

Observations on degenerate saddle point problems

Andrew V. Knyazev ¹

*Department of Mathematical Sciences
University of Colorado at Denver and Health Sciences Center
P.O. Box 173364, Campus Box 170, Denver, CO 80217-3364*

Abstract

We investigate degenerate saddle point problems, which can be viewed as limit cases of standard mixed formulations of symmetric problems with large jumps in coefficients. We prove that they are well-posed in a standard norm despite the degeneracy. By wellposedness we mean a stable dependence of the solution on the right-hand side. A known approach of splitting the saddle point problem into separate equations for the primary unknown and for the Lagrange multiplier is used. We revisit the traditional Ladygenskaya–Babuška–Brezzi (LBB) or inf–sup condition as well as the standard coercivity condition, and analyze how they are affected by the degeneracy of the corresponding bilinear forms. We suggest and discuss generalized conditions that cover the degenerate case. The LBB or inf–sup condition is necessary and sufficient for wellposedness of the problem with respect to the Lagrange multiplier under some assumptions. The generalized coercivity condition is necessary and sufficient for wellposedness of the problem with respect to the primary unknown under some other assumptions. We connect the generalized coercivity condition to the positiveness of the minimum gap of relevant subspaces, and propose several equivalent expressions for the minimum gap. Our results provide a foundation for research on uniform wellposedness of mixed formulations of symmetric problems with large jumps in coefficients in a standard norm, independent of the jumps. Such problems appear, e.g., in numerical simulations of composite materials made of components with contrasting properties.

Key words: Wellposedness, mixed, symmetric, saddle point, Lagrange multiplier, Ladygenskaya–Babuška–Brezzi (LBB) condition, inf–sup condition, coercivity, minimum gap between subspaces.

PACS: 46.15.Cc, 02.30.Sa, 02.60.Lj

1. Introduction

Degenerate saddle point problems, e.g., can be viewed as limit cases of mixed formulations of symmetric problems with large jumps in coefficients, corresponding to an infinite jump. We prove that the degeneracy does not affect the wellposedness in a standard norm under some natural assumptions, using ideas that are initiated by [3, 4, 5, 6, 7, 14, 15]. By wellposedness, contrary to illposedness, we mean a stable dependence of the solution on the right-hand side. Results of this paper provide a foundation for research on uniform wellposedness of mixed formulations of symmetric problems with large jumps in coefficients in a standard norm, independent of the jumps.

Email address: `andrew.knyazev@cudenver.edu` (Andrew V. Knyazev).

URL: `http://math.cudenver.edu/~aknyazev/` (Andrew V. Knyazev).

¹ Partially supported by the National Science Foundation award DMS-0612751.

The necessary and sufficient condition, e.g., [9, 10], of the standard wellposedness of an operator equation with an arbitrary right-hand side is the existence of a bounded inverse of the operator. We argue that in some practical cases the equation is degenerate, i.e. the inverse operator does not exist. Assuming that the right-hand side is in the operator range, a solution exists, but is not unique. To make the solution unique we factor out the operator null-space. This leads to a natural generalization, where boundedness of the pseudoinverse of the operator is used as the necessary and sufficient condition of wellposedness of a degenerate operator equation, by analogy with [13, 15].

With this idea in mind, we revisit necessary and sufficient conditions of wellposedness of an abstract mixed problem. In the symmetric case we consider here, the mixed problem can be interpreted as a variational saddle point problem. For generalized saddle point problems we refer the reader, e.g., to [11].

We start in Section 2 with a standard abstract symmetric mixed problem as in [9, 10]. By analogy with [14, 17], we

split the saddle point problem into two equations, for the primary unknown and for the Lagrange multiplier. This split is somewhat implicit in [9, 10]. The equation for the primary unknown is self-consistent, since here we eliminate the Lagrange multiplier from the mixed system using an orthogonal projector.

Following, e.g., [10], we discuss the traditional necessary and sufficient conditions of wellposedness, namely, the Ladygenskaya–Babuška–Brezzi (LBB) or *inf-sup* condition and the coercivity condition. The LBB or *inf-sup* condition, considered in Section 3, is necessary and sufficient for a stable dependence of the Lagrange multiplier on an arbitrary right-hand side.

We review the traditional point of view that the coercivity condition is a necessary and sufficient condition of wellposedness of the problem. In Section 4, an operator form of the dual variational problem without assuming the coercivity condition is considered. We examine the uniqueness of the solution and describe all possible multiple solutions for a given right-hand side. All admissible right-hand sides are determined. We formulate several equivalent necessary and sufficient conditions of wellposedness in terms of closedness of relevant subspaces. We also derive a geometrical condition—a positiveness of a *minimum gap* [12] between relevant closed subspaces.

A possible application of our theory is the Hellinger–Reissner formulation, e.g., [1], of nonhomogeneous Lamé equations for media with (almost) rigid inclusions, where the Lagrange multiplier is the displacement, and we get an operator equation for the stress on the closed subspace of divergence free (in a weak sense) stresses. Infinitely large Lamé coefficients λ and μ , in a subdomain, result in a null-space of the operator in the equation for the stress, so the inverse operator does not exist and the problem is not wellposed in a traditional sense. Our abstract geometrical condition of generalized wellposedness in this example is equivalent to a possibility of extension of displacements preserving the energy norm of the Lamé operator. It has been proved in [4, 7] that such an extension is possible under some assumptions. We expect that in the limit case of infinitely large Lamé coefficients λ and μ in a subdomain the pseudoinverse of the operator is bounded, which makes the problem wellposed for the stresses in the L_2 sense, i.e. the L_2 norm of the stress is stable even if the Lamé coefficients are large in a subdomain. We plan to address this application in the future.

2. Abstract symmetric saddle point problems

In this section we essentially follow well known arguments, e.g., [10], with some simplifications due to the symmetry of the saddle point problem and our unwillingness to introduce dual spaces. Straightforward manipulations, using a pair of complementary closed subspaces, allow us, as in [14, 17], to formulate separate equations for the primary unknown and for the Lagrange multiplier of the sad-

dle point problem; see, e.g., survey [8, Sec. 6] for similar matrix null-space methods. We start by formulating and investigating the problem using bilinear forms, and then repeat the arguments for operator-based formulations that are used in the last section of the paper.

2.1. Formulations using bilinear forms

Let \mathbf{H} and \mathbf{V} be two real Hilbert spaces with scalar products and norms denoted by $(\cdot, \cdot)_{\mathbf{H}}, \|\cdot\|_{\mathbf{H}}$ and $(\cdot, \cdot)_{\mathbf{V}}, \|\cdot\|_{\mathbf{V}}$ correspondingly. Let $a(\cdot, \cdot) : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : \mathbf{H} \times \mathbf{V} \rightarrow \mathbb{R}$ be two continuous bilinear forms with $a(\cdot, \cdot)$ symmetric and nonnegative definite. We consider the following problem: for a given $g \in \mathbf{H}$ and $f \in \mathbf{H}$ find $\sigma \in \mathbf{H}$, called the “primary unknown,” and $u \in \mathbf{V}$, called the “Lagrange multiplier,” such that

$$\begin{cases} a(\sigma, \epsilon) + b(\epsilon, u) = (g, \epsilon)_{\mathbf{H}}, \quad \forall \epsilon \in \mathbf{H}, \\ b(\sigma - f, v) = 0, \quad \forall v \in \mathbf{V}. \end{cases} \quad (1)$$

We place the right-hand side f “inside” of the form b as it allows us to take $f \in \mathbf{H}$, not to introduce the dual space \mathbf{V}' , and makes several statements somewhat simpler. We call (1) a saddle point problem, since equations (1) are the optimality conditions and their solution is a saddle point for the Lagrangian, e.g., [10], defined by $a(\sigma, \sigma) + 2b(\sigma - f, u) - 2(g, \sigma)_{\mathbf{H}}$.

