NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
Numer. Linear Algebra Appl. 2001; **00**:1–22 *Prepared using nlaauth.cls [Version: 2000/03/22 v1.0]*

Extension theorems for Stokes and Lamé equations for nearly incompressible media and their applications to numerical solution of problems with highly discontinuous coefficients

N. S. Bakhvalov[1], A. V. Knyazev[2,*] and R. R. Parashkevov[3]

[1] *Institute of Numerical Mathematics, Russian Academy of Sciences, Moscow, Russia*
[2] *Department of Mathematics, University of Colorado at Denver, USA*
WWW URL: `http://www-math.cudenver.edu/~aknyazev`
[3] *ExxonMobil Upstream Research Company, Houston, TX*

SUMMARY

We prove extension theorems in the norms described by Stokes and Lamé operators for the three-dimensional case with periodic boundary conditions. For the Lamé equations, we show that the extension theorem holds for nearly incompressible media, but may fail in the opposite limit, i.e. for case of absolutely compressible media. We study carefully the latter case and associate it with the Cosserat problem.

Extension theorems serve as an important tool in many applications, e.g., in domain decomposition and fictitious domain methods, and in analysis of finite element methods. We consider an application of established extension theorems to an efficient iterative solution technique for the isotropic linear elasticity equations for nearly incompressible media and for the Stokes equations with highly discontinuous coefficients. The iterative method involves a special choice for an initial guess and a preconditioner based on solving a constant coefficient problem. Such preconditioner allows the use of well-known fast algorithms for preconditioning.

Under some natural assumptions on smoothness and topological properties of subdomains with small coefficients, we prove convergence of the simplest Richardson method uniform in the jump of coefficients. For the Lamé equations, the convergence is also uniform in the incompressible limit.

Our preliminary numerical results for two-dimensional diffusion problems show fast convergence uniform in the jump and in the mesh size parameter.

KEY WORDS: Linear elasticity, incompressible, Lamé equations, Stokes equations, discontinuous coefficients, iterative methods, divergence-free vector functions, fictitious domain method, Cosserat problem, homogenization, extension theorems, Steklov-Poincaré operators

AMS(MOS) subject classifications. 65F10, 65N12, 73C02, 76D07.

Dedicated to Olof B. Widlund on the occasion of his 60th birthday.

---

[*] Correspondence to: Department of Mathematics, University of Colorado at Denver, P.O. Box 173364, Campus Box 170, Denver, CO 80217-3364. *E-mail*: `andrew.knyazev@cudenver.edu`

Contract/grant sponsor: NSF DMS; contract/grant number: 9501507

Accepted March 25, 2001

1. Introduction

Extension theorems play a crucial role in many applications, e.g., in the theory of domain decomposition, e.g., [58], and fictitious domain methods, e.g., [11], where they describe basic properties of the Steklov-Poincaré operators, and in homogenization, e.g., [59, 16, 15, 56, 34, 11]. In analysis of the finite element methods (FEM), they are needed to derive FEM error estimates uniform in the jump of coefficients, e.g., [37]. Extension theorems are also useful in a theory of partial differential equations (PDE's), e.g., [51], Appendix A, where they are proved for general Sobolev spaces in Lipschitz domains.

In the present paper, for the Stokes equations, we use an extension of solenoidal, or divergence-free, functions from W_2^1, see also [10, 31, 34, 3, 5, 58]. For the Lamé equations, a basic extension theorem in the periodic case is proved in [11]. Here, we show that the Lamé extension theorem holds for nearly incompressible media, i.e. a theorem of extension of functions from W_2^1 preserving the energy norm of the Lamé operator uniformly with respect to the parameter λ as $\lambda \to +\infty$ is proved. We find, however, that the Lamé extension theorem may fail in the opposite limit, i.e. for the unusual case of absolutely compressible media, which has important applications for gases. We study carefully the latter case, find the corresponding new null-space, and associate it with the famous Cosserat problem [25, 26, 53].

For simplicity, periodic boundary value problems for the Stokes equations, in Section 2, and the Lamé equations, in Section 3, with piece-wise constant coefficients are considered. Some other standard boundary value problems can be treated similarly, or can be reduced to the periodic case by using the fictitious/embedding domain method, see Section 6, an algebraic version of which is known as the capacitance matrix method and is popular in engineering, e.g., [57]. We note, however, that the periodic case is also very important in practice by itself as it appears naturally in traditional homogenization procedures for periodic media.

Let us highlight that existence of an extension theorem for a continuous problem gives a hope, but does not guarantee, that a similar extension theorem holds for a particular discretization of the continuous problem. Establishing discrete extension theorems for some typical discretization schemes is an important, highly technical and delicate next step in developing the theory. For the diffusion equation, discrete extension theorems are known in many cases, e.g., for the FEM, see [22, 62, 49, 55, 2, 50, 33, 58]. Discrete analogs of extension theorems in the present paper are apparently not yet known.

One interesting class of applications of extension theorems has been developed in [8, 38, 9, 14, 10, 17, 61, 1, 11, 4, 18, 19, 39, 3, 7, 5, 20, 21], where efficient preconditioned iterative methods for different boundary value problems for PDE's with large jumps in the coefficients have been proposed. For the corresponding discrete problems, analogous methods have been suggested, e.g., in [12, 7, 41], for traditional FEM and in [35] for mixed FEM. In the present paper, we only describe algorithms in a continuous case, for simplicity and to highlight that completely different discretization may successfully be used, in particular, so-called "mesh–free" methods, e.g., [54].

The main idea of our approach to iterative solution of stationary problems with highly discontinuous coefficients can be described as follows. We suggest using a standard preconditioned iterative solver, e.g., the preconditioned conjugate gradient method. The peculiarity is in the choices of the preconditioner and the initial approximation. The choice of the preconditioner is determined by the problem. Usually, a problem similar to the original one, but with constant, or slightly varying coefficients, needs to be solved on every step of the iterative method. Several very efficient algorithms are known for such problems, e.g., multilevel methods, and can be used for our preconditioning. We discuss some possibilities in Section 5. Because of the special choice of the preconditioner, the spectrum of the preconditioned system is located on a few, often just two, small and well-separated intervals; which is the ideal situation for the conjugate gradient method. The increase in the coefficient jump leads to the larger distance between the intervals, but does not affect their lengths. The special constructive choice of the initial guess, which we suggest, further improves the convergence as it put iterative errors in a special subspace, which is invariant with respect to the iterative operator. For some classes of problems, it has been shown that the increase in the coefficient jump does not cause deterioration in the convergence with respect to a natural coefficient-independent norm.

Some of domain decomposition methods converge similarly to our methods, i.e. the convergence is uniform with respect to the jump in coefficients in a coefficient-independent norm, if the domain decomposition is performed along the interface of the coefficient discontinuity. However, such version of the domain decomposition method may be hard to program for optimal design problems. In contrast, our approach is quite suitable for such problems, especially if the original domain is embedded in a torus using the fictitious domain method.

There is a recent increased attention to multigrid methods for problems with jumps in coefficients, where a typical goal is to obtain a uniform convergence in an “energy norm.” However, a uniform in the mesh size and in the jump of coefficients convergence with respect to the energy norm does not necessarily guarantee a similar convergence with respect to a coefficient independent norm. The energy norm itself does not control well the solution in subdomains with small coefficients, which may lead in some cases to physically meaningless large errors in these subdomains.

In the present paper, some results on linear elasticity from [11] are extended to incompressible and nearly incompressible medium. Most importantly, we finally present long overdue proofs of our results announced in [10].

Our results and ideas are gaining popularity in the engineering literature, see, e.g., [29, 30, 27, 28]. The iterative solver for the linear elasticity equations, suggested in [10] and analyzed in the present paper, has been independently rediscovered in two recent engineering publications [54, 52]. It has also been used to compute effective elastic moduli for incompressible material with a periodic system of pores in [6].

We also note that our iterative solver for the Lamé equations is simpler that that of [39], but our assumptions are more restrictive, see Remark 5.1.

We provide numerical results for the most simple FEM discretization of the two-dimensional diffusion equation, also analyzed in [7], in Section 7. While these tests are for the diffusion equation only, they still provide a good illustration of the main idea, described above, that leads to a uniform convergence. We find that our methods converge fast even for situations where our theory fails in the continuous case.

Numerical results for the Lamé equations, using the iterative methods we analyze in the present paper, can be found in [29, 54, 52, 6].

A preliminary version of the paper, without numerical results, has been published as a technical report UCD-CCM 120, 1997, at the Center for Computational Mathematics, University of Colorado at Denver.

2. **Stokes Equations**

We consider a three-dimensional periodic boundary value problem for Stokes equations in the form

$$\begin{aligned} 2\frac{\partial\,[\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})]}{\partial x_i} + \frac{\partial p}{\partial x_j} &= \frac{\partial f_{ij}}{\partial x_i}, \quad j=1,2,3, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{2.1}$$

where $\mathbf{u}=(u_1,u_2,u_3)^T$ and $\varepsilon_{ij}(\mathbf{u}) \overset{\text{def}}{=} \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$.

We assume the summation on repeating indices from 1 to 3. For our periodic boundary value problem we assume that all functions are defined on a three-dimensional unit torus $\mathbf{T}$ with Cartesian coordinate system. We define the unit torus to be the unit cube with all pairs of opposite faces identified. For a Lipschitz domain $\Omega \subseteq \mathbf{T}$ the following norms will be used throughout the paper:

$$\|\,\mathbf{w}\,\|^2_{[W_2^1(\Omega)]^3} \overset{\text{def}}{=} \int_\Omega \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}d\mathbf{x}, \quad \mathbf{w}\in[W_2^1(\Omega)]^3/R^3$$

and

$$\|\,\mathbf{w}\,\|_{\left[W_2^{1/2}(\partial\Omega)\right]^3} \overset{\text{def}}{=} \inf_{\substack{\mathbf{v}\in[W_2^1(\Omega)]^3 \\ \mathbf{v}|_{\partial\Omega}=\mathbf{w}}} \|\,\mathbf{v}\,\|_{[W_2^1(\Omega)]^3}, \quad \mathbf{w}\in\left[W_2^{1/2}(\partial\Omega)\right]^3/R^3.$$

Let $\mathbf{H}$ be the factor space with regard to the constants from R^3 of the space of *solenoidal* functions from $[W_2^1(\mathbf{T})]^3$, equipped with the norm $\|\cdot\|_{[W_2^1(\mathbf{T})]^3}$. Given $f_{ij}\in L_2(\mathbf{T})$, a weak solution of problem (2.1) is a function $\mathbf{u}\in\mathbf{H}$, satisfying

$$\Lambda(\mathbf{u},\mathbf{v}) = \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall\mathbf{v}\in\mathbf{H}, \tag{2.2}$$

where $\Lambda(\mathbf{u},\mathbf{v}) \overset{\text{def}}{=} 2\int_{\mathbf{T}}\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})d\mathbf{x}$. Let $\mathbf{D}\subset\mathbf{T}$ be a Lipschitz domain and its complement $\mathbf{T}\backslash\overline{\mathbf{D}}$ be also a domain. We assume that the viscosity coefficient $\mu()$ is piece-wise constant, and equals to $\mu>0$ and $\mu^*>0$ in $\mathbf{D}$ and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly. We will need two auxiliary results.

