Analysis of transmission problems on Lipschitz boundaries in stronger norms

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Abstract

For a model diffusion equation on a Lipschitz simply connected bounded domain with a small diffusion coefficient in a Lipschitz simply connected subdomain located strictly inside of the original domain, Knyazev [2003] studies asymptotic properties of the solution with respect to the small diffusion coefficient vanishing. This problem asymptotically turns into a diffusion problem with a cavity. One proof technique of this fact utilizes a reduction of the problem to the boundary of the inner subdomain, using a transmission condition. An analogous approach appears in studying domain decomposition methods without overlap, reducing the investigation to the surface that separates the subdomains and in theoretical foundation of a fictitious domain method, e.g., to prove a classical estimate that guaranties convergence of the solution of the fictitious domain problem to the solution with the cavity.

On a continuous level, the analysis is usually performed in a trace norm, e.g., the $H^{1/2}$ norm for second order elliptic equations. This norm appears naturally for Poincaré-Steklov operators, which are convenient to employ to formulate the transmission condition. A regularity theory of Poincaré-Steklov operators for Lipschitz domains is closely related to regularity of transmission and diffraction problems and properties of layer potentials on nonsmooth interfaces, and has been extensively studied. Taking advantage of the recent progress in the regularity theory for Lipschitz domains, Knyazev [2003] provides an analysis of the transmission condition in the $H^{1/2+\alpha}$ norm with $\alpha > 0$, for a simple model problem. This result leads to a convergence theory of the fictitious domain method for a second order elliptic PDE in an $H^{1+\alpha}$ norm, while the standard result is in an H^1 norm. For the case of Lipschitz domains that we consider, $\alpha < 1/2$. This is an extension to Lipschitz domains of a classical result Kopčenov [1974] for smooth domains.

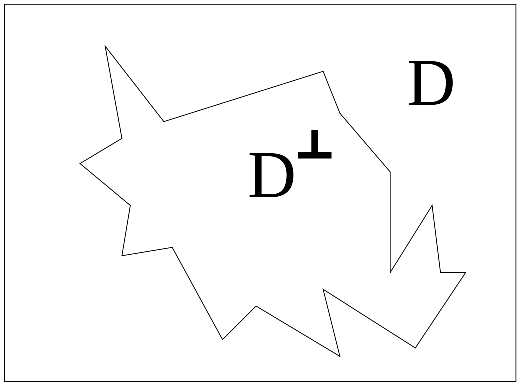
Knyazev [2003] proof relies on the regularity of the solution of the diffusion equation uniformly in the value of the small diffusion coefficient in the subdomain, established in Knyazev and Widlund [2003]. The solution is regular away from the boundary and the interface of the jump in the diffusion coefficient for a sufficiently smooth right-hand side. Under our simplifying assumption that the interface of the jump is strictly inside, the question of regularity can be reduced to studying layer potentials on the interface without boundary conditions, since the interface is then is a closed Lipschitz curve without selfintersection and it does not have any common (junction) points with the boundary, where the homogeneous Dirichlet boundary condition is enforced. We show in Knyazev and Widlund [2003] that the solution belongs to the space $H^{1/2+\alpha}$ norm with $1/2 > \alpha > 0$, uniformly in the small diffusion coefficient vanishing in the inner subdomain.

Contents

The presented results are published in

1. Knyazev and Widlund [2003]
2. Knyazev [2003]

Diffusion equation with the jump in the diffusion coefficient



□

$$\mathrm{div}\,(k\,\mathrm{grad}\,u-\phi)=0,\ u\in\overset{0}{H}{}^{1}(\Box),\ \phi\in(L_2(\Box))^2, \tag{1}$$

where □ is a Lipschitz simply connected bounded domain.

Let $\mathcal{D}\subset\Box$ be a Lipschitz connected domain and let the open set $\mathcal{D}^{\perp}$ be defined by the conditions:

$$\mathcal{D}\cap\mathcal{D}^{\perp}=\emptyset,\quad \bar{\mathcal{D}}\cup\bar{\mathcal{D}}^{\perp}=\bar{\Box}.$$

$\mathcal{D}^{\perp}$ is a Lipschitz simply connected domain strictly inside of □.

We assume that k is a piecewise constant function on $\Box$, and highly discontinuous :

$$k = \omega \text{ on } \mathcal{D}^{\perp},\ k = 1 \text{ on } \mathcal{D},\ \text{where } 0 < \omega \leq 1. \tag{2}$$

The interface Γ separates subdomains $\mathcal{D}$ and $\mathcal{D}^{\perp}$, where the diffusion coefficient takes different values.