We call a linear manifold, not necessarily closed, a “subspace” and a closed linear manifold a “closed subspace.” Let us introduce a special notation $\mathbf{N} \subseteq \mathbf{H}$ for the closed subspace, which is the null-space of the bilinear form $b(\cdot, \cdot)$ with respect to its first argument, i.e. $\mathbf{N} = \{\epsilon \in \mathbf{H} : b(\epsilon, v) = 0, \forall v \in \mathbf{V}\}$. Let us denote by $\mathbf{P} \equiv \mathbf{N}^{\perp} \subseteq \mathbf{H}$ the closed subspace which is \mathbf{H} -orthogonal (complementary) to \mathbf{N} . Closed subspaces \mathbf{N} and \mathbf{P} play important roles in this paper, so let us introduce an \mathbf{H} -orthogonal projector P on \mathbf{H} such that $\mathbf{N}(P) = \mathbf{N}$ and $\mathbf{R}(P) = \mathbf{P}$ and the complementary projector $P^{\perp} = I - P$ with $\mathbf{R}(P^{\perp}) = \mathbf{N}$ and $\mathbf{N}(P^{\perp}) = \mathbf{P}$, where by $\mathbf{R}(P)$ we denote the range of operator P and, with a slight abuse of the notation, by $\mathbf{N}(P)$ we denote the null-space of operator P . We assume throughout the paper, unless stated otherwise, that a bounded operator is defined everywhere on a corresponding space. As an orthogonal projector, operator $P : \mathbf{H} \rightarrow \mathbf{H}$ is bounded \mathbf{H} -selfadjoint, $P = P^*$, and satisfies $P = P^2$.

In the first equation of system (1), let us split it into two equations, by plugging separately $\epsilon = P\epsilon \in \mathbf{P}$ and $\epsilon = P^{\perp}\epsilon \in \mathbf{N}$ and using the fact that $b(P^{\perp}\epsilon, u) = 0, \forall \epsilon \in \mathbf{H}$. The second equation in system (1) has a simple equivalent geometric interpretation: $\sigma - f \in \mathbf{N}$, or $(\sigma - f, \epsilon)_{\mathbf{H}} = 0, \forall \epsilon \in \mathbf{P}$. We then rewrite system (1) in the following equivalent form:

$$\begin{cases} a(\sigma, \epsilon) + b(\epsilon, u) = (g, \epsilon)_{\mathbf{H}}, \quad \forall \epsilon \in \mathbf{P}, \\ a(\sigma, \epsilon) = (g, \epsilon)_{\mathbf{H}}, \quad \forall \epsilon \in \mathbf{N}, \\ (\sigma - f, \epsilon)_{\mathbf{H}} = 0, \quad \forall \epsilon \in \mathbf{P}. \end{cases} \quad (2)$$

Now we make an important observation that we can treat the first line in system (2) as an equation for the Lagrange multiplier u , given the primary unknown σ , i.e.

$$b(\epsilon, u) = (g, \epsilon)_{\mathbf{H}} - a(\sigma, \epsilon), \quad \forall \epsilon \in \mathbf{P}. \quad (3)$$

The last two lines in system (2) involve neither the Lagrange multiplier u , nor the bilinear form b , and can be used to determine the primary unknown σ :

$$\begin{cases} a(\sigma, \epsilon) = (g, \epsilon)_{\mathbf{H}}, \quad \forall \epsilon \in \mathbf{N}, \\ (\sigma - f, \epsilon)_{\mathbf{H}} = 0, \quad \forall \epsilon \in \mathbf{P}. \end{cases} \quad (4)$$

System (4) describes, e.g., [10], the optimality conditions of the constrained minimization problem $\inf \{a(\sigma, \sigma) - 2(g, \sigma)_{\mathbf{H}}\}, \sigma \in \mathbf{H} : (\sigma - f, \epsilon)_{\mathbf{H}} = 0, \forall \epsilon \in \mathbf{P}$.

2.2. Operator-based formulations

In addition to the formulations above involving bilinear forms, it is convenient to consider equivalent operator-based formulations. We associate with the forms a and b two linear continuous operators $A : \mathbf{H} \rightarrow \mathbf{H}$ and $B : \mathbf{H} \rightarrow \mathbf{V}$ defined by $(A\sigma, \epsilon)_{\mathbf{H}} = a(\sigma, \epsilon)$, $(B\sigma, v)_{\mathbf{V}} = b(\sigma, v)$, $\forall \sigma, \epsilon \in \mathbf{H}$, $v \in \mathbf{V}$. In this definition of A and B we follow a slightly simplified, e.g., [11, 17], rather than standard [10], approach, namely, we do not need dual spaces \mathbf{H}' and \mathbf{V}' . Now, we reformulate the main statements of subsection 2.1 using the just defined operators A and B . The following operator formulation

$$\begin{cases} A\sigma + B^*u = g \text{ in } \mathbf{H}, \\ B(\sigma - f) = 0 \text{ in } \mathbf{V} \end{cases} \quad (5)$$

is equivalent to the original problem (1) with the bilinear forms, where the adjoint operator $B^* : \mathbf{V} \rightarrow \mathbf{H}$ is defined, as usual, by $(\sigma, B^*v)_{\mathbf{H}} = (B\sigma, v)_{\mathbf{V}}$, $\forall \sigma \in \mathbf{H}$, $v \in \mathbf{V}$. The operator A is selfadjoint and nonnegative definite, $A = A^* \geq 0$ on \mathbf{H} since it is defined by the symmetric and nonnegative definite form a .

We notice that the second equation in system (5) has the same geometric interpretation as in the case of bilinear forms-based system (1): $\sigma - f \in \mathbf{N}(B)$. The null-space $\mathbf{N}(B) \subseteq \mathbf{H}$ and its \mathbf{H} -orthogonal complement $\mathbf{R}(B^*) \subseteq \mathbf{H}$ have already been denoted by \mathbf{N} and \mathbf{P} , correspondingly, and introduced together with the \mathbf{H} -orthogonal projector P on \mathbf{H} such that $\mathbf{N} = \mathbf{N}(P) = \mathbf{N}(B)$ and $\mathbf{P} = \mathbf{R}(P) = \mathbf{R}(B^*)$ in the previous subsection.