Proposition 2.1. *For any Lipschitz domain* $\Omega\subset\mathbf{T}$ *there exists a constant* $\kappa_1\equiv\kappa_1(\Omega)>0$, *such that for an arbitrary function* $\mathbf{v}\in[W_2^1(\Omega)]^3/R^3$ *a function* $\mathbf{w}\in[W_2^1(\Omega)]^3/R^3$ *can be found for which* $\int_\Omega\varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x}\ge\kappa_1\,\|\,\mathbf{w}\,\|^2_{\left[W_2^{1/2}(\partial\Omega)\right]^3}$ *and* $\mathbf{w}=\mathbf{v}+C\mathbf{x}$ *in* Ω *with some* 3×3 *matrix* $C=-C^T$, *not depending on* $\mathbf{x}\in\mathbf{T}$, *the vector of independent variables on the torus* $\mathbf{T}$.

Proof. The following version of a Korn type inequality is obtained in [11, Section 5]: for a Lipschitz domain Ω on the torus $\mathbf{T}$ there can be found a constant $\kappa_1\equiv\kappa_1(\Omega)>0$, for which for any function $\mathbf{v}\in[W_2^1(\Omega)]^3/R^3$ there exist a function $\mathbf{w}\in[W_2^1(\Omega)]^3/R^3$ such that $\int_\Omega\varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x}\ge\kappa_1\int_\Omega\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}d\mathbf{x}$ and $\mathbf{w}=\mathbf{v}+C\mathbf{x}$ in Ω with some 3×3 constant matrix $C=-C^T$.

By definition,

$$\begin{aligned}\int_\Omega \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} d\mathbf{x} &= \| \mathbf{w} \|^2_{[W_2^1(\Omega)]^3} \\ &\geq \| \mathbf{w} \|^2_{\left[W_2^{1/2}(\partial\Omega)\right]^3} ,\end{aligned}$$

and we come to the desired inequality. □

Proposition 2.2. *For any Lipschitz domain* $\Omega \subset \mathbf{T}$ *there exists a positive constant* $\theta(\Omega) < \infty$ *such that for an arbitrary function* $\mathbf{v} \in [W_2^{1/2}(\partial\Omega)]^3/R^3$ *under the condition* $\oint (\mathbf{v}, \mathbf{n}) d\partial\Omega = 0$, *there can be found a solenoidal extension* $\mathbf{v} \in [W_2^1(\Omega)]^3/R^3$, *for which*

$$\| \mathbf{v} \|^2_{[W_2^1(\Omega)]^3} \leq \theta(\Omega) \| \mathbf{v} \|^2_{[W_2^{1/2}(\partial\Omega)]^3} .$$

Proof. In [42], this proposition has been proven for a bounded Lipschitz domain $\Omega \subset R^3$. The proof is applicable for a Lipschitz domain on a torus as well.

The desired solenoidal extension can also be found explicitly in Ω as a weak solution of the following problem:

$$\Delta \mathbf{v} + \mathbf{grad}\, q = \mathbf{0},$$

$$\operatorname{div} \mathbf{v} = 0,$$

with the trace of $\mathbf{v}$ given on the boundary $\partial\Omega$. Then, the desired estimate of the lemma is simply the well-known stability estimate for the problem above, e.g., [32]. □

We are now ready to prove our main extension result for solenoidal functions. Announced in our earlier paper [10], it also appeared in several different forms in [31, 34, 3, 5, 58].

Lemma 2.1. *There exists a constant* $\kappa = \kappa(\mathbf{D}) > 0$, *such that for an arbitrary function* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$ *such that* $\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} \geq \kappa \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x}$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$ *with some* 3×3 *matrix* $C = -C^T$ *independent of* $\mathbf{x}$.

Proof. Let us apply Proposition 2.1 to the function $\mathbf{v}$ in the domain $\Omega = \mathbf{D}$. There exist a function $\mathbf{w}$ and a constant $\kappa_1 = \kappa_1(\mathbf{D})$, such that

$$\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} \geq \kappa_1 \| \mathbf{w} \|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

where $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, for some constant antisymmetric matrix C. By assumption, the function $\mathbf{v}$ is solenoidal on the torus $\mathbf{T}$, therefore it is solenoidal in $\mathbf{D}$. Then $\mathbf{w}$ is solenoidal in $\mathbf{D}$, and its trace on the boundary $\partial\mathbf{D}$ satisfies the following

$$\oint (\mathbf{w}, \mathbf{n}) d\partial\mathbf{D} = \int \operatorname{div} \mathbf{w}\, d\mathbf{D} = 0.$$

Next, we use Proposition 2.2 for the domain $\Omega = \mathbf{T} \backslash \overline{\mathbf{D}}$. For $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$ there exists a solenoidal extension $\mathbf{w} \in [W_2^1(\mathbf{T} \backslash \overline{\mathbf{D}})]^3/R^3$ such that

$$\| \mathbf{w} \|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \quad \leq \quad \theta \| \mathbf{w} \|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

where $\theta = \theta(\mathbf{T}\backslash\overline{\mathbf{D}})$.

Finally,

$$\| \mathbf{w} \|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \quad = \quad \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \frac{\partial w_i}{\partial x_j}\frac{w_i}{\partial x_j} \geq 2\int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x}. \tag{2.3}$$

In this way, the function $\mathbf{w}$ is solenoidal on the whole torus $\mathbf{T}$ and $\mathbf{w} \in [W_2^1(\mathbf{T})]^3/R^3$, i.e. $\mathbf{w} \in \mathbf{H}$; and the following estimate holds

$$\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x} \quad \geq \quad 2\frac{\kappa_1}{\theta}\int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x},$$

equivalent to the one we are looking for with

$$\frac{1}{\kappa} = 1 + \frac{\theta}{2\kappa_1}.$$

□

Proof.[Alternative proof of Lemma 2.1] One can prove Lemma 2.1 without using traces, as suggested by E. G. D'yakonov (private communication).

The first step is just the same as in the proof of Proposition 2.1. We use a Korn type inequality obtained in [11, Section 5]: for the given function $\mathbf{v} \in [W_2^1(\mathbf{D})]^3/R^3$ there exist a function $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$, such that

$$\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})d\mathbf{x} \quad \geq \quad \kappa_1 \| \mathbf{w} \|^2_{[W_2^1(\mathbf{D})]^3},$$

where $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$ with a constant antisymmetric matrix C. By assumption, the function $\mathbf{v}$ is solenoidal on the torus $\mathbf{T}$, thus it is solenoidal in $\mathbf{D}$, and then $\mathbf{w}$ is solenoidal in $\mathbf{D}$ as well.

The second step is different. As shown in [11], for any function $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$ there exists an extension, not necessarily solenoidal, $\mathbf{r} \in [W_2^1(\mathbf{T})]^3/R^3$, such that $\mathbf{r} = \mathbf{w}$ in $\mathbf{D}$ and $\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \leq C(\mathbf{D})\,\|\mathbf{w}\|^2_{[W_2^1(\mathbf{D})]^3}$.

Now, we set $\mathbf{w} = \mathbf{r} + \mathbf{s}$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$, where $\mathbf{s}$ is a weak solution to the following homogeneous Dirichlet problem:

$$\Delta(\mathbf{r}+\mathbf{s}) + \mathbf{grad}\, q = \mathbf{0}$$

$$\text{div}\,(\mathbf{r}+\mathbf{s}) = 0,$$

with $\mathbf{s} = \mathbf{0}$ on the boundary $\partial\mathbf{D}$. The boundary condition is chosen in such a way that we can extend $\mathbf{s}$ as zero inside $\mathbf{D}$. We have a stability estimate, e.g., [23, Chapter 2],

$$\|\mathbf{s}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \leq C_1(\mathbf{D})\,\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3},$$

which shows that

$$\begin{aligned}\|\mathbf{w}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} = \|\mathbf{s}+\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \quad &\leq \quad 2\left(\|\mathbf{s}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} + \|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}\right) \\ &\leq \quad 2(1+C_1)\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}.\end{aligned}$$

Combining with (2.3) and the previous estimates, we derive the desired inequality with $\kappa^{-1} = 1+\kappa_1^{-1}C(1+C_1)$. □

Let us now present an application of our extension theorem to convergence analysis of a preconditioned iterative solver for the Stokes equations. For preconditioning, we introduce

$$\Lambda^*(\mathbf{v},\mathbf{w}) \quad \overset{\text{def}}{=} \quad 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w})d\mathbf{x}, \qquad \forall \mathbf{v},\mathbf{w} \in \mathbf{H}, \tag{2.4}$$

and notice that the bilinear form $\frac{1}{\mu^*}\Lambda^*(\cdot,\cdot)$ describes a new scalar product on **H** and the corresponding norm, which is equivalent to the original norm $\|\cdot\|^2_{[W_2^1(\mathbf{T})]^3}$.

We consider the following iterative method for problem (2.2):

$$\Lambda^*(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v})+\Lambda(\mathbf{u}^n,\mathbf{v}) \quad = \quad \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v}\in\mathbf{H},\ n=0,1,\ldots, \tag{2.5}$$

where the initial guess $\mathbf{u}^0 \in \mathbf{H}$ is determined by the equation

$$\Lambda^*(\mathbf{u}^0,\mathbf{v}) \quad = \quad \int_{\mathbf{T}} g_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v}\in\mathbf{H}, \tag{2.6}$$

with

$$g_{ij} = \begin{cases} \text{an arbitrary function of } L_2(\mathbf{D}) & \text{in} \quad \mathbf{D}, \\ f_{ij} & \text{in} \quad \mathbf{T}\backslash\overline{\mathbf{D}}. \end{cases} \tag{2.7}$$

Using Lemma 2.1 and arguments analogous to those in [11], we prove the following

Lemma 2.2. *We define the subspace* $\mathbf{N}\subset\mathbf{H}$ *of functions* $\mathbf{w}\in\mathbf{H}$ *such that* $\mathbf{w}=C\mathbf{x}$ *in the domain* **D** *where* $\mathbf{x}\in\mathbf{T}$ *is the vector of independent variables on the torus* **T** *and* $C=-C^T$ *is any* 3×3 *matrix, independent of* **x**. *Let the subspace* $\mathbf{R}\subset\mathbf{H}$ *be defined as the orthogonal, with respect to the bilinear form* $\Lambda^*(\cdot,\cdot)$, *complement of* **N**, *i.e.*

$$\Lambda^*(\mathbf{v},\mathbf{w})=0, \quad \forall \mathbf{v}\in\mathbf{R},\mathbf{w}\in\mathbf{N}.$$

Then:

(a) for the initial guess $\mathbf{u}^0$, *we have* $\mathbf{u}^0-\mathbf{u}\in\mathbf{R}$ *where* **u** *is the true solution of our problem (2.2);*

(b) **R** *is an invariant subspace for the error propagation operator from* $\varepsilon^n=\mathbf{u}^n-\mathbf{u}$ *to* $\varepsilon^{n+1}=\mathbf{u}^{n+1}-\mathbf{u}$ *acting according to the rule:*

$$\Lambda^*(\frac{\varepsilon^{n+1}-\varepsilon^n}{\tau},\mathbf{v})+\Lambda(\varepsilon^n,\mathbf{v}) \quad = \quad 0, \quad \forall \mathbf{v}\in\mathbf{H},\ n=0,1,\cdots; \tag{2.8}$$