For the “right-hand side” ϕ, we assume that

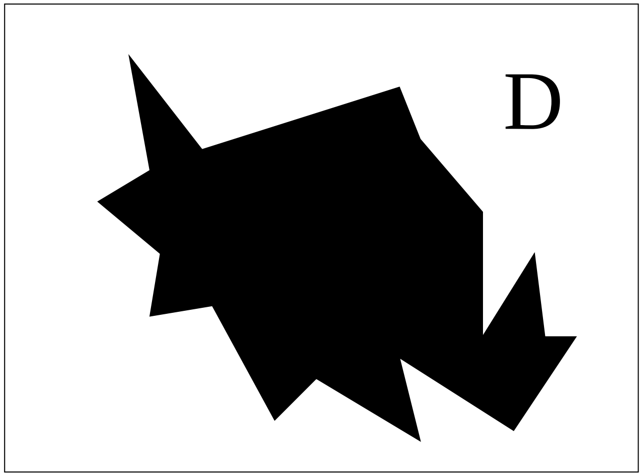
$$\phi \in (H^{\alpha}(\Box))^2,\ 0 \leq \alpha < 1/2$$

and that its restriction on $\mathcal{D}^{\perp}$ satisfies:

$$\phi|_{\mathcal{D}^{\perp}} = 0.$$

We need the later assumption as we want to fix ϕ and to have the solution u uniformly bounded as a function of $\omega \to 0$ at the same time.

The limit ($\omega = 0$) problem in $\mathcal{D}$ with the cavity $\mathcal{D}^{\perp}$



u_0 is defined in domain $\mathcal{D}$, satisfies the same differential equation as u in $\mathcal{D}$, i.e.

$$\operatorname{div}\left(\operatorname{grad} u_0 - \phi\right) = 0 \text{ in } \mathcal{D}, \tag{3}$$

and takes homogeneous Dirichlet boundary conditions on the external boundary $\partial\mathcal{D} \cap \partial\Box = \partial\Box$ of domain $\mathcal{D}$ and homogeneous Neumann boundary conditions on its internal boundary Γ.

H^1 boundedness of the solution uniform in ω

Here and throughout, let C be a generic constant, independent of ω, i.e. $C \neq C(\omega)$.

Theorem 1 *Bakhvalov and Knyazev [1994]*

$$\|u\|_{H^1(\Box)} \leq C\|\phi\|_{(L_2(\Box))^2}.$$

$H^1(\mathcal{D})$ convergence of the solution when $\omega \to 0$

Theorem 2 *Lions [1973], Bakhvalov and Knyazev [1994] Then*

$$\|u|_{\mathcal{D}} - u_0\|_{H^1(\mathcal{D})} \leq C\omega\,\|\phi|_{\mathcal{D}}\|_{(L_2(\mathcal{D}))^2}. \qquad (4)$$

$H^{1+\alpha}(\mathcal{D})$ regularity of u when $\omega \to 0$

For a *fixed* ω, a regularity result is known to hold, e.g., Escauriaza et al. [1992], Savaré [1998]. A similar regularity result, but *uniform* in ω, is established in Knyazev and Widlund [2003]:

Theorem 3 $\|u|_{\mathcal{D}}\|_{H^{1+\alpha}(\mathcal{D})}$ *is uniformly bounded in* ω, *with a fixed positive* α *up to a certain value* $\alpha_{\max}$, *in the following sense:*

$$\|u|_{\mathcal{D}}\|_{H^{1+\alpha}(\mathcal{D})} \leq C\|\phi\|_{(H^{\alpha}(\square))^2}, \; \alpha \in (0, \alpha_{\max}), \tag{5}$$

where $\alpha_{\max} = 1/2$.

$H^{1+\alpha}(\mathcal{D})$ regularity of u_0

In the limit $\omega = 0$, which corresponds to a Neumann boundary value problem on $\mathcal{D}$, according to Theorem 2, the regularity result also holds with $\alpha_{max} = 1/2$; see, e.g., Kenig [1994]:

Theorem 4 *$u_0 \in H^{1+\alpha}(\mathcal{D})$ with a fixed positive α up to a certain value α_{max}, in the following sense:*

$$\|u_0\|_{H^{1+\alpha}(\mathcal{D})} \leq C\|\phi\|_{(H^{\alpha}(\Box))^2}, \ \alpha \in (0, \alpha_{max}), \tag{6}$$

where $\alpha_{max} = 1/2$.

$H^{1+\alpha}(\mathcal{D})$ convergence of the solution when $\omega \to 0$

Theorem 5 *Knyazev [2003] For a small enough nonnegative ω, we have*

$$\|u|_{\mathcal{D}} - u_0\|_{H^{1+\alpha}(\mathcal{D})} \leq C\omega \|\phi|_{\mathcal{D}}\|_{(H^{\alpha}(\mathcal{D}))^2}, \ \alpha \in (0, \alpha_{\max}), \tag{7}$$

where $\alpha_{\max} = 1/2$.

The proof is based on a regularity theory of Steklov-Poincaré operators for Lipschitz domains Costabel [1988], Heuer and Stephan [2000], McLean [2000].

Conclusions

- Novel regularity results uniform in the jump in the coefficients are obtained Knyazev and Widlund [2003]
- Convergence theory of the fictitious domain method in stronger norms is developed for the diffusion equation Knyazev [2003]

Future research directions

- Convergence of the flux in stronger norms when $\omega \to \infty$
- Linear elasticity with jumps in elasticity moduli

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