We split the first equation in system (5) in two orthogonal parts corresponding to \mathbf{N} and \mathbf{P} , using that $PB^*u = B^*u$ and $P^\perp B^*u = 0$, since $\mathbf{R}(B^*) \subseteq \mathbf{P}$. We replace B with P , since they share the same null-space, in the second equation

in system (5) to get the following equivalent form of system (2):

$$\begin{cases} PA\sigma + B^*u = Pg \text{ in } \mathbf{H}, \\ P^\perp A\sigma = P^\perp g \text{ in } \mathbf{H}, \\ P(\sigma - f) = 0 \text{ in } \mathbf{H}. \end{cases} \quad (6)$$

We notice that the first line in system (6) is an equation for the Lagrange multiplier u , given the primary unknown σ , as in (3), i.e. $B^*u = P(g - A\sigma)$.

We next discuss the necessary and sufficient conditions from [10] of wellposedness of the problem and make it clear why one can find weaker necessary and sufficient conditions. To simplify our arguments, we take advantage in the rest of the paper of the split of the original system into separate equations for the Lagrange multiplier u and the primary unknown σ that we have described in this section. It is important to realize, however, that we have not made any substitutions, neither in the solutions u and σ , nor in the right-hand sides f and g . So whatever statements we next prove concerning the dependence of the solutions u and σ on the right-hand sides f and g , these statements are equally applicable to both the separate equations and to the original system in either bilinear form- or operator-based context.

3. Inf-sup or LBB condition

In this section, we discuss a traditional assumption, being recently referred to as *Ladygenskaya-Babuška-Brezzi (LBB) condition*, see Babuška and Aziz [2], Brezzi and Fortin [10], Ladyzhenskaya [16], that the range of operator $B : \mathbf{H} \rightarrow \mathbf{V}$, denoted by $\mathbf{R}(B)$, is closed. The closedness of a range of a closed operator is ultimately connected to the boundedness of the operator (pseudo-)inverse, e.g., [12].

In our specific situation, operator B is bounded with the closed domain \mathbf{H} and, thus, is closed, so its (pseudo-)inverse $B^{-1} : \mathbf{R}(B) \rightarrow \mathbf{H}/\mathbf{N}(B)$ is also closed. It is necessary to use a factor-space here to define the inverse, since the standard operator inverse $B^{-1} : \mathbf{R}(B) \rightarrow \mathbf{H}$ does not exist if $\mathbf{N}(B)$ is nontrivial. We note that $\mathbf{N}(B)$ is closed and that the factor-space $\mathbf{H}/\mathbf{N}(B)$ is a Hilbert space, as is \mathbf{H} . In a Hilbert space, a convenient set of representants for the classes in the factor-space is simply the corresponding orthogonal complement, e.g., $\mathbf{H}/\mathbf{N}(B)$ is isometrically isomorphic to $\mathbf{P} = (\mathbf{N}(B))^\perp \subseteq \mathbf{H}$, so we set $\|\sigma\|_{\mathbf{H}/\mathbf{N}(B)} = \|P\sigma\|_{\mathbf{H}}$. The subspace $\mathbf{R}(B)$ is the domain of the closed operator $B^{-1} : \mathbf{R}(B) \rightarrow \mathbf{H}/\mathbf{N}(B)$ therefore, $\mathbf{R}(B)$ is closed if and only if $B^{-1} : \mathbf{R}(B) \rightarrow \mathbf{H}/\mathbf{N}(B)$ is bounded. Closedness of $\mathbf{R}(B)$ is equivalent to closedness of $\mathbf{R}(B^*)$, so all the arguments above can be equivalently reformulated for the adjoint operator B^* and its (pseudo-)inverse.

When written in terms of inequalities involving the bilinear form b :

$$\begin{aligned} \inf_{\sigma \in \mathbf{H}} \sup_{v \in \mathbf{V}} \frac{b(\sigma, v)}{\|\sigma\|_{\mathbf{H}/\mathbf{N}(B)} \|v\|_{\mathbf{V}}} &= \inf_{\sigma \in \mathbf{H}} \frac{\|B\sigma\|_{\mathbf{V}}}{\|\sigma\|_{\mathbf{H}/\mathbf{N}(B)}} \\ &= \frac{1}{\|B^{-1}\|_{\mathbf{R}(B) \rightarrow \mathbf{H}/\mathbf{N}(B)}} > 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \inf_{v \in \mathbf{V}} \sup_{\sigma \in \mathbf{H}} \frac{b(\sigma, v)}{\|\sigma\|_{\mathbf{H}} \|v\|_{\mathbf{V}/\mathbf{N}(B^*)}} &= \inf_{v \in \mathbf{V}} \frac{\|B^*v\|_{\mathbf{H}}}{\|v\|_{\mathbf{V}/\mathbf{N}(B^*)}} \\ &= \frac{1}{\|B^{-*}\|_{\mathbf{R}(B^*) \rightarrow \mathbf{V}/\mathbf{N}(B^*)}} > 0, \end{aligned}$$

the LBB condition is also known as the *inf-sup* condition, see Babuška and Aziz [2], Brezzi and Fortin [10], where $\mathbf{V}/\mathbf{N}(B^*)$ means the factor-space of \mathbf{V} with respect to the closed subspace $\mathbf{N}(B^*)$. We implicitly assume that the arguments in the inf-sup formulas above and throughout the paper do not make both the numerator and the denominator vanish. In Ladyzhenskaya [16], the inf-sup condition does not appear to be explicitly formulated, instead, closedness of a range of the gradient operator is investigated in connection with wellposedness of the diffusion equation.

We note that the induced norms of an operator and its adjoint are equal, so both inf-sup expressions above are equal to the same constant that we call c_b . If at least one of the spaces \mathbf{H} or \mathbf{V} is finite dimensional then the value c_b is positive automatically, so it becomes important how c_b depends on some parameters, e.g., on the dimension.

Let us mention that in many practical applications the space \mathbf{V} can be naturally defined such that $\mathbf{N}(B^*) = \{0\}$, so the latter inf-sup expression of the LBB condition takes the form

$$\inf_{v \in \mathbf{V}} \sup_{\sigma \in \mathbf{H}} \frac{b(\sigma, v)}{\|\sigma\|_{\mathbf{H}} \|v\|_{\mathbf{V}}} = c_b > 0,$$

which can be most often seen in publications on the subject. We now contribute our own equivalent formulations of the LBB condition.

Lemma 3.1 *Subspaces $\mathbf{R}(B) \subseteq \mathbf{V}$ and $\mathbf{R}(BB^*) \subseteq \mathbf{V}$ are closed simultaneously. Moreover, if either of them is closed we have $\mathbf{R}(BB^*) = \mathbf{R}(B)$.*

Proof. If $BB^*v = 0$ then $(B^*v, B^*v)_{\mathbf{H}} = 0$, i.e. $B^*v = 0$, which proves that $\mathbf{N}(BB^*) = \mathbf{N}(B^*)$. Taking an orthogonal complement to both parts gives $\overline{\mathbf{R}(BB^*)} = \overline{\mathbf{R}(B)}$ as the operator BB^* is selfadjoint. Trivially, $\mathbf{R}(BB^*) \subseteq \mathbf{R}(B)$. If the range $\mathbf{R}(BB^*)$ is closed then $\overline{\mathbf{R}(B)} = \overline{\mathbf{R}(BB^*)} = \mathbf{R}(BB^*) \subseteq \mathbf{R}(B)$, but clearly $\mathbf{R}(B) \subseteq \overline{\mathbf{R}(B)}$, which proves closedness of $\mathbf{R}(B) = \mathbf{R}(BB^*)$.