(c) on this invariant subspace, we have

$$\kappa\,\frac{\mu}{\mu^*}\,\Lambda^*(\mathbf{v},\mathbf{v})\le\Lambda(\mathbf{v},\mathbf{v}) \quad \le \quad \max\{\frac{\mu}{\mu^*},1\}\,\Lambda^*(\mathbf{v},\mathbf{v}), \quad \forall \mathbf{v}\in\mathbf{R}.$$

Proposition 2.3. *The following equality holds:*

$$\Lambda^*(\mathbf{v},\mathbf{w}) \quad = \quad \Lambda(\mathbf{v},\mathbf{w}), \quad \forall \mathbf{v}\in\mathbf{H},\mathbf{w}\in\mathbf{N}.$$

Proof. We note that $\varepsilon_{ij}(\mathbf{w}) = 0$ in the domain $\mathbf{D}$ when $\mathbf{w} \in \mathbf{N}$. Therefore,

$$\begin{aligned}\Lambda^*(\mathbf{v},\mathbf{w}) = 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} &= 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x} = \\ 2\int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \mu(\mathbf{x})\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x} &= 2\int_{\mathbf{T}} \mu(\mathbf{x})\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} = \Lambda(\mathbf{v},\mathbf{w}).\end{aligned}$$

□

Proof.[Lemma 2.2]

(a) According to the definition of the space $\mathbf{R}$, we need to verify the equality $\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{w}) = 0, \mathbf{w} \in \mathbf{N}$, which is equivalent to the equality $\Lambda^*(\mathbf{u},\mathbf{w}) = \Lambda(\mathbf{u},\mathbf{w})$. The last equality is true by Proposition 2.3.

(b) Let us consider the bounded linear operator $A : \mathbf{H} \to \mathbf{H}$ defined by

$$\Lambda^*(A\mathbf{v},\mathbf{w}) = \Lambda(\mathbf{v},\mathbf{w}), \quad \forall \mathbf{v},\mathbf{w} \in \mathbf{H}.$$

The error propagation operator from ε^n to ε^{n+1} is $I - \tau A$, where I is the identity operator, i.e. $\varepsilon^{n+1} = (I - \tau A)\varepsilon^n$. The statement needed to be verified is equivalent to the statement that $\mathbf{R}$ is an invariant subspace for the operator A. If $\Lambda^*(\mathbf{v},\mathbf{w}) = 0,\ \ \mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{N}$, then $\Lambda(\mathbf{v},\mathbf{w}) = 0$ by Proposition 2.3 and, therefore, $\Lambda^*(A\mathbf{v},\mathbf{w}) = 0$ by the definition of the operator A.

(c) The right inequality is true for any function $\mathbf{v} \in \mathbf{H}$ and can be proven directly. Let us verify the left inequality. For any $\mathbf{v} \in \mathbf{H}$, we have

$$\begin{aligned}\Lambda(\mathbf{v},\mathbf{v}) &= 2\mu \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} + 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &\geq 2\mu \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x}.\end{aligned}$$

Also, by Lemma 2.1, we can find a function $\mathbf{w} \in \mathbf{H}$ such that $\mathbf{v} - \mathbf{w} \in \mathbf{N}$ and

$$\begin{aligned}\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} &= \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} \\ &\geq \kappa \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} = \frac{\kappa}{2\mu^*}\Lambda^*(\mathbf{w},\mathbf{w}) .\end{aligned}$$

Taking into account that $\mathbf{v} \in \mathbf{R}$, and using the definition of the subspace $\mathbf{R}$, we can derive $\Lambda^*(\mathbf{v} - \mathbf{w}, \mathbf{w}) = 0$, and therefore, $\Lambda^*(\mathbf{w},\mathbf{w}) = \Lambda^*(\mathbf{v},\mathbf{v}) + \Lambda^*(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}) \geq \Lambda^*(\mathbf{v},\mathbf{v})$. □

As a direct consequence of Lemma 2.2 (cf. [11]), we obtain the following

Theorem 2.1. *Let $\mu \geq \mu^* > 0$ and $\tau = \frac{\mu^*}{\mu}$. Then the sequence of approximations $\{\mathbf{u}^n\}$ given by the method (2.5) with the initial guess computed from (2.6) satisfies the following convergence rate estimate:*

$$\Lambda^*(\mathbf{u}^n - \mathbf{u}, \mathbf{u}^n - \mathbf{u}) \quad \leq \quad q^{2n}\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{u}^0 - \mathbf{u}), \quad 0 \leq q = 1 - \kappa < 1.$$

Proof. Using notations of Lemma 2.2, we write the identity $\varepsilon^{n+1} = (I - \tau A)\varepsilon^n$, where $\varepsilon^{n+1}, \varepsilon^n \in \mathbf{R}$. The operator $I - \tau A : \mathbf{R} \to \mathbf{R}$ is selfadjoint and is a contraction in the scalar product $\Lambda^*(\cdot,\cdot)$, which follows from statement (c) from Lemma 2.2 that in operator form reads $\kappa\mu/\mu^* I \leq A \leq \mu/\mu^* I$. For example, when $\tau = \mu^*/\mu$ we obtain $0 \leq I - \tau A \leq qI$, which had to be proven. □

The theorem shows uniform, with respect to $\mu^*, 0 < \mu^* \leq \mu$, convergence of the method. The next theorem demonstrates that the initial error in our iterative method is uniformly bounded under some natural assumptions required to have a possibility of taking the limit $\mu^* \to 0$, cf. [11].

Theorem 2.2. *Let $\mu \geq \mu^* > 0$, where μ is fixed. Let also*

$$tr\{(F_{\mathbf{D}} - G_{\mathbf{D}})C_{\mathbf{D}}\} = 0, \tag{2.9}$$

where $F_{\mathbf{D}}$ and $G_{\mathbf{D}}$ are averages in $\mathbf{D}$ of the matrices $f = (f_{ij})$ and $g = (g_{ij})$, and $C_{\mathbf{D}}$ is an arbitrary 3×3 antisymmetric matrix, $C_{\mathbf{D}} = -C_{\mathbf{D}}^T$, such that there exists a continuous single-value branch of the multi-place function $C_{\mathbf{D}}\mathbf{x}\,|_{\mathbf{D}}$ restricted on $\mathbf{D}$, see [11].

Then, the initial guess computed from (2.6) satisfies the following estimate:

$$\frac{1}{\mu^*}\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{u}^0 - \mathbf{u}) \quad \leq \quad const\left(\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9} + \|f\|^2_{[L_2(\mathbf{T})]^9}\right),$$

where here and below const denotes a generic constant independent of μ^*.

Proof. We write

$$\Lambda(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}) = \Lambda(\mathbf{u}^0, \mathbf{v}) - \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H} \tag{2.10}$$

and write the vector $\mathbf{v} \in \mathbf{H}$ as an orthogonal sum with respect to $\Lambda^*(\cdot,\cdot)$,

$$\mathbf{v} = \mathbf{v}_N + \mathbf{v}_R, \;\; \mathbf{v}_N \in \mathbf{N}, \; \mathbf{v}_R \in \mathbf{R}.$$

We will now show that all terms containing $\mathbf{v}_N$ in (2.10) vanish.

For the left hand side, $\Lambda(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}_N) = 0$, because of Proposition 2.3, as $\mathbf{u}^0 - \mathbf{u} \in \mathbf{R}$ by Lemma 2.2 and $\mathbf{v}_N \in \mathbf{N}$.

For the right hand side, using Proposition 2.3, we rewrite the first term

$$\Lambda(\mathbf{u}^0, \mathbf{v}_N) = \Lambda^*(\mathbf{u}^0, \mathbf{v}_N) = \int_{\mathbf{T}} g_{ij}\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x},$$

using the definition (2.6) of $\mathbf{u}^0$ for the last equality. Making that substitution, we obtain the right hand side of (2.10) in the form

$$\int_{\mathbf{T}}(g_{ij} - f_{ij})\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x} = \int_{\mathbf{D}}(g_{ij} - f_{ij})\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x}$$

as $\mathbf{v}_N \in \mathbf{N}$, and this value is zero owing to condition (2.9).

Thus, we just proved that it is sufficient to take $\mathbf{v} \in \mathbf{R}$ in (2.10). Lemma 2.2 states that $\mathbf{u}^0 - \mathbf{u} \in \mathbf{R}$ and the symmetric bilinear form $\Lambda(\cdot,\cdot)$ is coercive and bounded on the subspace $\mathbf{R}$ with respect to the form $\dfrac{\mu}{\mu^*}\Lambda^*(\cdot,\cdot)$ with constants κ and 1 respectively.

We now estimate the norm, with respect to the $\dfrac{1}{\mu^*}\Lambda^*(\cdot,\cdot)$ scalar product, of the linear functional of the right hand side of (2.10). We will not use the fact that $\mathbf{v} \in \mathbf{R}$ in our arguments below.

For the first term,

$$|\Lambda(\mathbf{u}^0, \mathbf{v})|^2 \leq \Lambda(\mathbf{u}^0, \mathbf{u}^0)\Lambda(\mathbf{v}, \mathbf{v}) \leq \left(\frac{\mu}{\mu^*}\right)^2 \Lambda^*(\mathbf{u}^0, \mathbf{u}^0)\Lambda^*(\mathbf{v}, \mathbf{v}),$$

where

$$\frac{1}{\mu^*}\Lambda^*(\mathbf{u}^0, \mathbf{u}^0) \leq const\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9}. \tag{2.11}$$

For the second term,

$$\left|\int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}\right|^2 \leq \|f\|^2_{[L_2(\mathbf{T})]^9}\|\mathbf{v}\|^2_{[W_2^1(\mathbf{T})]^3} \leq \|f\|^2_{[L_2(\mathbf{T})]^9}\frac{const}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v}),\ \mathbf{v}\in\mathbf{H},$$

with a constant *const* independent of μ^*.

We now combine the two estimates to obtain

$$\left|\Lambda(\mathbf{u}^0,\mathbf{v})-\int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}\right|^2 \leq const\left(\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9}+\|f\|^2_{[L_2(\mathbf{T})]^9}\right)\frac{1}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v}).$$

Therefore, problem (2.10) is well-posed in the subspace **R** of **H**, and the stability estimate holds. □

Remark 2.1. *The coefficient $\mu(\mathbf{x})$ can be variable in* **D**. *Let $\kappa\chi\mu\leq\mu(\mathbf{x})\leq\mu$ for almost all $\mathbf{x}\in\mathbf{D}$, where χ and μ are positive constants. Then Theorem 2.1 holds with $q=1-\kappa\chi<1$. Actually, only statement (c) of Lemma 2.2 changes and the corresponding inequalities can be replaced by*

$$\kappa\chi\frac{\mu}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v})\leq\Lambda(\mathbf{v},\mathbf{v})\leq\max\{\frac{\mu}{\mu^*},1\}\Lambda^*(\mathbf{v},\mathbf{v}),\ \ \mathbf{v}\in\mathbf{R}.$$

Remark 2.2. *One can relax the requirements for connectivity of* **D** *and* $\mathbf{T}\backslash\overline{\mathbf{D}}$. *Let* **D** *consists of finite number of Lipschitz domains* $\mathbf{D}_i$ *with non-intersecting closures and each connected component of* $\mathbf{T}\backslash\overline{\mathbf{D}}$ *satisfies the condition (L). We can change the formulation of Lemma 2.1 in such a way that* $\mathbf{w}=\mathbf{v}+\mathbf{c}_i+C_i\mathbf{x}$ *in* $\mathbf{D}_i$ *with some constant vectors* $\mathbf{c}_i$ *and* 3×3 *matrices* $C_i=-C_i^T$ *different in every connected component* $\mathbf{D}_i$. *Then Lemma 2.1 is valid and hence, Theorem 2.1 (compare to [8, 9]).*

3. Lamé Equations

Following [9, 11], we consider a three-dimensional periodic boundary value problem for the isotropic linear elasticity (Lamé) equations:

$$2\frac{\partial\left[\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\right]}{\partial x_i}+\frac{\partial[\lambda(\mathbf{x})\operatorname{div}\mathbf{u}]}{\partial x_j} = \frac{\partial f_{ij}}{\partial x_i},\ \ j=1,2,3, \tag{3.1}$$

where $\mathbf{u}=(u_1,u_2,u_3)^T$ and $\varepsilon_{ij}(\mathbf{u})\overset{\text{def}}{=}\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]$.

