Extension theorems for Lamé equations for nearly incompressible media with applications to numerical solution of problems with highly discontinuous coefficients

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Abstract

We prove in Bakhvalov et al. [2002] extension theorems in the energy norms described by Stokes and Lamé operators for the three-dimensional case with periodic boundary conditions typical in homogenization procedures for periodic media. For the Lamé equations, we show that the extension theorem holds for nearly incompressible media, but may fail in the opposite limit, i.e. for case of absolutely compressible media. We study carefully the latter case and associate it with the Cosserat problem.

Extension theorems serve as an important tool in many applications, e.g., in domain decomposition and fictitious domain methods, and in analysis of finite element methods.

We consider an application of established extension theorems to an efficient iterative solution technique for the isotropic linear elasticity equations for nearly incompressible media and for the Stokes equations with highly discontinuous coefficients in a periodic cell. The iterative method involves a special choice for an initial guess and a preconditioner based on solving a constant coefficient problem. Such preconditioner allows the use of well-known fast algorithms for preconditioning. Under some natural assumptions on smoothness and topological properties of subdomains with small coefficients, we prove in Bakhvalov et al. [2002] convergence of the simplest Richardson method uniform in the jump of coefficients. For the Lame equations, the convergence is also uniform in the incompressible limit, cf. Bakhvalov and Knyazev [1992, 1994]. Our iterative solver for the linear elasticity equations has been used to compute effective elastic moduli for incompressible material with a periodic system of pores in Bakhvalov et al. [1996] and independently rediscovered in Michel et al. [1999].

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The talk is based on Bakhvalov et al. [2002].

Stokes equations

We consider a three-dimensional periodic boundary value problem for Stokes equations in the form

$$\begin{aligned} 2\frac{\partial\left[\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\right]}{\partial x_i} + \frac{\partial p}{\partial x_j} &= \frac{\partial f_{ij}}{\partial x_i}, \quad j = 1,2,3, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\varepsilon_{ij}(\mathbf{u}) \overset{\text{def}}{=} \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$. For our periodic boundary value problem we assume that all functions are defined on a three-dimensional unit torus $\mathbf{T}$ with Cartesian coordinate system.

Let **H** be the factor space with regard to the constants from R^3 of the space of *solenoidal* functions from $[W_2^1(\mathbf{T})]^3$, equipped with the norm $\|\cdot\|_{[W_2^1(\mathbf{T})]^3}$. Given $f_{ij} \in L_2(\mathbf{T})$, a weak solution of problem (1) is a function $\mathbf{u} \in \mathbf{H}$, satisfying

$$\Lambda(\mathbf{u},\mathbf{v}) \quad = \quad \int_{\mathbf{T}} f_{ij} \frac{\partial v_j}{\partial x_i} d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}, \tag{2}$$

where $\Lambda(\mathbf{u},\mathbf{v}) \overset{\text{def}}{=} 2\int_{\mathbf{T}} \mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})d\mathbf{x}$.

Let $\mathbf{D} \subset \mathbf{T}$ be a Lipschitz domain and its complement $\mathbf{T}\backslash\overline{\mathbf{D}}$ be also a domain. We assume that the viscosity coefficient $\mu()$ is piece-wise constant, and equals to $\mu > 0$ and $\mu^* > 0$ in $\mathbf{D}$ and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly.

Extension of solenoidal functions

Announced in our earlier paper Bakhvalov and Knyazev [1992], it also appeared in several different forms in Bakhvalov [1995, 1999], D'yakonov [1996], Jikov et al. [1994], Quarteroni and Valli [1999].

Theorem 1 *There exists a constant* $\kappa = \kappa(\mathbf{D}) > 0$, *such that for an arbitrary function* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$ *such that* $\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x} \geq \kappa \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x}$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$ *with some* 3×3 *matrix* $C = -C^T$ *independent of* $\mathbf{x}$.