To prove the inverse statement, assuming that $\mathbf{R}(B)$ is closed, we invoke the orthogonal decomposition argument²: $\mathbf{H} = \mathbf{R}(B^*) \oplus (\mathbf{R}(B^*))^\perp = \mathbf{R}(B^*) \oplus \mathbf{N}(B)$ since $\mathbf{R}(B)$ and thus $\mathbf{R}(B^*)$ are closed. Multiplying this equality by B gives $\mathbf{R}(B) = B\mathbf{H} = B(\mathbf{R}(B^*) \oplus \mathbf{N}(B)) = B\mathbf{R}(B^*) = \mathbf{R}(BB^*)$. \square

² This proof is suggested by an anonymous referee

We use the previous lemma to introduce $(BB^*)^{-1} : \mathbf{R}(BB^*) \rightarrow \mathbf{V}/\mathbf{N}(B^*)$ in the next Lemma 3.2. It is necessary to use the factor-space $\mathbf{V}/\mathbf{N}(B^*)$ here, since the standard inverse $(BB^*)^{-1} : \mathbf{R}(BB^*) \rightarrow \mathbf{V}$ does not exist if $\mathbf{N}(B^*)$ is nontrivial.

Lemma 3.2 *Closedness of $\mathbf{R}(B) \subseteq \mathbf{V}$ is equivalent to boundedness of the operator $(BB^*)^{-1} : \mathbf{R}(BB^*) \rightarrow \mathbf{V}/\mathbf{N}(B^*)$.*

Proof. By Lemma 3.1, closedness of $\mathbf{R}(B) \subseteq \mathbf{V}$ is equivalent to closedness of $\mathbf{R}(BB^*) \subseteq \mathbf{V}$. We use several well-known statements on closed operators, e.g., [12], applied to the operator BB^* , that we have already reviewed in the second paragraph of this section for the operator B . The operator BB^* is bounded and has the closed domain \mathbf{V} , so the operator is closed and its (pseudo-)inverse $(BB^*)^{-1} : \mathbf{R}(BB^*) \rightarrow \mathbf{V}/\mathbf{N}(B^*)$ with the domain $\mathbf{R}(BB^*) \subseteq \mathbf{V}$ is also closed. The domain $\mathbf{R}(BB^*) \subseteq \mathbf{V}$ of the closed operator $B^{-1} : \mathbf{R}(BB^*) \rightarrow \mathbf{H}/\mathbf{N}(B)$ is closed if and only if the operator is bounded. \square

If $\mathbf{R}(B)$ is closed then, using Lemmas 3.1 and 3.2, $\mathbf{R}(B) = \mathbf{R}(BB^*)$ and we can derive the following useful formula

$$P = B^*(BB^*)^{-1}B : \mathbf{H} \rightarrow \mathbf{H}. \quad (7)$$

Indeed, we first note that $\mathbf{R}((BB^*)^{-1}) \subseteq \mathbf{V}/\mathbf{N}(B^*)$ is multiplied by B^* in (7), so the product is independent of the choice of a representant from the equivalence class $\mathbf{V}/\mathbf{N}(B^*)$ and, thus, is correctly defined. Second, right-hand side of (7) is a linear and bounded operator as a product of linear and bounded operators. Moreover, it is an *orthogonal projector* on \mathbf{H} since it is selfadjoint and idempotent, and has the null-space the same as the orthoprojector P has.

If the LBB condition is not satisfied, i.e. $\mathbf{R}(B)$ is not closed, then the domain of definition of the operator $B^*(BB^*)^{-1}B$ is the subspace $\mathbf{R}(B^*) \oplus \mathbf{N}(B)$, which is not closed, and formula (7), where P is the orthogonal projector on \mathbf{H} with $\mathbf{N}(P) = \mathbf{N}(B)$, clearly does not hold.

Let us note that in the case of finite dimensional spaces \mathbf{H} and \mathbf{V} the range $\mathbf{R}(B)$ is evidently closed, the operator $(B^*)^+ = (BB^*)^+B$ is the well-known *Moore–Penrose pseudo inverse* of B^* , and $P = B^*(B^*)^+$ is the well known formula for the orthogonal projector onto the range of B^* .

If σ is an exact solution of system (5), then u in (5) can be found from the equation $B^*u = -A\sigma + g \in \mathbf{R}(B^*)$. If σ is an approximate solution of system (5) such that the condition $A\sigma - g \in \mathbf{R}(B^*)$, which is necessary and sufficient for the existence of u , does not hold, then u can be computed from the projected equation $B^*u = P(-A\sigma + g) \in \mathbf{P}$. Both the original and the projected equations for u are wellposed by the LBB assumption, i.e. $\mathbf{R}(B^*) = \mathbf{P}$ and

$$\|u\|_{\mathbf{V}/\mathbf{N}(B^*)} \leq \frac{\|a\|}{c_b} \|\sigma\|_{\mathbf{H}} + \frac{1}{c_b} \|g\|_{\mathbf{H}}.$$

Whether the LBB assumption is necessary for wellposedness of the equation for u depends on if the set of all possible right-hand sides $g - A\sigma$ gives the whole subspace $\mathbf{R}(B^*)$,

see [10]. For example, in a practically important case $g = 0$ we have $B^*u = -A\sigma = -PA\sigma \in \mathbf{R}(PA) \subseteq \mathbf{R}(P)$. If the latter inclusion is strict, it opens up an opportunity for a weaker, compared to the original LBB, assumption of wellposedness of the above equation for u .

In the present paper, however, we are concerned with finding σ , not u . The LBB condition for the bilinear form b appears to be of no importance for our results in the next section where we analyze wellposedness of system (5) with respect to the σ unknown only, assuming that the u unknown is of no interest, or can be found for a given σ using some *postprocessing*.

4. Coercivity conditions

4.1. The standard coercivity condition

We finally get to the main topic of the paper: an assumption on A which is a condition of wellposedness of (5) with respect to σ . For the reader's convenience, we briefly repeat the necessary notation and the system of equations for σ to make this section self-consistent. Let \mathbf{H} be a real Hilbert space and P be an orthoprojector in \mathbf{H} with a null-space $\mathbf{N}(P) = \mathbf{N}$ and a range $\mathbf{R}(P) \equiv \mathbf{P}$ —we emphasize that the range of any orthoprojector in a Hilbert space is closed. Let A be a linear and bounded operator such that $0 \leq A^* = A$ on \mathbf{H} . The last two lines in system (6) represent an operator form of system (4); they do not involve the Lagrange multiplier u or the operator B and determine the primary unknown $\sigma \in \mathbf{H}$:

$$\begin{cases} P^\perp(A\sigma - g) = 0 & \text{in } \mathbf{H}, \\ P(\sigma - f) = 0 & \text{in } \mathbf{H}, \end{cases} \quad (8)$$

where $g \in \mathbf{H}$ and $f \in \mathbf{H}$ are given and $P^\perp \equiv I - P$. We can also replace system (8) with the following equivalent single equation:

$$P^\perp A|_{\mathbf{N}} \psi = P^\perp g - P^\perp A P f \in \mathbf{N}, \quad \sigma = \psi + P f, \quad (9)$$