We re-define the space $\mathbf{H}\overset{\text{def}}{=}[W_2^1(\mathbf{T})]^3/R^3$, i.e. we no longer require vectors in **H** to be solenoidal. Then a weak solution to problem (3.1), given $f_{ij}\in L_2(\mathbf{T})$, is a function $\mathbf{u}\in\mathbf{H}$ such that

$$\Lambda(\mathbf{u},\mathbf{v})=\int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x},\quad \forall\mathbf{v}\in\mathbf{H}, \tag{3.2}$$

where $\Lambda(\mathbf{u},\mathbf{v})\overset{\text{def}}{=}\int_{\mathbf{T}}[2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})+\lambda(\mathbf{x})\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}]d\mathbf{x}$. We assume that Lamé coefficients $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are piece-wise constant and equal to (λ,μ) and (λ^*,μ^*) in **D** and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly. Also, we assume

$$\mu,\mu^*>0,K\equiv\lambda+\frac{2}{3}\mu>0,K^*\equiv\lambda^*+\frac{2}{3}\mu^*>0,0<\eta\leq\frac{\mu}{\mu^*}\frac{K^*}{K}\leq\eta^{-1}, \tag{3.3}$$

for some constant $\eta\leq1$.

Lemma 3.1. *Under the assumptions (3.3), we have the following estimates:*

$$0 < \alpha \leq \frac{2\mu \sum_{i,j} |a^i_j|^2 + \lambda |\sum_i a^i_i|^2}{2\mu^* \sum_{i,j} |a^i_j|^2 + \lambda^* |\sum_i a^i_i|^2} \leq \beta,$$

$$\alpha \equiv \min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}, \beta \equiv \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}, \frac{\alpha}{\beta} \geq \eta > 0,$$

where a^i_j, $i,j = 1,2,3$ *are arbitrary real numbers, not all equal to zero.*

Proof. First, we re-write the fraction in the form $\dfrac{\mu x + Ky}{\mu^* x + K^* y}$, where

$$x = 2\sum_{i,j} |a^i_j - \frac{1}{3}\delta_{ij} \sum_k a^k_k|^2, \quad \text{and} \quad y = |\sum_i a^i_i|^2.$$

Then, we use the well-known inequalities

$$\min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\} \leq \frac{\mu x + Ky}{\mu^* x + K^* y} \leq \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\},$$

which are valid for $\mu, \mu^* > 0, x, y \geq 0$, and $K, K^* > 0$ to obtain the result. □

Corollary 3.1. *The constants* α *and* β *are actually bounds for the ratio of the squares of the following energy norms:*

$$0 < \alpha \leq \frac{\int_{\mathbf{D}} [2\mu \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) + \lambda |\mathrm{div}\, \mathbf{w}|^2] d\mathbf{x}}{\int_{\mathbf{D}} [2\mu^* \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) + \lambda^* |\mathrm{div}\, \mathbf{w}|^2] d\mathbf{x}} \leq \beta, \quad \forall \mathbf{w} \in [W^1_2(\mathbf{D})]^3 / R^3.$$

Proof. Noting that $\varepsilon_{ij}(\mathbf{w}) \in L_2(\mathbf{D})$ and applying Lemma 3.1 with $a^i_j = \varepsilon_{ij}(\mathbf{w}(\mathbf{x}))$, $\mathbf{x} \in \mathbf{D}$ almost everywhere, we obtain the estimates by integrating the numerator and the denominator over $\mathbf{D}$. □

Let us define on $\mathbf{H}$ the following bilinear form:

$$\Lambda^*(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) + \lambda^* \int_{\mathbf{T}} \mathrm{div}\, \mathbf{u}\, \mathrm{div}\, \mathbf{v}\, d\mathbf{x},$$

and let the forms $\Lambda^*_{\mathbf{D}}(\mathbf{u}, \mathbf{v})$ and $\Lambda^*_{\mathbf{T} \setminus \overline{\mathbf{D}}}(\mathbf{u}, \mathbf{v})$ be defined in the same way, except for the domain of integration.

Theorem 3.1. *Let* $\dfrac{K^*}{\mu^*} \geq \delta > 0$. *Then, there exist a positive constant* $\kappa = \kappa(\delta, \mathbf{D})$, *such that for every* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$, *such that* $\Lambda^*_{\mathbf{D}}(\mathbf{w}, \mathbf{w}) \geq \kappa \Lambda^*(\mathbf{w}, \mathbf{w})$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$, *where* $C = -C^T$ *is some* 3×3 *matrix independent of* $\mathbf{x}$.

Proof. Let us choose a fixed function $\mathbf{w}_f \in \mathbf{H}$, such that $\oint (\mathbf{w}_f, \mathbf{n})\, d\partial\mathbf{D} = 1$. For example, if the domain $\mathbf{D}$ does not wrap around the torus $\mathbf{T}$ in the direction of x_1, one can choose $\mathbf{w}_f = (x_1, x_1, x_1)$. Then, by Proposition 2.1, there exist a constant $\kappa_1 = \kappa_1(\mathbf{D})$, such that for $\forall \mathbf{v} \in \mathbf{H}$ there exist a function $\mathbf{w} \in [W^1_2(\mathbf{D})]^3 / R^3$, such that

$$\int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) \geq \kappa_1 \|\mathbf{w}\|^2_{[W^{1/2}_2(\partial\mathbf{D})]^3},$$

and $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, where $C = -C^T$ is some 3×3 matrix independent of $\mathbf{x}$. Setting $\mathbf{w}_s = \mathbf{w} - \mathbf{w}_f \oint (\mathbf{w},\mathbf{n})\, d\partial\mathbf{D}$ in $\mathbf{D}$, we have $\mathbf{w}_s \in [W_2^1(\mathbf{D})]^3/R^3$ and $\oint (\mathbf{w}_s,\mathbf{n})\, d\partial\mathbf{D} = 0$. Then, by Proposition 2.2, there exists a solenoidal extension $\mathbf{w}_s \in [W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3/R^3$ satisfying

$$\|\mathbf{w}_s\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \le \theta \|\mathbf{w}_s\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

with some positive constant $\theta = \theta(\mathbf{T}\backslash\overline{\mathbf{D}})$. Now we can extend $\mathbf{w}$ from $\mathbf{D}$ into $\mathbf{T}\backslash\overline{\mathbf{D}}$ by $\mathbf{w} = \mathbf{w}_s + \mathbf{w}_f \oint (\mathbf{w},\mathbf{n})\, d\partial\mathbf{D}$, and then $\mathbf{w} \in \mathbf{H}$ with

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w},\mathbf{w}) \le 2\left(\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_s,\mathbf{w}_s) + \Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_f,\mathbf{w}_f)\Phi(\mathbf{w})\right),$$

where $\Phi(\mathbf{w}) = |\oint (\mathbf{w},\mathbf{n})\, d\partial\mathbf{D}|^2$. Let us estimate each term on the right-hand side separately. First,

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_s,\mathbf{w}_s) = 2\mu^* \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{w}_s)\varepsilon_{ij}(\mathbf{w}_s) d\mathbf{x} \le \mu^* \|\mathbf{w}_s\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \le$$

$$\theta\mu^* \|\mathbf{w}_s\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3} \le 2\theta\mu^* \{\|\mathbf{w}\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3} + c_1\Phi(\mathbf{w})\} \le$$

$$2\theta\mu^* \{\frac{1}{\kappa_1} \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} + c_1\Phi(\mathbf{w})\},$$

with $c_1 = \|\mathbf{w}_f\|^2_{[W_2^{1/2}(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}$. In the above inequalities we have used that $\operatorname{div} \mathbf{w}_s = 0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$, and $\mathbf{w}_s = \mathbf{w} - \mathbf{w}_f \oint (\mathbf{w},\mathbf{n})\, d\partial\mathbf{D}$ on $\partial\mathbf{D}$.

Second, using $\lambda^* \le K^*$, we have

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_f,\mathbf{w}_f) = 2\mu^* c_2 + \lambda^* c_3 \le 2\mu^* c_2 + K^* c_3,$$

with $c_2 = \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{w}_f)\varepsilon_{ij}(\mathbf{w}_f) d\mathbf{x}$ and $c_3 = \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} |\operatorname{div} \mathbf{w}_f|^2\, d\mathbf{x}$.

Finally, using the Cauchy inequality, we get

$$\Phi(\mathbf{w}) = |\oint (\mathbf{w},\mathbf{n})\, d\partial\mathbf{D}|^2 = |\int_{\mathbf{D}} \operatorname{div} \mathbf{w}\, d\mathbf{x}|^2 \le mes(\mathbf{D}) \int_{\mathbf{D}} |\operatorname{div} \mathbf{w}|^2\, d\mathbf{x}.$$

Combining the above estimates, we obtain

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w},\mathbf{w}) \le 2\mu^{\$} \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w}) d\mathbf{x} + \lambda^{\$} \int_{\mathbf{D}} |\operatorname{div} \mathbf{w}|^2\, d\mathbf{x},$$

where $\mu^{\$} = \dfrac{2\mu^*\theta}{\kappa_1}$ and $\lambda^{\$} = (4\mu^*(\theta c_1 + c_2) + 2K^* c_3) mes(\mathbf{D})$.

Now, by setting $\mu = \mu^{\$}$ and $\lambda = \lambda^{\$}$ in the assumptions of Corollary 3.1 and taking into account $\dfrac{K^*}{\mu^*} \ge \delta > 0$, we get

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w},\mathbf{w}) \le \beta^{\$} \Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}),$$

where $\beta^{\$} = 2\max\left\{\dfrac{\theta}{\kappa_1}, \dfrac{1}{\delta} mes(\mathbf{D}) \left[2\theta\left\{\dfrac{1}{3\kappa_1 mes(\mathbf{D})} + c_1\right\} + 2c_2 + \delta c_3\right]\right\}$, which is equivalent to the desired inequality with $\kappa = \dfrac{1}{1+\beta^{\$}}$. □

Remark 3.1. *A weaker version of Theorem 3.1 has been proven in [11], where the constant* $\kappa \to 0$ *in the incompressible limit, i.e. when* $\dfrac{K^*}{\mu^*} \to +\infty$. *In contrast, the constant* κ *in Theorem 3.1 depends only on the lower bound on* $\dfrac{K^*}{\mu^*}$ *and thus does not deteriorate in the incompressible limit.*

Next, we show that the result of Theorem 3.1 is sharp in the sense that, in general, $\kappa \to 0$ when $\dfrac{K^*}{\mu^*} \to 0$.