Lamé Equations

We consider a three-dimensional periodic boundary value problem for the isotropic linear elasticity (Lamé) equations:

$$2\frac{\partial\left[\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\right]}{\partial x_i}+\frac{\partial[\lambda(\mathbf{x})\operatorname{div}\mathbf{u}]}{\partial x_j} = \frac{\partial f_{ij}}{\partial x_i},\quad j=1,2,3, \tag{3}$$

where $\mathbf{u}=(u_1,u_2,u_3)^T$ and $\varepsilon_{ij}(\mathbf{u})\stackrel{\text{def}}{=}\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right].$

We re-define the space $\mathbf{H} \overset{\text{def}}{=} [W_2^1(\mathbf{T})]^3/R^3$, i.e. we no longer require vectors in $\mathbf{H}$ to be solenoidal. Then a weak solution to problem (3), given $f_{ij} \in L_2(\mathbf{T})$, is a function $\mathbf{u} \in \mathbf{H}$ such that

$$\Lambda(\mathbf{u},\mathbf{v}) = \int_{\mathbf{T}} f_{ij} \frac{\partial v_j}{\partial x_i} d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}, \tag{4}$$

where $\Lambda(\mathbf{u},\mathbf{v}) \overset{\text{def}}{=} \int_{\mathbf{T}} [2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) + \lambda(\mathbf{x})\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}]d\mathbf{x}$. We assume that Lamé coefficients $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are piece-wise constant and equal to (λ,μ) and (λ^*,μ^*) in $\mathbf{D}$ and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly. Also, we assume

$$\mu,\mu^* > 0, K \equiv \lambda + \frac{2}{3}\mu > 0, K^* \equiv \lambda^* + \frac{2}{3}\mu^* > 0, 0 < \eta \le \frac{\mu}{\mu^*}\frac{K^*}{K} \le \eta^{-1}, \tag{5}$$

for some constant $\eta \le 1$.

Extension theorem in the Lamé energy norm. Incompressible limit.

Let us define on **H** the following bilinear form:

$$\Lambda^*(\mathbf{u},\mathbf{v}) \overset{\text{def}}{=} 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) + \lambda^* \int_{\mathbf{T}} \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, d\mathbf{x},$$

and let the forms $\Lambda^*_{\mathbf{D}}(\mathbf{u},\mathbf{v})$ and $\Lambda^*_{\mathbf{T}\setminus\overline{\mathbf{D}}}(\mathbf{u},\mathbf{v})$ be defined in the same way, except for the domain of integration.

Theorem 2 *Let* $\dfrac{K^*}{\mu^*} \geq \delta > 0$. *Then, there exist a positive constant* $\kappa = \kappa(\delta,\mathbf{D})$, *such that for every* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$, *such that* $\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) \geq \kappa\Lambda^*(\mathbf{w},\mathbf{w})$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* **D**, *where* $C = -C^T$ *is some* 3×3 *matrix independent of* **x**.

A weaker version of Theorem 2 has been proven in Bakhvalov and Knyazev [1994], where the constant $\kappa \to 0$ in the incompressible limit.

Extension may fail in the compressible limit

Next, we show that the result of Theorem 2 is sharp in the sense that, in general, $\kappa \rightarrow 0$ when $\dfrac{K^*}{\mu^*} \rightarrow 0$.

Theorem 3 *If the domain* $\mathbf{D}$ *does not wrap around the torus* $\mathbf{T}$*, then there exist a nontrivial function* $\mathbf{v} \in \mathbf{H}$*, such that for any function* $\mathbf{w} \in \mathbf{H}$*, such that* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$ *for some* 3×3 *matrix* $C = -C^T$ *the ratio* $\dfrac{\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})}{\Lambda^*(\mathbf{w},\mathbf{w})} \rightarrow 0$ *as* $\dfrac{K^*}{\mu^*} \rightarrow 0$*. Namely, an example of such vector function is* $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ *in* $\mathbf{D}$ *and extended as a function in* $[W_2^1(\mathbf{T})]^3$ *outside* $\mathbf{D}$.