where in (9) we take a restriction of the operator $P^\perp A$ on its invariant closed subspace \mathbf{N} , and we are looking for a solution $\psi \in \mathbf{N}$. Then the necessary and sufficient condition of wellposedness of problem (9) for an *arbitrary* $g \in \mathbf{H}$ is, clearly, that the range of $P^\perp A|_{\mathbf{N}}$ is \mathbf{N} . This leads to the traditional assumption, see [10], $a(\sigma, \sigma) \geq c_a > 0, \forall \sigma \in \mathbf{N}, \|\sigma\|_{\mathbf{H}} = 1$ or, in an operator form, $A \geq c_a I$ on $\mathbf{N} \subseteq \mathbf{H}$, since A is selfadjoint nonnegative. Thus, this assumption is also *necessary and sufficient* [9, 10] for wellposedness of system (5) with respect to σ for *an arbitrary* $g \in \mathbf{H}$. In the rest of the section, we analyze the scenario, where A is selfadjoint nonnegative on \mathbf{H} , but may be degenerate on \mathbf{N} , so we impose necessary restrictions on $g \in \mathbf{H}$, and determine a generalized coercivity condition that covers the case of the degeneracy.

4.2. Existence, uniqueness, and wellposedness

Before we investigate the existence and uniqueness of the solution σ , we prove the following technical, but important, lemma.

Lemma 4.1 *Let P be an orthoprojector in \mathbf{H} with a null-space $\mathbf{N}(P) = \mathbf{N}$ and a range $\mathbf{R}(P) \equiv \mathbf{P} = \mathbf{N}^\perp$, and A be a linear and bounded operator such that $0 \leq A^* = A$ on \mathbf{H} . Then*

$$\mathbf{N}(P^\perp A) \cap \mathbf{N} = \mathbf{N}(A) \cap \mathbf{N}, \quad (10)$$

$$\{\mathbf{N}(P^\perp A) \cap \mathbf{N}\}^\perp = \overline{\mathbf{R}(A) + \mathbf{P}}, \quad (11)$$

$$\mathbf{N} = (\mathbf{N}(P^\perp A) \cap \mathbf{N}) \oplus \overline{P^\perp \mathbf{R}(A)}. \quad (12)$$

Proof. We first verify (10). It follows from $\mathbf{N}(P^\perp A) \supseteq \mathbf{N}(A)$ that the right-hand side of (10) is included in the left-hand side. To prove the reverse inclusion, let $\varphi \in \mathbf{N}$ and $P^\perp A\varphi = 0$, then $0 = (P^\perp A\varphi, \varphi) = (A\varphi, \varphi) = \|A^{1/2}\varphi\|^2$ (recall that $A \geq 0$). Then $A^{1/2}\varphi = 0$ and $A\varphi = 0$. Therefore, equality (10) holds.

Equality (11) follows from (10), by substituting $\mathbf{N}(A) = \mathbf{F}^\perp$ and $\mathbf{R}(A) = \mathbf{F}$ in the well-known simple identity $\mathbf{F}^\perp \cap \mathbf{P}^\perp = (\mathbf{F} + \mathbf{P})^\perp$ and noting that $(\overline{\mathbf{R}(A) + \mathbf{P}})^{\perp\perp} = \overline{\mathbf{R}(A) + \mathbf{P}} = \overline{\mathbf{R}(A) + \mathbf{P}}$ by properties of the closure.

Finally, to obtain the second term in the orthogonal decomposition (12) of \mathbf{N} we see that by (11) $\{\mathbf{N}(P^\perp A) \cap \mathbf{N}\}^\perp \cap \mathbf{N} = \overline{\mathbf{R}(A) + \mathbf{P}} \cap \mathbf{N}$; at the same time

$$\begin{aligned} \overline{\mathbf{R}(A) + \mathbf{P}} \cap \mathbf{N} &= \overline{P^\perp \mathbf{R}(A) + \mathbf{P}} \cap \mathbf{N} \\ &= (\overline{P^\perp \mathbf{R}(A)} \oplus \mathbf{P}) \cap \mathbf{N} = \overline{P^\perp \mathbf{R}(A)}, \end{aligned}$$

which completes the proof of the lemma. \square

We start with the solution uniqueness.

Lemma 4.2 *Suppose that for some fixed $g \in \mathbf{H}$ and $f \in \mathbf{H}$ there exists a solution σ of (8). Then it is unique provided that $\mathbf{N}(A) \cap \mathbf{N} = \{0\}$; otherwise, all possible solutions yield the hyperplane $\sigma + \{\mathbf{N}(A) \cap \mathbf{N}\}$ and there exists the unique normal (with minimal norm in \mathbf{H}) solution of (8) that can be also defined as a common element of the above hyperplane and the closed subspace $\mathbf{R}(A) + \mathbf{P}$, which is the set of all normal solutions for all possible f and g .*

Proof. All solutions of (8) with $g = f = 0$ constitute the closed subspace $\mathbf{N}(P^\perp A) \cap \mathbf{N}$ (may be 0-dimensional), which by (10) is the same as $\mathbf{N}(A) \cap \mathbf{N}$. Hence, all solutions of (8) with the given g and f , provided that there exists at least one solution σ , constitute the hyperplane $\sigma + \mathbf{N}(A) \cap \mathbf{N}$. It is known that each closed hyperplane in a Hilbert space has a unique element with the minimal norm, i.e. the element that is orthogonal to the directing closed subspace $\mathbf{N}(A) \cap \mathbf{N}$ of the hyperplane. The orthogonal complement to the directing closed subspace is already given by (11). \square

In the rest of the subsection we use the following equation equivalent to (9):

$$(P^\perp A + P)\sigma = P^\perp g + P f. \quad (13)$$

The assumptions on the right-hand side of the system (8) which ensure the existence of a solution are rather standard and follow from (13) easily.

Lemma 4.3 *For any $f \in \mathbf{H}$ there exists a solution of (8) if and only if $g \in \mathbf{R}(A) + \mathbf{P}$, i.e. $P^\perp g + Pf \in P^\perp \mathbf{R}(A) + \mathbf{P} = \mathbf{R}(A) + \mathbf{P}$.*

Proof. The subspace (not necessarily closed) $P^\perp \mathbf{R}(A) + \mathbf{P}$ is simply the range of the operator $P^\perp A + P$ of equation (13). \square

The subspace $\mathbf{R}(A) + \mathbf{P}$ that appears in Lemmas 4.2 and 4.3 plays the central role in the following necessary and sufficient conditions of wellposedness.

Theorem 4.1 *The following statements are equivalent:*

- (i) *The subspace $\mathbf{R}(A) + \mathbf{P}$ is closed.*
- (ii) *The subspace $A\mathbf{N} + \mathbf{P}$ is closed.*
- (iii) *The subspace $P^\perp \mathbf{R}(A)$ is closed.*
- (iv) *The subspace $P^\perp A\mathbf{N}$ is closed.*
- (v) *Problem (13) with $f \in \mathbf{H}$ and $g \in \mathbf{R}(A) + \mathbf{P}$ is well-posed in the factor-space, $\|\sigma\|_{\mathbf{H}/\{\mathbf{N}(A) \cap \mathbf{N}\}} \leq c(\|g\| + \|f\|)$, or, equivalently, $\|\sigma\| \leq c(\|g\| + \|f\|)$ for the normal solution $\sigma \in \mathbf{R}(A) + \mathbf{P}$.*

Proof. (1) \Leftrightarrow (3) We have $\mathbf{R}(A) + \mathbf{P} = P^\perp \mathbf{R}(A) \oplus \mathbf{P}$.