Lemma 3.2. *If the domain* $\mathbf{D}$ *does not wrap around the torus* $\mathbf{T}$, *then there exist a nontrivial function* $\mathbf{v} \in \mathbf{H}$, *such that for any function* $\mathbf{w} \in \mathbf{H}$, *such that* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$, *for some* 3×3 *matrix* $C = -C^T$ *the ratio* $\dfrac{\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})}{\Lambda^*(\mathbf{w},\mathbf{w})} \to 0$ *as* $\dfrac{K^*}{\mu^*} \to 0$. *Namely, an example of such vector function is* $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ *in* $\mathbf{D}$ *and extended as a function in* $[W^1_2(\mathbf{T})]^3$ *outside* $\mathbf{D}$.

Proof. First of all, the fact that function $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ in $\mathbf{D}$ can be extended to the whole torus in $[W^1_2(\mathbf{T})]^3$ follows from extension arguments in [11]. For this function, $\varepsilon(\mathbf{v})$ is a spherical tensor in $\mathbf{D}$, i.e. $\varepsilon_{ij}(\mathbf{v}) = \Psi(\mathbf{x})\delta_{ij}$ in $\mathbf{D}$, namely, with $\Psi(\mathbf{x}) \equiv 1$. For any $\mathbf{w} \in \mathbf{H}$, such that $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, we have $\varepsilon_{ij}(\mathbf{w}) = \varepsilon_{ij}(\mathbf{v}) = \Psi(\mathbf{x})\delta_{ij}$ in $\mathbf{D}$. Then, on one hand

$$\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) = 2\mu^* \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})\, d\mathbf{x} + \lambda^* \int_{\mathbf{D}} |\mathrm{div}\, \mathbf{v}|^2\, d\mathbf{x} = 9K^* \int_{\mathbf{D}} \Psi^2(\mathbf{x}) d\mathbf{x}.$$

On the other hand, we have

$$\Lambda^*(\mathbf{w},\mathbf{w}) = 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x} + \lambda^* \int_{\mathbf{T}} |\mathrm{div}\, \mathbf{w}|^2\, d\mathbf{x} =$$

$$\mu^* \int_{\mathbf{T}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} d\mathbf{x} + (\lambda^* + \mu^*) \int_{\mathbf{T}} |\mathrm{div}\, \mathbf{w}|^2\, d\mathbf{x} > (\lambda^* + \mu^*) \int_{\mathbf{T}} |\mathrm{div}\, \mathbf{w}|^2\, d\mathbf{x},$$

after integrating by parts. Now we use the assumption $K^* \geq 0$ to replace $\lambda^* + \mu^* \geq \frac{1}{3}\mu^*$, and integrate over the smaller domain $\mathbf{D}$ to obtain

$$\Lambda^*(\mathbf{w},\mathbf{w}) > \frac{1}{3}\mu^* \int_{\mathbf{D}} |\mathrm{div}\, \mathbf{w}|^2\, d\mathbf{x} = 3\mu^* \int_{\mathbf{D}} \Psi^2(\mathbf{x}) d\mathbf{x},$$

where we have used the identities $\mathrm{div}\, \mathbf{w} = \mathrm{div}\, \mathbf{v} = 3\Psi(\mathbf{x})$ in $\mathbf{D}$. Therefore,

$$\frac{\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})}{\Lambda^*(\mathbf{w},\mathbf{w})} < 3\frac{K^*}{\mu^*}.$$

□

In Lemma 3.2, we give only an example of a function $\mathbf{v}$. In the next section, we find all such functions.

Remark 3.2. *If* $\mathbf{D}$ *wraps around the torus* $\mathbf{T}$ *at least in one direction, it seems possible to prove Theorem 3.1 even if* $\dfrac{K^*}{\mu^*} \to 0$. *One approach is to try to estimate* $\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})$ *from below directly using arguments similar to those applied to other types of boundary value problems in [13]. Other possibilities are using results of [53] on Cosserat eigenproblem, or adopting arguments from [43]. We discuss it in some more details at the end of the next section.*

The rest of this section is very similar to the last part of the previous section.

Lemma 3.3. *Define the space* $\mathbf{N} \subset \mathbf{H}$ *of functions* $\mathbf{v} \in \mathbf{H}$ *of the type* $\mathbf{v} = C\mathbf{x}$ *in the domain* $\mathbf{D}$ *where* $\mathbf{x} \in \mathbf{T}$ *is the vector of independent variables on the torus* $\mathbf{T}$*, and* $C = -C^T$ *is some* 3×3 *matrix, independent of* $\mathbf{x}$*. Let the subspace* $\mathbf{R} \subset \mathbf{H}$ *be defined by the formula* $\Lambda^*(\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{v} \in \mathbf{R}, \mathbf{w} \in \mathbf{N}$.

Then:
a) for the initial guess $\mathbf{u}^0$*, we have* $\mathbf{u}^0 - \mathbf{u} \in \mathbf{R}$*;*
(b) $\mathbf{R}$ *is an invariant subspace for the error propagation operator from* $\varepsilon^n = \mathbf{u}^n - \mathbf{u}$ *to* $\varepsilon^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}$ *acting by formula (2.5) according to the rule:*

$$\Lambda^*\left(\frac{\varepsilon^{n+1} - \varepsilon^n}{\tau}, \mathbf{v}\right) + \Lambda(\varepsilon^n, \mathbf{v}) \quad = \quad 0 \quad \forall \mathbf{v} \in \mathbf{H},\ n = 0, 1, \ldots; \tag{3.4}$$

(c) if $\dfrac{K^*}{\mu^*} \geq \delta > 0$*, then on this invariant subspace we have*

$$\kappa\, \alpha\, \Lambda^*(\mathbf{v}, \mathbf{v}) \leq \Lambda(\mathbf{v}, \mathbf{v}) \quad \leq \quad \max\{\beta, 1\}\, \Lambda^*(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{R},$$

where $\alpha = \min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$, $\beta = \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$*, and* κ *is determined in Theorem 3.1.*

Proposition 3.1. *The following equality holds:*

$$\Lambda^*(\mathbf{v}, \mathbf{w}) \quad = \quad \Lambda(\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{N}.$$

Proof. We note that $\varepsilon_{ij}(\mathbf{w}) = 0$ and $\operatorname{div} \mathbf{w} = 0$ in the domain $\mathbf{D}$ when $\mathbf{w} \in \mathbf{N}$, and therefore,

$$\begin{aligned}
\Lambda^*(\mathbf{v}, \mathbf{w}) \quad &= \quad 2\mu^* \int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x} + \lambda^* \int_{\mathbf{T}} \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{w}\, d\mathbf{x} \quad = \\
& 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x} + \lambda^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{w}\, d\mathbf{x} \quad = \\
& \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \{2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) + \lambda(\mathbf{x}) \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{w}\}\, d\mathbf{x} \quad = \\
& \int_{\mathbf{T}} \{2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) + \lambda(\mathbf{x}) \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{w}\}\, d\mathbf{x} \quad = \quad \Lambda(\mathbf{v}, \mathbf{w}).
\end{aligned}$$

□

Proof.[Lemma 3.3]
(a) and (b) can be proven exactly like in Lemma 2.2, except we need to use Proposition 3.1 now.
(c) The right inequality is true for all functions $\mathbf{v} \in \mathbf{H}$ and can be proven directly by Corollary 3.1. The left inequality can be shown as follows. For any $\mathbf{v} \in \mathbf{H}$ we have

$$\begin{aligned}
\Lambda(\mathbf{v}, \mathbf{v}) \quad &= \quad \int_{\mathbf{T}} \{2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) + \lambda(\mathbf{x})|\operatorname{div} \mathbf{v}|^2\}\, d\mathbf{x} \quad \geq \\
& \int_{\mathbf{D}} \{2\mu\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) + \lambda|\operatorname{div} \mathbf{v}|^2\}d\mathbf{x} \quad \geq \\
& \alpha \int_{\mathbf{D}} \{2\mu^*\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) + \lambda^*|\operatorname{div} \mathbf{v}|^2\}d\mathbf{x} \quad = \quad \alpha\Lambda^*_{\mathbf{D}}(\mathbf{v}, \mathbf{v}),
\end{aligned}$$

by Corollary 3.1. Next, by Theorem 3.1 there exist a function $\mathbf{w} \in \mathbf{H}$, such that $\mathbf{v} - \mathbf{w} \in \mathbf{N}$ and $\Lambda^*_{\mathbf{D}}(\mathbf{v}, \mathbf{v}) = \Lambda^*_{\mathbf{D}}(\mathbf{w}, \mathbf{w}) \geq \kappa\Lambda^*(\mathbf{w}, \mathbf{w})$. Noting that $\mathbf{v} \in \mathbf{R}$ and using the definition of the space $\mathbf{R}$, we get $\Lambda^*(\mathbf{v} - \mathbf{w}, \mathbf{w}) = 0$, and therefore, $\Lambda^*(\mathbf{w}, \mathbf{w}) = \Lambda^*(\mathbf{v}, \mathbf{v}) + \Lambda^*(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}) \geq \Lambda^*(\mathbf{v}, \mathbf{v})$. □

Lemma 3.3 implies the following

Theorem 3.2. *Let* $\frac{K^*}{\mu^*} \geq \delta > 0$, $\beta = \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\} \geq 1$, *and* $\tau = \frac{1}{\beta}$. *Then the sequence of approximations* $\{\mathbf{u}^n\}$ *given by the method (2.5) with the initial guess computed from (2.6) satisfies the following convergence rate estimate:*

$$\Lambda^*(\mathbf{u}^n - \mathbf{u}, \mathbf{u}^n - \mathbf{u}) \quad \leq \quad q^{2n}\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{u}^0 - \mathbf{u}), \quad 0 \leq q = 1 - \kappa\eta < 1.$$

Proof. The proof is the same as in Theorem 2.1, except that we use Lemma 3.3 now. □

We can also prove an estimate for the initial error, analogous to that derived in the previous section for the Stokes equations.

4. Null-Space Corresponding to the Absolutely Compressible Media

In this section, we characterize a null-space of the quadratic form $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$, $\mathbf{v} \in [W_2^1(\mathbf{D})]^3$ with $K^* = 0$.

To that end, we need to introduce some notations. Let $\mathbf{D}$ be a Lipschitz domain on the torus $\mathbf{T}$ that *does not wrap around the torus*. $\mathcal{D}(\mathbf{D})$ denotes the space of functions in $C^\infty(\mathbf{D})$ with support in $\mathbf{D}$ equipped with the standard topology, and $\mathcal{D}'(\mathbf{D})$ denotes the space of all continuous linear functionals (generalized functions, or distributions) on $\mathcal{D}(\mathbf{D})$. We shall use the same notation, $\mathcal{D}'(\mathbf{D})$, for spaces of generalized vector/tensor functions with finite number of components as well.