If $\mathbf{D}$ wraps around the torus $\mathbf{T}$ at least in one direction, it seems possible to prove Theorem 2 even if $\dfrac{K^*}{\mu^*} \rightarrow 0$.

Compressible limit null space and Cosserat eigenfunctions

Theorem 4 *The kernel of the quadratic form* $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$, $\mathbf{v} \in [W_2^1(\mathbf{D})]^3$, *with* $K^* = 0$, *in addition to the standard rigid body motions (translations and rotations), includes the 4-dimensional space spanned by*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix};\ \begin{pmatrix} \frac{1}{2}(x_1^2 - x_2^2 - x_3^2) \\ x_1x_2 \\ x_1x_3 \end{pmatrix};\ \begin{pmatrix} x_1x_2 \\ \frac{1}{2}(x_2^2 - x_1^2 - x_3^2) \\ x_2x_3 \end{pmatrix};\ \begin{pmatrix} x_1x_3 \\ x_2x_3 \\ \frac{1}{2}(x_3^2 - x_1^2 - x_2^2) \end{pmatrix}$$

These are the four eigenfunctions of the Cosserat eigenproblem with Neumann boundary conditions corresponding to the eigenvalue K^*=0, see Mikhlin [1973]. For a sphere, those functions were found in 1901 by Eugène and François Cosserat Cosserat and Cosserat [1901a,b].

Iterative linear solver

$$\Lambda^*\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v}\right)+\Lambda(\mathbf{u}^n,\mathbf{v}) \quad = \quad \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v}\in\mathbf{H}, \;\; n=0,1,\ldots, \tag{6}$$

where the initial guess $\mathbf{u}^0 \in \mathbf{H}$ is determined by the equation

$$\Lambda^*(\mathbf{u}^0,\mathbf{v}) \quad = \quad \int_{\mathbf{T}} g_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v}\in\mathbf{H}, \tag{7}$$

with

$$g_{ij}=\begin{cases} \text{an arbitrary function of } L_2(\mathbf{D}) & \text{in} \quad \mathbf{D}, \\ f_{ij} & \text{in} \quad \mathbf{T}\backslash\overline{\mathbf{D}}. \end{cases} \tag{8}$$

Convergence rate of iterations

Theorem 5 *Let* $\dfrac{K^*}{\mu^*} \geq \delta > 0$, $\beta = \max\{\dfrac{\mu}{\mu^*}, \dfrac{K}{K^*}\} \geq 1$, *and* $\tau = \dfrac{1}{\beta}$. *Then the sequence of approximations* $\{\mathbf{u}^n\}$ *given by the method (6) with the initial guess computed from (7) satisfies the following convergence rate estimate:*

$$\Lambda^*(\mathbf{u}^n - \mathbf{u}, \mathbf{u}^n - \mathbf{u}) \quad \leq \quad q^{2n}\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{u}^0 - \mathbf{u}), \quad 0 \leq q = 1 - \kappa\eta < 1.$$

κ *is the constant from extenstion Theorem 2 and* η *is defined in assumption (5).*

Conclusions

- Novel extension results uniform in the incompressible limit are obtained for the Lamé equations Bakhvalov et al. [2002]
- Extension in the compressible limit is analysed and connected to the Cosserat eigenproblem Bakhvalov et al. [2002]
- Convergence theory of iterations uniform in the jump in the coefficients and in the incompressible limit is developed Bakhvalov et al. [2002]

Future research directions

- Convergence of the FEM uniform in the jump in Lamé coefficients, by analogy with Knyazev and Widlund [2003] for the diffusion equation.
- Regularity of linear elasticity with jumps in elasticity moduli uniform in the jump, extening results for the diffusion equation Knyazev [2003].

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