(1) \Leftrightarrow (2) The subspace $P^\perp \mathbf{R}(A) \oplus \mathbf{P} = \mathbf{R}(A) + \mathbf{P}$ is the range of the operator $P^\perp A + P$. The range of a bounded operator is closed if and only if the range of the conjugate operator is closed.

(2) \Leftrightarrow (4) Using the same arguments as above, $A\mathbf{N} + \mathbf{P} = P^\perp A\mathbf{N} \oplus \mathbf{P}$.

(1) \Leftrightarrow (5) The operator $P^\perp A + P$ is bounded and defined everywhere on a Hilbert space, thus it is closed. Therefore, the (pseudo)inverse operator

$$(P^\perp A + P)^{-1} : \mathbf{R}(P^\perp A + P) \rightarrow \mathbf{H}/\{\mathbf{N}(A) \cap \mathbf{N}\}$$

is closed. It is bounded if and only if its domain of definition $\mathbf{R}(P^\perp A + P)$ is closed. A normal solution is a convenient representant of a factor-class in a Hilbert space. \square

4.3. Generalized coercivity conditions

Statements (1)–(4) in Theorem 4.1 may not be so easily verifiable in practice, so we want to find a somewhat easier assumption that generalizes the standard coercivity assumption $A \geq c_a I$ on $\mathbf{N} \subseteq \mathbf{H}$, which itself does not hold if the operator A vanishes on a nontrivial subspace of $\mathbf{N} \subseteq \mathbf{H}$.

Let us return back to equation (9). We remind the reader that the first equation in (8) is equivalent to the orthogonal expansion $\sigma = \psi + Pf$, where $\psi = P^\perp \sigma \in \mathbf{N}$. This and the second equation in (8) lead to (9) that we present here, introducing a special notation $K = P^\perp A|_{\mathbf{N}}$, in the equivalent form

$$K\psi = \phi, \psi \in \overline{P^\perp \mathbf{R}(A)}, \phi = P^\perp g - P^\perp APf \in \overline{P^\perp \mathbf{R}(A)}. \quad (14)$$

under the assumption that $g \in \mathbf{R}(A) + \mathbf{P}$.

The operator K is bounded, selfadjoint, and nonnegative definite on \mathbf{N} , where $\mathbf{N} \subseteq \mathbf{H}$ inherits the scalar product and the norm of \mathbf{H} , so there exists a bounded, selfadjoint,

and nonnegative definite square root \sqrt{K} on \mathbf{N} . Applying the inf-sup condition to the operator \sqrt{K} on \mathbf{N} , by direct analogy with Lemmas 3.1 and 3.2 and their proofs, we have that $\mathbf{N}(\sqrt{K}) = \mathbf{N}(K)$ and

Theorem 4.2 *The following statements are equivalent:*

- (i) *The subspace $\mathbf{R}(\sqrt{K}) \subseteq \mathbf{N}$ is closed.*
- (ii) *The subspace $\mathbf{R}(K) \subseteq \mathbf{N}$ is closed.*
- (iii) *The inf-sup condition for the operator \sqrt{K} on \mathbf{N}*

$$\inf_{\epsilon \in \mathbf{N}} \sup_{\sigma \in \mathbf{N}} \frac{(\sqrt{K}\epsilon, \sigma)_{\mathbf{N}}}{\|\epsilon\|_{\mathbf{N}/\mathbf{N}(K)} \|\sigma\|_{\mathbf{N}}} = \inf_{\epsilon \in \mathbf{N}} \frac{\|\sqrt{K}\epsilon\|_{\mathbf{N}}}{\|\epsilon\|_{\mathbf{N}/\mathbf{N}(K)}} \equiv \frac{1}{\sqrt{\rho}} > 0 \quad (15)$$

holds.

- (iv) *The norm of the operator $K^{-1} : \mathbf{R}(K) \rightarrow \mathbf{N}/\mathbf{N}(K)$ is equal to $\rho < \infty$.*

Moreover, under either of the assumptions we have $\mathbf{R}(\sqrt{K}) = \mathbf{R}(K)$.

Noticing that $\mathbf{R}(K) = P^\perp A\mathbf{N}$, we immediately see that statements (4) in Theorem 4.1 and (2) in Theorem 4.2 are the same, so all statements of Theorems 4.1 and 4.2 are equivalent. Our last goals in this subsection are to present statement (3) of Theorem 4.2 in original terms, so that it resembles the coercivity condition, and to bound the norm of the solution in terms of the norms of the right-hand sides, using statement (4) of Theorem 4.2.

Theorem 4.3 *For any $g \in \mathbf{R}(A) + \mathbf{P}$ the following assumption*

$$A \geq \frac{1}{\rho} I \text{ on the subspace } P^\perp \mathbf{R}(A) \quad (16)$$

with a (finite) constant $\rho > 0$ is necessary and sufficient for the normal solution σ with $P^\perp \sigma \in P^\perp \mathbf{R}(A)$ to exist and to be unique and continuous in $f \in \mathbf{H}$ and $g \in \mathbf{R}(A) + \mathbf{P}$. Moreover, assumption (16) implies

$$\|\sigma\|^2 \leq \|f\|^2 + \rho^2 \|g - APf\|^2. \quad (17)$$

Proof. First, we note that inequality (16) on the subspace $P^\perp \mathbf{R}(A)$ is equivalent to the same inequality on its closure $\overline{P^\perp \mathbf{R}(A)}$ because of the continuity of A and the scalar product. Second, as $(\epsilon, K\epsilon) = (\epsilon, P^\perp A\epsilon) = (\epsilon, A\epsilon)$ for all $\epsilon \in \overline{P^\perp \mathbf{R}(A)} \subseteq \mathbf{N}$, inequality (16) is also equivalent to

$$K \geq \frac{1}{\rho} I \text{ on the closed subspace } \overline{P^\perp \mathbf{R}(A)}. \quad (18)$$

Now we show that (18) is equivalent to (15), which is condition (3) of Theorem 4.2. For the numerator in (15), we have $\|\sqrt{K}\epsilon\|_{\mathbf{N}}^2 = (\epsilon, K\epsilon)$. To handle the denominator in (15), we remind the reader the orthogonal decomposition $\mathbf{N} = (\mathbf{N}(P^\perp A) \cap \mathbf{N}) \oplus \overline{P^\perp \mathbf{R}(A)}$ stated as (12) and proved in Lemma 4.1. Splitting $\epsilon \in \mathbf{N}$ according to this orthogonal decomposition, we see that its first component—from $\mathbf{N}(P^\perp A) \cap \mathbf{N} = \mathbf{N}(K)$ —vanishes both in the numerator, since it is in the null-space of K , and in the denominator

of (15), by the definition of the factor-norm, which gives (18), where only the second component—from $\overline{P^\perp \mathbf{R}(A)}$ —survives.