Let

$$\mathrm{curl} = \begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix}, \ \mathbf{grad} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

For composite operations, we simply use rules of matrix operations, e.g., the strain tensor can be written as

$$\varepsilon(\mathbf{v}) = \frac{1}{2}\left(\mathbf{grad}\,\mathbf{v}^T + (\mathbf{grad}\,\mathbf{v}^T)^T\right)$$

and the second order differential operator $\mathrm{curl}\left((\mathrm{curl}\,\tau)^T\right)$, acting on a given second order tensor τ, is

$$\mathrm{curl}\left((\mathrm{curl}\,\tau)^T\right) =$$

$$\begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}\right]^T.$$

Let us recall a necessary condition for a symmetric tensor to be a symmetric part of a gradient of a vector, i.e. the Saint-Venant compatibility conditions for a strain tensor:

Proposition 4.1. *Let* $\mathbf{v} \in \mathcal{D}'(\mathbf{D})$ *be a generalized vector function with three components. Then* $\mathrm{curl}\left((\mathrm{curl}\,\varepsilon(\mathbf{v}))^T\right) = 0$ *(in a sense of distributions).*

Proof. In $\mathcal{D}'(\mathbf{D})$ we can interchange an order of differentiation, thus the well-know identity $\mathrm{curl}\,\mathbf{grad} = 0$ holds.

Using standard rules of matrix operations, we get

$$\begin{array}{rcccccl} 2\mathrm{curl}\left((\mathrm{curl}\,\varepsilon(\mathbf{v}))^T\right) & = & \mathrm{curl}\left(((\mathrm{curl}\,\mathbf{grad})\mathbf{v}^T)^T\right) & + & \mathrm{curl}\left((\mathrm{curl}(\mathbf{grad}\,\mathbf{v}^T)^T)^T\right) & = & \\ & & 0 & + & (\mathrm{curl}\,\mathbf{grad})(\mathrm{curl}\,\mathbf{v})^T & = & 0. \end{array}$$

□

Lemma 4.1. *The kernel of the quadratic form* $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v}),\ \mathbf{v}\in[W^1_2(\mathbf{D})]^3$, *with* $K^*=0$, *in addition to the standard rigid body motions (translations and rotations), consists of the following 4-dimensional space:*

$$\operatorname{span}\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix};\begin{pmatrix}\frac12(x_1^2-x_2^2-x_3^2)\\x_1x_2\\x_1x_3\end{pmatrix};\begin{pmatrix}x_1x_2\\\frac12(x_2^2-x_1^2-x_3^2)\\x_2x_3\end{pmatrix};\begin{pmatrix}x_1x_3\\x_2x_3\\\frac12(x_3^2-x_1^2-x_2^2)\end{pmatrix}\right\}.$$

Proof. We note that the following decomposition of the strain tensor

$$\varepsilon=\frac13\varepsilon_{ii}I+\varepsilon^d,\ I=\delta_{ij},$$

into a spherical part, $\frac13\varepsilon_{ii}I=\Psi(\mathbf{x})I$ with a function $\Psi(\mathbf{x})=\frac13\varepsilon_{ii}$, and a deviatoric part ε^d leads to

$$\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})=K^*\int_{\mathbf{D}}[\varepsilon_{ii}(\mathbf{v})]^2d\mathbf{x}+2\mu^*\int_{\mathbf{D}}\varepsilon^d_{ij}(\mathbf{v})\varepsilon^d_{ij}(\mathbf{v})d\mathbf{x},\ \mathbf{v}\in[W^1_2(\mathbf{D})]^3.$$

When $K^*=0$, the kernel of the quadratic form is, therefore, the same as the kernel of the first-order differential operator $\varepsilon^d(\mathbf{v})$ acting on $\mathbf{v}\in[W^1_2(\mathbf{D})]^3$. In the rest of the proof, we shall find the kernel of $\varepsilon^d(\mathbf{v})$ in a larger space, namely, $\mathbf{v}\in\mathcal{D}'(\mathbf{D})$; and we shall see that all generalized functions of the kernel in $\mathcal{D}'(\mathbf{D})$ are, actually, smooth functions in the Sobolev space we need, i.e. $[W^1_2(\mathbf{D})]^3$. Such an approach is used in [60] to find the kernel of $\varepsilon(\mathbf{v})$.

When $\varepsilon^d(\mathbf{v})=0$, the decomposition of the strain tensor given above shows that $\varepsilon(\mathbf{v})$ is a spherical tensor, $\varepsilon=\Psi(\mathbf{x})I$. For a spherical tensor, the statement of Proposition 4.1 simplifies to

$$\operatorname{curl}\left((\operatorname{curl} I\,\Psi(\mathbf{x}))^T\right)=-\operatorname{curl}^2\Psi(\mathbf{x})=0,$$

which is equivalent to equations

$$\frac{\partial^2\Psi}{\partial x_i\partial x_j}=0,\ i,j=1,2,3,$$

i.e. the distribution Ψ has all its second order derivatives vanishing. Thus, $\Psi(\mathbf{x})\in\operatorname{span}\{1,x_1,x_2,x_3\}$ necessarily.

Now, we need to find $\mathbf{v}\in\mathcal{D}'(\mathbf{D})$ as a solution of the equation $\varepsilon(\mathbf{v})=\Psi(\mathbf{x})I$. Direct computation shows that plugging every given basis vector of the subspace for $\mathbf{v}$ from the statement of the lemma into $\varepsilon(\mathbf{v})$ indeed produces the corresponding function $\Psi(\mathbf{x})=1,\ x_1,\ x_2$, or x_3. We note that for a given generalized function Ψ the solution $\mathbf{v}\in\mathcal{D}'(\mathbf{D})$ of the equation $\varepsilon(\mathbf{v})=\Psi(\mathbf{x})I$ is unique up to a rigid body motion, as if $\mathbf{w}$ is another solution, then $\varepsilon(\mathbf{v}-\mathbf{w})=0$ and the standard arguments, e.g., [60], can be applied. Thus, the subspace of the statement of the lemma is the kernel of the operator $\varepsilon^d(\mathbf{v})$ in the space of distributions. Every generalized function in the subspace is smooth and can be considered as an element of the Sobolev space $[W^1_2(\mathbf{D})]^3$ as well, which completes the proof. □

Remark 4.1. *The four vector functions in Lemma 4.1 are the four eigenfunctions of the Cosserat eigenproblem with Neumann boundary conditions corresponding to the eigenvalue* K^**=0, see [53]. For a sphere, those functions were found in 1901 by Eugène and François Cosserat [25, 26]. The Cosserat eigenproblem has attracted a significant attention recently, see [24, 47, 48, 44, 45, 40, 46].*

In the next lemma, we show that if we fix the function $\mathbf{v}$ in Theorem 3.1 and allow the constant κ depend on $\mathbf{v}$, then functions from the kernel, and only those, can cause trouble.

Lemma 4.2. *Let* $\mathbf{v} \in \mathbf{H}$ *be fixed and* $K^*/\mu^* \longrightarrow 0$. *Then there exists a constant* $\kappa(\mathbf{v}) > 0$, *such that*

$$\frac{\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \kappa \quad \text{as} \quad \frac{K^*}{\mu^*} \longrightarrow 0,$$

if and only if $\mathbf{v}$ *does not belong to the null-space of* $\Lambda^*_{\mathbf{D}}(\cdot,\cdot)$ *with* $K^* = 0$.

Proof. If $\mathbf{v}$ is in the kernel, then $\varepsilon_{ij}(\mathbf{v})$ is a spherical tensor and all the arguments of Lemma 3.2 apply. If $\mathbf{v}$ is not in the kernel, then the deviatoric part of the strain tensor $\varepsilon^d(\mathbf{v}) \neq 0$; and we have

$$\frac{\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \frac{\int_{\mathbf{D}} 2\mu^* \varepsilon^d_{ij}(\mathbf{v})\varepsilon^d_{ij}(\mathbf{v})\, d\mathbf{x}}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \frac{\int_{\mathbf{D}} 2\mu^* \varepsilon^d_{ij}(\mathbf{v})\varepsilon^d_{ij}(\mathbf{v})\, d\mathbf{x}}{(3K^* + 2\mu^*)\|\mathbf{v}\|^2_{[W^1_2(\mathbf{T})]^3}}. \tag{4.1}$$

To obtain the last inequality, we use

$$\Lambda^*(\mathbf{v},\mathbf{v}) = \mu^* \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} + (\lambda^* + \mu^*) \int_{\mathbf{T}} |\mathrm{div}\, \mathbf{v}|^2\, d\mathbf{x} \leq$$

$$\mu^* \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} + 3(\lambda^* + \mu^*) \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} = (3K^* + 2\mu^*)\|\mathbf{v}\|^2_{[W^1_2(\mathbf{T})]^3}.$$

We complete the proof by taking the limit $\dfrac{K^*}{\mu^*} \longrightarrow 0$ in (4.1). □

Remark 4.2. *Let now the domain* $\mathbf{D}$ *wrap around the torus* $\mathbf{T}$ *at least in one direction. Then none of the functions of Lemma 4.1 can be realized, and the null-space of* $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$ *with* $K^* = 0$ *may consist only of some standard rigid body motions (translations and rotations). This gives us hope that for such a domain* $\mathbf{D}$ *the statement of Theorem 3.1 holds even if* $\dfrac{K^*}{\mu^*} \longrightarrow 0$. *The proof is outside of the scope of the paper; however, we want to highlight some arguments.*

Without loss of generality, we set $\mu^* = 1$. *Let* $1 \geq K^* \geq 0$. *We notice that the term* $\Lambda^*(\mathbf{w},\mathbf{w})$ *is uniformly (in* K^**) equivalent to* $\int_{\mathbf{T}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x}$. *Thus, if we could prove that*

$$\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) \geq C \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x}$$

uniformly in K^*, *then we would be able to use Theorem 3.1 in its current form to conclude that it holds when* $K^* \longrightarrow 0$ *as well, provided that* $\mathbf{D}$ *wraps around the torus.*

In the worst case, when $K^* = 0$, *the estimate above takes the form*

$$\int_{\mathbf{D}} \varepsilon^d_{ij}(\mathbf{w})\varepsilon^d_{ij}(\mathbf{w})\, d\mathbf{x} \geq C \int_{\mathbf{D}} \varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{w})\, d\mathbf{x}, \ \forall \mathbf{w} \in [W^1_2(\mathbf{D})]^3.$$

We can ignore rigid body motions here as they are in the null-space of both quadratic forms.

One can try to establish the inequality directly using arguments similar to those applied to other types of boundary value problems in [13]. A simpler possibility is adopting arguments used in [43], pp. 28-34, to prove the Korn-type inequality.

