign}(\lambda_j)\}.$$

The **polar decomposition** of a symmetric matrix can be written as

$$A = |A| \text{sign}(A) = \text{sign}(A) |A|.$$

# Inverse of the matrix absolute value as an ideal SPD preconditioner

Let  $T = |A|^{-1}$ . The preconditioned linear system  $TAx = Tb$  is

$$\text{sign}(A)x = |A|^{-1} b.$$

The matrix  $TA = \text{sign}(A)$  has only two distinct eigenvalues:  $-1$  and  $1$ .

⇒ The Preconditioned Minimal Residual method converges to the exact solution in at most two steps.

- $T = |A|^{-1}$  is an *ideal* SPD preconditioner for a symmetric indefinite linear system
- Construction of the *exact*  $|A|^{-1}$  is generally prohibitively expensive
- **Our idea:** Construct a *practical* SPD preconditioner  $T$  as an *approximation* to the ideal preconditioner, i.e.,

$$T \approx \alpha |A|^{-1}.$$

# Absolute value preconditioning

We call  $T$  an **absolute value preconditioner** for a nonsingular symmetric indefinite linear system  $Ax = b$ , if

$$T \approx \alpha |A|^{-1}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

*In what sense do we understand “ $\approx$ ”? How to measure the quality of  $T$ ?*

- For a general symmetric indefinite linear system, construct  $T$  to attain a relatively small norm

$$\|T - \alpha |A|^{-1}\|.$$

This can be done, e.g., by approximating the action of a *matrix function*  $f(A) = |A|^{-1}$  on a vector. Typically too costly.

- For *mesh problems*, construct  $T$ , such that  $|A|$  and  $T^{-1}$  are *spectrally equivalent*, i.e.,

$$\delta_0(v, T^{-1}v) \leq (v, |A|v) \leq \delta_1(v, T^{-1}v), \quad \forall v \in \mathbb{R}^n,$$

where  $\delta_1 \geq \delta_0 > 0$  are independent of  $h$ .

# Spectrally equivalent absolute value preconditioning

## Theorem

Let us be given a symmetric indefinite  $A \in \mathbb{R}^{n \times n}$ , an SPD  $T \in \mathbb{R}^{n \times n}$ , and constants  $\delta_1 \geq \delta_0 > 0$ , such that

$$\delta_0(v, T^{-1}v) \leq (v, |A|v) \leq \delta_1(v, T^{-1}v), \quad \forall v \in \mathbb{R}^n.$$

Then all the eigenvalues of  $TA$  are located in the union of two intervals

$$[-\delta_1, -\delta_0] \cup [\delta_0, \delta_1].$$

Can we construct an actual example of an absolute value preconditioner?

# Absolute value preconditioners for strictly (block) diagonally dominant matrices

Matrix  $A = \{A_{ij}\} \in \mathbb{R}^{n \times n}$ ,  $i, j = 1, \dots, s$ , is strictly block diagonally dominant if

$$(\|A_{ii}^{-1}\|)^{-1} > \sum_{\substack{j=1 \\ j \neq i}}^s \|A_{ij}\|, \quad i = 1, \dots, s.$$

## Theorem

Let  $A$  be a strictly block diagonally dominant symmetric indefinite matrix, such that

$$\delta (\|A_{ii}^{-1}\|)^{-1} \geq \sum_{\substack{j=1 \\ j \neq i}}^s \|A_{ij}\|, \quad i = 1, \dots, s,$$

for a fixed  $\delta \in [0, 1)$ . Let  $T = \text{diag}\{|A_{11}|^{-1}, |A_{22}|^{-1}, \dots, |A_{ss}|^{-1}\}$ . Then all the eigenvalues of the matrix  $TA$  are located in the union of intervals

$$\{y \in \mathbb{R} : |y + 1| \leq \delta\} \cup \{y \in \mathbb{R} : |y - 1| \leq \delta\}.$$

# Absolute value preconditioner for the model problem

Consider the “**shifted Laplacian**” equation on a unit square with Dirichlet boundary conditions and a relatively small shift value,

$$-\Delta u(x, y) - c^2 u(x, y) = f(x, y), \quad (x, y) \in \Omega = (0, 1) \times (0, 1) \\ u|_{\Gamma} = 0.$$

The discretization of the boundary value problem using a standard 5-point FD stencil on a uniform mesh leads to the linear system

$$(L - c^2 I)x = b.$$

- The shifted negative discrete Laplace operator  $L - c^2 I$  is, in general, symmetric and indefinite. We assume it to be nonsingular
- As a preconditioner, choose  $T \approx \alpha |L - c^2 I|^{-1}$   
(in the sense of the spectral equivalence of  $T^{-1}$  and  $|L - c^2 I|$ )
- Use a geometric MG approach to construct  $w = Tr$

# Two-grid absolute value preconditioner for the model problem

## Algorithm (Two-grid absolute value preconditioner)

Input  $r$ , output  $w = Tr$ .