We conclude that (16) is equivalent to (15), which is condition (3) of Theorem 4.2, and thus, to all statements of Theorems 4.1 and 4.2. Finally, if (16) holds then the subspace $\mathbf{R}(P^\perp A)$ is closed, the operator $K : P^\perp \mathbf{R}(A) \mapsto \overline{P^\perp \mathbf{R}(A)}$ is an isomorphism and problem (14) is wellposed for $f \in \mathbf{H}$ and $g \in \mathbf{R}(A) + \mathbf{P}$, i.e.

$$\|\psi\| \leq \rho \|P^\perp g - P^\perp A P f\| \leq \rho \|g - A P f\| \quad (19)$$

by Theorem 4.2. Estimate (17) follows from $\sigma = \psi + P f$ and (19) due to the statement of Lemma 4.2 that the normal solution $\sigma \in \overline{\mathbf{R}(A) + \mathbf{P}}$, that is, $\psi \in \overline{P^\perp \mathbf{R}(A)} = P^\perp \mathbf{R}(A) + \mathbf{P} = \mathbf{P}^\perp \cap (\mathbf{P}^\perp \cap \mathbf{N}(A))^\perp$ is the corresponding part of the orthogonal expansion $\sigma = \psi + P f$ for the normal solution. \square

4.4. Minimum gap between subspaces

The rest of the section concerns the case where the range of A is closed, so assumption (16) can be equivalently reformulated using the minimum gap between some relevant subspaces. We first find a simple way to check if the range of A is closed.

Lemma 4.4 *Condition*

$$A \geq \frac{1}{\rho_D} I \text{ on the subspace } \mathbf{R}(A) \equiv \mathbf{D} \quad (20)$$

with a (finite) constant $\rho_D > 0$ is equivalent to closedness of \mathbf{D} .

Proof. The operator A is a linear, bounded, and everywhere defined. Thus, it is closed and its inverse $A^{-1} : \mathbf{D} \rightarrow \mathbf{H}/\mathbf{N}(A)$ is also closed. Boundedness of the inverse is equivalent, on the one hand, to condition (20) and, on the other hand, to closedness of \mathbf{D} . \square

Now we are ready to present a simplified version of the necessary and sufficient condition of wellposedness (16), assuming that the range of A is closed.

Theorem 4.4 *Let the range $\mathbf{R}(A) \equiv \mathbf{D}$ be closed, the orthogonal projector on \mathbf{D} be denoted by D , and the constant $\rho_D > 0$ be defined by (20). Then inequality (16) is equivalent to the inequality*

$$\kappa \equiv \inf_{\psi \in P^\perp \mathbf{D}} \frac{\|D\psi\|}{\|\psi\|} > 0. \quad (21)$$

In particular, (20) and (21) lead to (16) with $\rho = \rho_D/\kappa^2$.

Proof. We have $\mathbf{R}(P^\perp A) = P^\perp \mathbf{R}(A) = P^\perp \mathbf{D}$, i.e. the subspaces indicated in (16) and (21) coincide. Now the main assertion of the Lemma is a consequence of relations $\frac{1}{\rho_D} (D\psi, D\psi) \leq (A\psi, \psi) = (AD\psi, D\psi) \leq \|A\| (D\psi, D\psi)$ which hold for an arbitrary $\psi \in \mathbf{H}$. \square

The next two lemmas provide alternative assumptions, equivalent to (21), which are necessary and sufficient for wellposedness, assuming that the range of A is closed. It is important to have a choice of a criterion that may be easier

to check in a practical application. For aesthetic reasons we denote $\mathbf{N} \equiv \mathbf{P}^\perp$.

Lemma 4.5 *Let D and P be orthogonal projectors onto closed subspaces \mathbf{D} and \mathbf{P} , and let $D^\perp = I - D$ and $P^\perp = I - P$ be orthogonal projectors onto the orthogonal complements \mathbf{D}^\perp and \mathbf{P}^\perp , respectively. The following statements are equivalent:*

- (i) *The subspace $P^\perp \mathbf{D}$ is closed.*
- (ii) *The subspace $\mathbf{D} + \mathbf{P}$ is closed.*
- (iii) *The subspace $\mathbf{D}^\perp + \mathbf{P}^\perp$ is closed.*
- (iv) *The subspace $P \mathbf{D}^\perp$ is closed.*

Proof. (1) \Leftrightarrow (2) The subspace $P^\perp \mathbf{D}$ is closed iff the subspace $P^\perp \mathbf{D} \oplus \mathbf{P} = \mathbf{D} + \mathbf{P}$ is closed as the terms are orthogonal in the first expression.

(2) \Leftrightarrow (3) By Theorem IV-4.8 of [12], a sum of closed subspaces in a Hilbert space is closed if and only if the sum of their orthogonal complements is closed.

(3) \Leftrightarrow (4) Using the same arguments as above, $\mathbf{P}^\perp + \mathbf{D}^\perp = \mathbf{P}^\perp \oplus P \mathbf{D}^\perp$. \square

Lemma 4.6 *Using the notation of Lemma 4.5, the following equalities hold:*

$$\begin{aligned} \inf_{\psi \in \mathbf{P}, \psi \notin \mathbf{D}} \frac{\text{dist}\{\psi; \mathbf{D}\}}{\text{dist}\{\psi; \mathbf{D} \cap \mathbf{P}\}} &= \inf_{\psi \in \mathbf{D}, \psi \notin \mathbf{P}} \frac{\text{dist}\{\psi; \mathbf{P}\}}{\text{dist}\{\psi; \mathbf{P} \cap \mathbf{D}\}} \\ &= \inf_{\psi \in \mathbf{D}^\perp, \psi \notin \mathbf{P}^\perp} \frac{\text{dist}\{\psi; \mathbf{P}^\perp\}}{\text{dist}\{\psi; \mathbf{P}^\perp \cap \mathbf{D}^\perp\}} \\ &= \inf_{\psi \in \mathbf{P}^\perp, \psi \notin \mathbf{D}^\perp} \frac{\text{dist}\{\psi; \mathbf{D}^\perp\}}{\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}} \\ &= \inf_{\psi \in P^\perp \mathbf{D}} \frac{\|D\psi\|}{\|\psi\|} = \inf_{\psi \in D^\perp \mathbf{P}} \frac{\|P\psi\|}{\|\psi\|} \\ &= \inf_{\psi \in P \mathbf{D}^\perp} \frac{\|D^\perp \psi\|}{\|\psi\|} = \inf_{\psi \in D \mathbf{P}^\perp} \frac{\|P^\perp \psi\|}{\|\psi\|}. \end{aligned}$$

Moreover, each statement in the previous Lemma is equivalent to the positiveness $\kappa > 0$ in (21).

Proof. The first three equalities are derived in Section IV-4 of [12] on the *minimum gap* between subspaces, along with a statement that positiveness of the minimum gap between two given subspaces is a necessary and sufficient condition of the sum of the subspaces, in our case, $\mathbf{D} + \mathbf{P}$, to be closed. We now prove that

$$\inf_{\psi \in \mathbf{P}^\perp, \psi \notin \mathbf{D}^\perp} \frac{\text{dist}\{\psi; \mathbf{D}^\perp\}}{\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}} = \inf_{\psi \in P^\perp \mathbf{D} \setminus \{0\}} \frac{\|D\psi\|}{\|\psi\|}.$$

All other equalities can be then trivially derived from the previous ones just by interchanging P and D .