Yet another choice is applying results of [53], where it is proved that the value $K^* = 0$ *corresponds to the extreme eigenvalue of the Cosserat eigenproblem with second-type boundary conditions, and this eigenvalue is isolated and that of finite multiplicity, see the previous remark. We cut our domain to get a new domain that does not wrap around the torus so results of [53] can be immediately applied. Now, functions on our original domain form a subspace in the space of functions on the new domain. This subspace has only a trivial intersection with the finite-dimensional eigenspace found in Lemma 4.1. Then, the desired estimate can be obtained using known theory of the Rayleigh–Ritz method, e.g., [36].*

5. Solving The Preconditioner Problem

In order to compute the initial guess (2.6), and at any iteration step of the process (2.5), we need to find a solution $\mathbf{u} \in [W_2^1(\mathbf{T})]^3/R^3$ to Stokes or Lamé equations

$$\begin{array}{rcll} \mu\Delta\mathbf{u} + \mathbf{grad}\, p &=& \mathbf{g}\,, & \\ \mathrm{div}\ \mathbf{u} &=& 0 & \text{(Stokes)}, \\ \mathrm{div}\ \mathbf{u} &=& \dfrac{p}{\lambda+\mu} & \text{(Lamé)}. \end{array} \tag{5.1}$$

The following theorem was first suggested by Kobelkov in [38] for a specially constructed finite-difference approximation to (5.1).

Theorem 5.1. *The solution to problem (5.1) is given by*

$$\mu\mathbf{u} = \Delta^{-1}\mathbf{g} - \Delta^{-1}\mathbf{grad}\,\mathrm{div}\,\Delta^{-1}\mathbf{g}$$

for the Stokes equations, and by

$$\mu\mathbf{u} = \Delta^{-1}\mathbf{g} - \Delta^{-1}\mathbf{grad}\,\frac{\lambda+\mu}{\lambda+2\mu}\,\mathrm{div}\,\Delta^{-1}\mathbf{g}$$

for the Lamé equations.

Proof. After applying the div operator on both sides of the first equation in (5.1), and using the identity $\Delta\mathrm{div} = \mathrm{div}\Delta$ for our periodic case, we get a Poisson equation for p, or $\dfrac{\lambda+2\mu}{\lambda+\mu}p$, correspondingly. Then we plug the solution to the Poisson equation back into the first equation. □

Remark 5.1. *In the above derivations we never used that λ is a constant, and therefore, the theorem holds for variable λ in the interval $-\frac{2}{3}\mu \le \lambda(\mathbf{x}) \le +\infty$. Thus, it is workable to use a preconditioner with variable λ, as it was suggested by Kobelkov in [39] for a somewhat more complicated method. In [39], that makes possible removing the assumption*

$$0 < \eta \le \frac{\mu}{\mu^*}\frac{K^*}{K} \le \eta^{-1},$$

which we need in (3.3). In the present paper, we do not explore effects of using a preconditioner with variable λ in our iterative solver (2.5) applied to the Lamé equations.

Once problem (5.1) is reduced to six scalar periodic boundary value problems for the Laplace operator, one can apply a variety of efficient solvers, like separation of variables, or multilevel methods. Also, if we interchange the places of Δ^{-1} and **grad**, the number of scalar equations can be reduced to four.

One can directly apply the Fourier method to problem (5.1): find the unknown scalars $c^j_{n_1,n_2,n_3}$, $j = 1,2,3$ and c_{n_1,n_2,n_3}, such that

$$u_j(\mathbf{x}) \approx \sum_{n_1,n_2,n_3} c^j_{n_1,n_2,n_3} \exp\left(2\pi i(n_1x_1 + n_2x_2 + n_3x_3)\right),$$

$$p(\mathbf{x}) \approx \sum_{n_1,n_2,n_3} c_{n_1,n_2,n_3} \exp\left(2\pi i(n_1x_1 + n_2x_2 + n_3x_3)\right).$$

There is also another version of the above method. We find, by solving a system of ordinary differential equation, unknown functions $c^j_{n_1,n_2}(x_3)$, $j = 1,2,3$ and $c_{n_1,n_2}(x_3)$, such that

$$u_j(\mathbf{x}) \approx \sum_{n_1,n_2} c^j_{n_1,n_2}(x_3)\exp\left(2\pi i(n_1x_1+n_2x_2)\right),$$

$$p(\mathbf{x}) \approx \sum_{n_1,n_2} c_{n_1,n_2}(x_3)\exp\left(2\pi i(n_1x_1+n_2x_2)\right).$$

The latter method is applicable not only on a torus, but also on a cylinder with homogeneous boundary conditions specified at the top and the bottom.

6. Method of Fictitious Domains

One can obtain an iterative method for solving problem (2.1) with $\mu(\mathbf{x}) = 0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$ (and for problem (3.1) with $\mu(\mathbf{x}) = \lambda(\mathbf{x}) = 0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$) by letting $\mu^* \to 0$ in (2.5)–(2.6) (and $\lambda^* \to 0$ with $\mu^*/\lambda^* = \mu/\lambda$ in the case of problem (3.1)).

Similarly, by letting $\mu \to +\infty$ (and $\lambda \to +\infty$ with $\mu^*/\lambda^* = \mu/\lambda$ in the case of problem (3.1)), we obtain an iterative method for solving problem (2.1) with $\dfrac{1}{\mu} = 0$ in $\mathbf{D}$ (and in case of problem (3.1) with $1/\mu = 1/\lambda = 0$ in $\mathbf{D}$). These types of problems arise (cf. [11]) when the method of fictitious domains is applied to boundary value problems of the first and second kind.

Mixed boundary value problems can be reduced to boundary value problems on a cylinder by applying the fictitious domains method. For example, let us consider the Lamé equations with coefficients λ and μ in $\mathbf{D}$, where $\mathbf{D}$ is a brick. Let the homogeneous Dirichlet boundary conditions be specified on the horizontal faces of $\mathbf{D}$ and the Neumann boundary conditions everywhere else. Then we consider a fictitious brick $\blacksquare$ containing $\mathbf{D}$, and having the same height, but larger length and depth. On $\blacksquare$ we consider the Lamé equations with the homogeneous Dirichlet boundary conditions specified on the horizontal faces and periodic boundary conditions at other pairs of opposite faces. We choose some small coefficient λ^*, while keeping $\mu^*/\lambda^* = \mu/\lambda$. Specifying periodic boundary conditions is equivalent to identifying the corresponding faces of $\blacksquare$, i.e. we have re-formulated a problem on a brick to one on a cylinder. Now we can apply all results from Section 3 to the problem on a cylinder. In this way, by letting in (2.5)–(2.6) $\mu^* \to 0$ and $\lambda^* \to 0$, we obtain an efficient iterative technique for the original mixed boundary value problem with convergence rate uniform in λ as $\lambda \to +\infty$.

Remark 6.1. *If the original problem possesses (odd or even) symmetry with respect to one or more coordinate planes, then the solution will have the same symmetry. In that case, all approximations in (2.5)–(2.6) will have the symmetry property. This allows us to solve the problem only on a part of the torus by imposing on the planes of symmetry boundary conditions of the third or fourth kind. The above statements hold also for the problem on a cylinder instead of the torus.*

7. Numerical Results

In this section, we provide numerical results for the most simple FEM discretization of the diffusion equation in two dimensions, also analyzed in [7]. While these tests are for the diffusion equation only, they still provide a good illustration of a uniform convergence with respect to the jump in coefficients.

We find that our methods converge fast even for situations where our theory fails in the continuous case.

Numerical results for the Lamé equations, using the iterative methods we analyze in the present paper, can be found in [29, 54, 52, 6]. We still do not have our own numerical results for the Lamé equations, however, see tests by others in [29].

Our set of numerical tests was performed for the standard Galerkin finite element method applied to the diffusion equation with homogeneous Dirichlet boundary conditions, as described in [37], in a square divided into two rectangles, or four squares, see Figure 1.

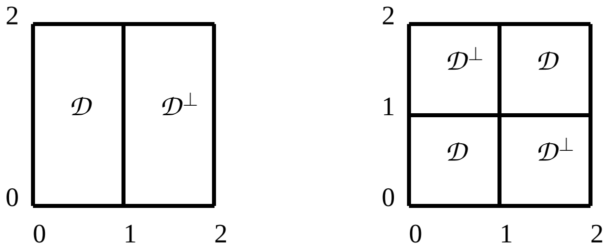

Figure 1. Model domains I and II for the diffusion equation

The diffusion coefficient equals one in $\mathcal{D}$ and a constant ω on $\mathcal{D}^{\perp}$ with ω ranging between 1 and 10^{-3} in different tests. Uniform meshes with size ranging between 1/2 and 1/32 were used. The discrete L_2 and H^1 norms of the errors were measured.

We note that Domain I satisfies our assumption of being Lipschitz, while Domain II does not as its boundary has a point of self-intersection. Thus, our theory predicts that our method should converge uniformly in ω and the mesh size for Domain I, but may have some convergence deterioration for Domain II as ω and the mesh size go to zero.

The Richardson method, i.e. method (2.5) adapted for the FEM discretization of the diffusion equation, the preconditioned steepest descent method, and the preconditioned conjugate gradient method were tested with a random initial guess and with the special initial guess, found similarly to (2.6), see [12] for the description of the algebraic analogs of (2.5) and (2.6). A direct solver of the discrete Laplacian was used as a preconditioner.

We first observed the convergence deterioration of the Richardson and the steepest descent method with a the random initial guess, when ω decreases, even for a fixed mesh size. Namely, a fourfold decrease of ω quadruples the number of iterations, exactly as the standard convergence theory predicts. The convergence was uniform in the mesh size, though.

The Richardson method with the conservative choice of the iterative step and our special initial guess for Domain I indeed converged uniformly in ω and the mesh size. For $\omega = 1/2$, the convergence was linear with the error reduction factor per iteration $q \approx 0.2$. The convergence slowed down to $q \approx 0.5$ for $\omega = 1/32$, but did not noticeably change for smaller ω's. We observed somewhat faster convergence of the first couple of iterations.

Our special initial guess for Domain I was so special that in several numerical experiments the

Richardson method with the optimal iterative step, the steepest descent, and the conjugate gradient method all converged in one step for all tested values of ω and the mesh size.

With the special initial guess for Domain II, the Richardson method and the steepest descent behaved similarly to each other. When the mesh size was fixed, the convergence was uniform in ω without noticeable slowdown for ω's smaller than 1/8. However, the convergence was not uniform in the mesh size - the fourfold refinement of the mesh doubled the number of iterations.

As our theoretical convergence results do not depend on ω, but are based on extension theorems, such numerical results illustrate the fact that there exists a mesh extension theorem for Domain II in spite of the failure of the continuous extension theorem, but with a mesh-dependent constant κ_h slowly decreasing with the mesh size h, see [55] and references there.

Finally, the most interesting results were obtained for the conjugate gradient method in Domain II. The speed of convergence was uniform in ω no matter whether or no the special initial guess was used. Moreover, the number of iterations were about the same for different initial guesses and ω's. The convergence was clearly super-linear and the error was reduced by the factor of 10^{-15} in just 6 iterations for $h = 1/4$ and 10 iterations for $h = 1/16$.

That showed, again, that our technique does not provide convergence truly uniform in ω and the mesh size for Domain II. On the other hand, the practical convergence of the conjugate gradient method was so fast that the lack of the uniformity became a purely theoretical issue.

ACKNOWLEDGEMENTS

The numerical tests presented were performed by Joe Cullen, David Duran, Bret Gunneson, Tom Huntley, and Constantine Kelesoglou of CU-Denver, as their final project for the class on iterative solvers taught by the second author.

The authors are also indebted to Ivo Babuška for pointing out to papers [54, 52].