- ① **Pre-smoothing.** Apply  $\nu$  pre-smoothing steps,

$$w^{(i+1)} = w^{(i)} + M^{-1}(r - Lw^{(i)}), \quad i = 0, \dots, \nu - 1, \quad w^{(0)} = 0.$$

This step results in the pre-smoothed vector  $w^{pre} = w^{(\nu)}$ ,  $\nu \geq 1$ .

- ② **Coarse grid correction.** Restrict  $r - Lw^{pre}$  to the coarse grid ( $R$ ), multiply it by  $|L_H - c^2 I_H|^{-1}$ , prolongate to the fine grid ( $P$ ), and add to  $w^{pre}$ ,

$$w^{cgc} = w^{pre} + P |L_H - c^2 I_H|^{-1} R (r - Lw^{pre}).$$

- ③ **Post-smoothing.** Apply  $\nu$  post-smoothing steps,

$$w^{(i+1)} = w^{(i)} + M^{-*}(r - Lw^{(i)}), \quad i = 0, \dots, \nu - 1, \quad w^{(0)} = w^{cgc}.$$

Return  $w = w^{(\nu)}$ .

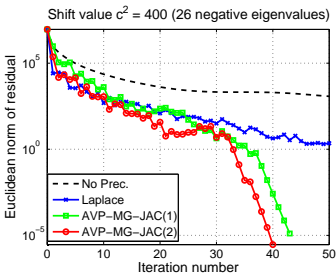
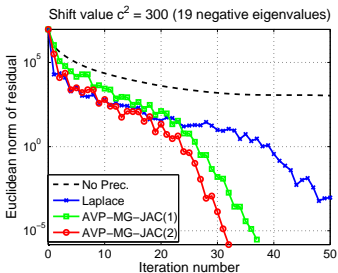
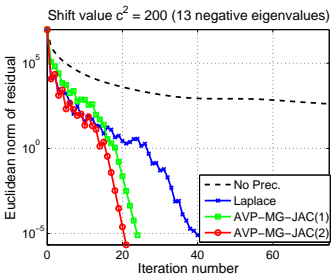
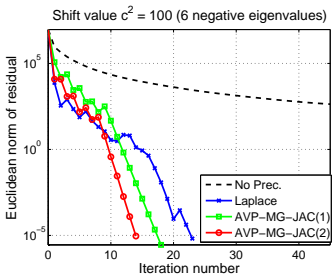
# MG absolute value preconditioner for the model problem

- The resulting preconditioner  $T$  is SPD under mild assumptions on the smoother, restriction and prolongation
- **In practice, we use the MG extension of the two-grid algorithm (“V-cycle”)**
- Compare our MG absolute value preconditioner with the preconditioner based on the inverse of the Laplacian (A. Bayliss, C. Goldstein, E. Turkel, 1983)

In our numerical tests

- Standard coarsening scheme. The coarsest grid is of the mesh size  $2^{-4}$ , and the finest grid is of the mesh size  $2^{-7}$
- Full weighting for the restriction and piecewise multilinear interpolation for the prolongation
- The smoother:  $\omega$ -damped Jacobi,  $\omega = 4/5$

# MG absolute value preconditioner for the model problem



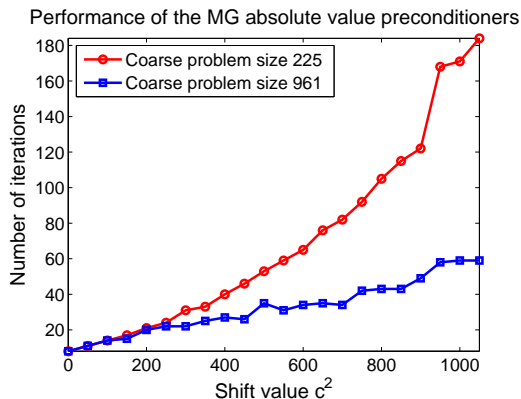
# Absolute value preconditioner for the model problem

Number of steps performed to achieve the decrease by the factor  $10^{-8}$  in the error norm.

	$h = 2^{-7}$	$h = 2^{-8}$	$h = 2^{-9}$	$h = 2^{-10}$
$c^2 = 100$	15	14	14	14
$c^2 = 200$	21	21	21	21
$c^2 = 300$	31	32	32	30
$c^2 = 400$	40	39	40	40

**Table:** Mesh-independent convergence of PMINRES with the MG absolute value preconditioner

# MG absolute value preconditioner for the model problem



**Figure:** Performance of the MG absolute value preconditioners for the model problem with different shift values. The problem size  $n = (2^7 - 1)^2 \approx 1.6 \times 10^4$ . The number of negative eigenvalues varies from 0 to 75. The initial error norm is decreased by  $10^{-8}$ .

# Conclusions, current and future work

## Conclusions

- 1 Introduced (SPD) absolute value preconditioning for symmetric indefinite linear systems
- 2 Constructed several examples of absolute value preconditioners
- 3 Good results for the “shifted Laplacian” with a relatively small shift value

## Current and future work

- Absolute value preconditioning for nonsymmetric linear systems, eigenvalue and singular value problems
- Algebraic multilevel absolute value preconditioning

Thank you!