We first notice that in the right-hand side we can apply the inf to the closure $\overline{P^\perp \mathbf{D}} \setminus \{0\}$ as well, because a norm is a continuous function,

$$\inf_{\psi \in P^\perp \mathbf{D} \setminus \{0\}} \frac{\|D\psi\|}{\|\psi\|} = \inf_{\psi \in \overline{P^\perp \mathbf{D}} \setminus \{0\}} \frac{\|D\psi\|}{\|\psi\|}.$$

We have, $\overline{P^\perp \mathbf{D}} = \mathbf{P}^\perp \cap (\mathbf{P}^\perp \cap \mathbf{D}^\perp)^\perp$ as $\mathbf{N}(D P^\perp) = \mathbf{P} \oplus (\mathbf{P}^\perp \cap \mathbf{D}^\perp)$. The latter can be checked directly.

We always have $\text{dist}\{\psi; \mathbf{D}^\perp\} = \|D\psi\|$. If $\psi \in \overline{P^\perp \mathbf{D}} = \mathbf{P}^\perp \cap (\mathbf{P}^\perp \cap \mathbf{D}^\perp)^\perp \subseteq (\mathbf{P}^\perp \cap \mathbf{D}^\perp)^\perp$, we also have $\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\} = \|\psi\|$. Thus,

$$\frac{\text{dist}\{\psi; \mathbf{D}^\perp\}}{\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}} = \frac{\|D\psi\|}{\|\psi\|}, \psi \in \overline{P^\perp \mathbf{D}} \setminus \{0\}.$$

Finally, using the orthogonal representation $\mathbf{P}^\perp = (\mathbf{P}^\perp \cap \mathbf{D}^\perp) \oplus \overline{P^\perp \mathbf{D}}$, every $\varphi \in \mathbf{P}^\perp$ can be written as the orthogonal sum $\varphi = (\varphi - \psi) \oplus \psi$, where $\varphi - \psi \in \mathbf{P}^\perp \cap \mathbf{D}^\perp$, $\psi \in \overline{P^\perp \mathbf{D}}$. Then $\text{dist}\{\psi; \mathbf{D}^\perp\} = \text{dist}\{\varphi; \mathbf{D}^\perp\}$ and also $\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\} = \text{dist}\{\varphi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}$; so the value of the ratio

$$\frac{\text{dist}\{\psi; \mathbf{D}^\perp\}}{\text{dist}\{\psi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}} = \frac{\text{dist}\{\varphi; \mathbf{D}^\perp\}}{\text{dist}\{\varphi; \mathbf{D}^\perp \cap \mathbf{P}^\perp\}}$$

does not depend on $\varphi - \psi$ and its two infimum values, taken with respect to $\psi \in \overline{P^\perp \mathbf{D}} \setminus \{0\}$ and $\varphi \in \mathbf{P}^\perp$, $\varphi \notin \mathbf{D}^\perp$, coincide. \square

Finally, we notice that $g = 0$ if we apply a saddle point approach to diffusion or linear elasticity equations. Indeed, in the Hellinger–Reissner formulation of nonhomogeneous Lamé equations, our σ represents the stress tensor, the Lagrange multiplier u is the displacement, and if we also introduce the strain ϵ by the strain-displacement relation $\epsilon = -B^*u$, then the first line in system (5) becomes $A\sigma - \epsilon = g$, which is the constitutive equation (3-D Hooke’s law), where of course $g = 0$. The second line in (5) is the equilibrium equation, where all body and traction forces are represented by $f \neq 0$. The assumption $g = 0$ allows us to look for even weaker conditions of wellposedness that we plan to investigate in the future.

Acknowledgments The author thanks Ivo Babuška and Franco Brezzi for discussions. This work has been stimulated by collaboration with Nikolai S. Bakhvalov. The author thanks an anonymous referee, who has made numerous useful suggestions to improve the original version of the paper, and CU-Denver students Donald McCuan and Christopher Harder for proofreading the paper.

References

[1] D. N. Arnold and R. Winther. Mixed finite elements for elasticity. *Numer. Math.*, 92(3):401–419, 2002.

[2] I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. B. Kellogg.

[3] N. S. Bakhvalov and A. V. Knyazev. A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations. *Soviet Math. Doklady*, 41(3):481–485, 1990.

[4] N. S. Bakhvalov and A. V. Knyazev. Fictitious domain methods and computation of homogenized properties of composites with a periodic structure of essentially

different components. In Gury I. Marchuk, editor, *Numerical Methods and Applications*, pages 221–276. CRC Press, Boca Raton, 1994.

[5] N. S. Bakhvalov and A. V. Knyazev. Preconditioned iterative methods in a subspace for linear algebraic equations with large jumps in the coefficients. In D. Keyes and J. Xu, editors, *Domain Decomposition Methods in Science and Engineering*, volume 180 of *Contemporary Mathematics*, pages 157–162. American Mathematical Society, Providence, 1994.

[6] N. S. Bakhvalov, A. V. Knyazev, and G. M. Kobel’kov. Iterative methods for solving equations with highly varying coefficients. In Roland Glowinski, Yuri A. Kuznetsov, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 197–205, Philadelphia, PA, 1991. SIAM.

[7] N. S. Bakhvalov, A. V. Knyazev, and R. R. Parashkevov. Extension theorems for Stokes and Lamé equations for nearly incompressible media and their applications to numerical solution of problems with highly discontinuous coefficients. *Numerical Linear Algebra with Applications*, 9(2):115–139, 2002.

[8] M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.

[9] F. Brezzi. On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers. *RAIRO Anal. Numer.*, 2:129–151, 1974.

[10] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer–Verlag, New York, 1991.

[11] P. Ciarlet, Jr., J. Huang, and J. Zou. Some observations on generalized saddle-point problems. *SIAM J. Matrix Anal. Appl.*, 25(1):224–236, 2003.

[12] T. Kato. *Perturbation Theory for Linear Operators*. Springer–Verlag, New–York, 1976.

[13] A. V. Knyazev. Analysis of transmission problems on Lipschitz boundaries in stronger norms. *J. Numer. Math.*, 11(3):225–234, 2003.

[14] A. V. Knyazev. Iterative solution of PDE with strongly varying coefficients: algebraic version. In R. Beauwens and P. de Groen, editors, *Iterative Methods in Linear Algebra*, pages 85–89, Amsterdam, 1992. Elsevier.

[15] A. V. Knyazev and O. Widlund. Lavrentiev regularization + Ritz approximation = uniform finite element error estimates for differential equations with rough coefficients. *Mathematics of Computation*, 72:17–40, 2003.

[16] O. A. Ladyzhenskaya. *The boundary value problems of mathematical physics*, volume 49 of *Applied Mathematical Sciences*. Springer–Verlag, New York, 1985.

[17] J. Xu and L. Zikatanov. Some observations on Babuška and Brezzi theories. *Numer. Math.*, 94(1):195–202, 2003.