REFERENCES

1. Zh. Arbash and G. M. Kobel'kov. Numerical methods for solving elasticity theory problems with strongly varying coefficients. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 8(5):371–383, 1993.
2. I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM Journal on Numerical Analysis*, 28(3):624–661, 1991.
3. N. S. Bakhvalov. Solution of the Stokes nonstationary problems by the fictitious domain method. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 10(3):163–172, 1995.
4. N. S. Bakhvalov. Fictitious domains methods for solving elasticity problems. *ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik (Journal of Applied Mathematics and Mechanics)*, 76(Suppl. 1):223–226, 1996.
5. N. S. Bakhvalov. Efficient methods for solving stiff multidimensional multiparameter problems. *Computational Mathematics and Mathematical Physics*, 39(12):1938- -1966, 1999.
6. N. S. Bakhvalov, K. Y. Bogachev, and M. E. Eglit. Numerical calculation of effective elastic moduli for incompressible porous material. *Mechanics of Composite Materials*, 32(5):399–405, 1996.
7. N. S. Bakhvalov, K. Yu. Bogachëv, and Zh. F. Mètr. An efficient algorithm for solving stiff elliptic problems with applications to the method of fictitious domains. *Computational Mathematics and Mathematical Physics*, 39(6):884–896, 1999.
8. N. S. Bakhvalov and A. V. Knyazev. A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations. *Soviet Mathematics. Doklady*, 41(3):481–485, 1990.
9. N. S. Bakhvalov and A. V. Knyazev. Efficient computation of averaged characteristics of composites of a periodic structure of essentially different materials. *Soviet Mathematics. Doklady*, 42(1):57–62, 1991.
10. N. S. Bakhvalov and A. V. Knyazev. An efficient iterative method for solving the Lamé equations for almost incompressible media and Stokes' equations. *Soviet Mathematics. Doklady*, 44(1):4–9, 1992.

11. N. S. Bakhvalov and A. V. Knyazev. Fictitious domain methods and computation of homogenized properties of composites with a periodic structure of essentially different components. In Gury I. Marchuk, editor, *Numerical Methods and Applications*, pages 221–276. CRC Press, Boca Raton, 1994.
12. N. S. Bakhvalov and A. V. Knyazev. Preconditioned iterative methods in a subspace for linear algebraic equations with large jumps in the coefficients. In D. Keyes and J. Xu, editors, *Domain Decomposition Methods in Science and Engineering*, volume 180 of *Contemporary Mathematics*, pages 157–162. American Mathematical Society, Providence, 1994. Proceedings of the Seventh International Conference on Domain Decomposition, October 27-30, 1993, held at the Pennsylvania State University.
13. N. S. Bakhvalov, A. V. Knyazev, and M. E. Eglit. An inequality of the Korn type and orthogonal decompositions in spaces of matrices. *Russian Academy of Sciences. Doklady. Mathematics*, 48(1):127– 129, 1994.
14. N. S. Bakhvalov, A. V. Knyazev, and G. M. Kobel'kov. Iterative methods for solving equations with highly varying coefficients. In Roland Glowinski, Yuri A. Kuznetsov, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 197–205, Philadelphia, PA, 1991. SIAM.
15. N. S. Bakhvalov and G. P. Panasenko. *Homogenization: Averaging of Processes in Periodic Media. Mathematical Problems in the Mechanics of Composite Materials*, volume 36 of *Mathematics and Its Applications. Soviet Series*. Kluwer Academic Publishers, Dordrecht/Boston/London, 1989.
16. Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.
17. K. Yu. Bogachev. Iterative methods of solving main boundary value problems for second-order quasilinear elliptic equations in complexly shaped domains. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 7(4):281–298, 1992.
18. K. Yu. Bogachev. Iterative method for solving quasilinear elliptic equations with widely varying coefficients singularly depending on two parameters. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 10(1):9–32, 1995.
19. K. Yu. Bogachev. Justification of the method of fictitious domains for the solution of mixed boundary value problems for quasilinear elliptic equations. *Moscow University Mathematics Bulletin*, 51(3):12–17, 1996.
20. K. Yu. Bogachev. Efficient algorithm for solving stiff elliptic problems with large parameters. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 14(6):479–493, 1999.
21. K. Yu. Bogachev. An efficient algorithm for solving elliptic problems with large parameters. *Computational Mathematics and Mathematical Physics*, 40(3):383– 395, 2000.
22. J. H. Bramble, J. E. Pasciak, and A. H. Schatz. An iterative method for elliptic problems on regions partitioned into substructures. *Mathematics of Computation*, 46(174):361–369, 1986.
23. F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer–Verlag, New York, 1991.
24. E. V. Chizhonkov. Application of the Cossera [Cosserat] spectrum to the optimization of a method for solving the Stokes problem. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 9(3):191–199, 1994.
25. E. Cosserat and F. Cosserat. Sur une application des fonctions potentielles de la théorie de l'élasticité. *Comptes Rendus de l'Académie des Sciences (Paris)*, 133:210– 213, 1901.
26. E. Cosserat and F. Cosserat. Sur une point critique particulier de la solution des équations de l'élasticité dans le cas où les efforts sur la frontière sont données. *Comptes Rendus de l'Académie des Sciences (Paris)*, 133:382– 384, 1901.
27. G. C. A. DeRose and A. R. Diaz. Single scale wavelet approximations in layout optimization. *Structural Optimization*, 18(1):1–11, 1999.
28. G. C. A. DeRose and A. R. Diaz. Solving three-dimensional layout optimization problems using fixed scale wavelets. *Computational Mechanics*, 25(2-3):274–285, 2000.
29. Giuseppe DeRose. *Solving Topology Optimization Problems Using Wavelet-Galerkin Techniques*. PhD thesis, Michigan State University, Mechanical Engineering Department, 2440 Engineering Building, East Lansing MI 48824, 1998.
30. Alejandro R. Díaz. A wavelet-Galerkin scheme for analysis of large-scale problems on simple domains. *International Journal for Numerical Methods in Engineering*, 44(11):1599–1616, 1999.
31. Eugene G. D'yakonov. *Optimization in solving elliptic problems*. CRC Press, Boca Raton, FL, 1996. Translated from the 1989 Russian original, Translation edited and with a preface by Steve McCormick.
32. Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier–Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.
33. G. Haase and S. V. Nepomnyaschikh. Explicit extension operators on hierarchical grids. *East-West Journal of Numerical Mathematics*, 5(4):231–248, 1997.
34. V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosifyan].
35. A. V. Knyazev. Iterative solution of PDE with strongly varying coefficients: algebraic version. In R. Beauwens and P. de Groen, editors, *Iterative Methods in Linear Algebra*, pages 85–89, Amsterdam, 1992. Elsevier. Proceedings IMACS Symp. Iterative Methods in Linear Algebra, Brussels, 1991.
36. A. V. Knyazev. New estimates for Ritz vectors. *Mathematics of Computation*, 66(219):985–995, 1997.
37. A. V. Knyazev and O. Widlund. Lavrentiev regularization + Ritz approximation = uniform finite element error estimates for differential equations with rough coefficients. *Mathematics of Computation*, posted on July 13, 2001, S0025-5718-01-

01378-3 (to appear in print),
http://www.ams.org/journal-getitem?pii=S0025-5718-01-01378-3

38. G. M. Kobel'kov. Efficient methods for solving elasticity theory equations. *Soviet Journal of Numerical Analysis and Mathematical Modelling*, 6(5):361–375, 1991.
39. G. M. Kobel'kov. On the solution of the boundary value problem for the diffusion equation with highly varying coefficient. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 11(6):487–495, 1996.
40. Alexander Kozhevnikov. A history of the Cosserat spectrum. In *The Maz′ya anniversary collection, Vol. 1 (Rostock, 1998)*, pages 223–234. Birkhäuser, Basel, 1999.
41. Yu. A. Kuznetsov. New iterative methods for singular perturbed positive definite matrices. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 15(1):65–71, 2000.
42. O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Gordon and Breach, New York, 1969.
43. V. G. Litvinov. *Motion of a nonlinearly viscous fluid*. "Nauka", Moscow, 1982. (In Russian).
44. Wensen Liu and Xanthippi Markenscoff. The discrete Cosserat eigenfunctions for a spherical shell. *Journal of Elasticity*, 52(3):239–255, 1998/99.
45. Wensen Liu, Xanthippi Markenscoff, and Michael Paukshto. The Cosserat spectrum theory for two-dimensional thermoelastic problems. *Journal of Thermal Stresses*, 22(2):225–239, 1999.
46. Wensen Liu and Allen Plotkin. Application of Cosserat-spectrum theory to the weakly compressible Stokes flow past a sphere. *Journal of Engineering Mathematics*, 38(2):155–172, 2000.
47. X. Markenscoff and M. Paukshto. On the cavities and rigid inclusions correspondence and the Cosserat spectrum. *Mathematische Nachrichten*, 177:183–188, 1996.
48. Xanthippi Markenscoff and Michael Paukshto. The Cosserat spectrum in the theory of elasticity and applications. *The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences*, 454(1970):631–643, 1998.
49. A. M. Matsokin. Norm-preserving prolongation of mesh functions. *Soviet Journal of Numerical Analysis and Mathematical Modelling*, 3(2):137–149, 1988.
50. A. M. Matsokin and S. V. Nepomnyashchikh. Fictitious component method using extension operator. *Siberian Journal of Computer Mathematics*, 1(1):31–45, 1992.
51. William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
52. J. C. Michel, H. Moulinec, and P. Suquet. Effective properties of composite materials with periodic microstructure: a computational approach. *Computer Methods in Applied Mechanics and Engineering*, 172(1- 4):109–143, 1999.
53. S. G. Mikhlin. The spectrum of a family of operators in the theory of elasticity. *Russian Mathematical Surveys*, 28:45–88, 1973.
54. H. Moulinec and P. Suquet. A numerical method for computing the overall response of nonlinear composites with complex microstructure. *Computer Methods in Applied Mechanics and Engineering*, 157(1- 2):69–94, 1998.
55. S. V. Nepomnyaschikh. Mesh theorems on traces, normalizations of function traces and their inversion. *Soviet Journal of Numerical Analysis and Mathematical Modelling*, 6(3):223–242, 1991.
56. O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, volume 26 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1992.
57. Alejandro Pares-Sierra and Geoffrey K. Vallis. A fast semi-direct method for the numerical solution of nonseparable elliptic equations in irregular domains. *Journal of Computational Physics*, 82(2):398–412, 1989.
58. A. Quarteroni and A. Valli. *Domain decomposition methods for partial differential equations*. Oxford University Press, Oxford, 1999.
59. Enrique Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1980.
60. Roger Temam. *Problèmes mathématiques en plasticité*, volume 12 of *Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science]*. Gauthier-Villars, Paris, 1983.
61. T. V. Veretentseva, G. M. Kobel'kov, and V. G. Sushko. On approximate solutions to some problems with rapidly varying coefficients. *Moscow University Computational Mathematics and Cybernetics*, (3):18–23, 1993.
62. Olof B. Widlund. An extension theorem for finite element spaces with three applications. In Wolfgang Hackbusch and Kristian Witsch, editors, *Numerical Techniques in Continuum Mechanics*, pages 110–122, Braunschweig/Wiesbaden, 1987. Notes on Numerical Fluid Mechanics, v. 16, Friedr. Vieweg und Sohn. Proceedings of the Second GAMM-Seminar, Kiel, January, 1986.