On an iterative method for finding lower eigenvalues

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Abstract – We present the analysis of the errors involved in approximate orthogonalization with respect to previously found eigenvectors in preconditioned iterations of a subspace for simultaneous determination of a cluster of eigenvalues and the corresponding eigenvectors of a large sparse symmetrical eigenvalue problem.

1. INTRODUCTION

Efficient preconditioned iterative methods have been lately gaining acceptance for solving partial symmetric sparse eigenvalue problems arising in grid discretization of spectral problems in mathematical physics (see, e.g., [1], the review [4], and the references therein). In addition to conventional preconditioned vector iterations for calculation (in turns) of the eigenvectors, there have been developed [3] preconditioned iterations of a subspace for simultaneous calculation of a group of the eigenvectors corresponding to a cluster of eigenvalues: using the subspace $H_p^n$ one constructs a new $p$-dimensional subspace $H_p^{n+1}$ and thereby determines more exact approximations to $\lambda_p$ and $U_p$, where $\lambda_p$ is the $p$-th, in increasing order, eigenvalue of the problem:

$$ Lu = \lambda M u, \quad L \in C(H), \quad M \in C(H) $$

(1.1)

while $U_p$ is the corresponding eigen subspace

$$ C(H) = \{ A: A \in B(H,H), A = A^* > 0 \} $$

$H$ is the $N$-dimensional Euclidean space with the inner product $(u,v)$. It has been proved that

$$ \beta^n = \beta(H_p^n) = \max_{u \in H_p^n, u \neq 0} \left\{ \frac{(Lu,u)}{(Mu,u)} \right\} $$

(1.2)

converges to $\lambda_p$ at a rate of the geometric progression with the ratio of

$$ q_p = q = q \left( \delta, \lambda_p, \frac{\beta^0}{\lambda_p}, \frac{\beta^0}{\gamma_p}, \frac{\lambda_p}{\gamma_p} \right) < 1 $$

provided $\beta^0 < \gamma_p$, where

$$ \gamma_p = \lambda_{p+1} > \lambda_p, \quad \delta = \delta_0 / \delta_1, \quad \delta_0 B \leq L \leq \delta_1 B, \quad \delta_0 > 0 $$

(1.3)

$$ H_p^{n+1} = R_p^n H_p^n, \quad R_p^n = E - \gamma B^{-1}(L - \beta^n M) $$

(1.4)

When implementing the method, it is necessary to solve systems of the form $Bu_p^{n+1} = g^n$ $p$-times at each iteration. In the case of grid problems an appropriate choice of the operator $B$ is usually associated with the spectral equivalence of the operators $L$ and $B$, and the operator $B$ can be selected out of known samples.
It is also easy to calculate the corresponding approximation to $U_{\lambda_i}$ in the subspace $H^0$. Using orthogonalization with respect to some approximately determined eigen subspaces, one can construct similar iterative methods of finding all $\lambda_i, i \leq p$, and the corresponding subspaces $U_{\lambda_i}$. The convergence rates of these methods are equal to those of geometric progressions with a ratio of $q \leq \sigma_p = q$. Thus, the availability of some closely-spaced $\lambda_i$ with $i \leq p$ does not decrease the convergence rate of the iterative method being discussed. The urgency of the problem of finding iterative methods possessing the above property has been pointed out, for instance, by G. I. Marchuk and V. I. Lebedev.

This paper is concerned with studying the convergence of the method mentioned above, with allowance made for the fact that orthogonalization with respect to the previously determined eigen subspaces is approximate. This study is, to a considerable extent, analogous to that made for vector iterations in [1] where more detailed proofs are available. The reader frightened by cumbersome calculations typical of the direct error analysis is referred to [1,4] where a backward error analysis is available, though for a somewhat more tedious orthogonalization procedure. The case of the singular operator $M$ of unixed sign is also studied in [1,4].

Here we present the results of numerical experiments on implementing the method under consideration for model difference problems.

This work has been published in Russian (see [2]), the main results have been discussed in [1] without detailed proofs.

2. DETERMINATION OF $\lambda_p$

For $A \in C(H)$ let $H_A$ be an Euclidean space that differs from $H$ only in the inner product

$$(u,v)_{H_A} = (Au,v), \quad \|u\|_{H_A}^2 = (Au,u).$$

By $u_1, u_2, \ldots, u_N$ we shall denote an orthonormal basis $H_M$ comprising the eigenvectors of (1.1):

$$L u_i = \lambda_i M u_i, \quad i = 1, \ldots, N$$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N. \quad (2.1)$$

Let $\omega_0$ be a subset of the indices $i$ for which there have been found the approximations $\tilde{u}_i$ to $u_i$, where

$$\|u_i - \tilde{u}_i\|_L \leq \varepsilon_{L,i}, \quad \|u_i - \tilde{u}_i\|_M \leq \varepsilon_{M,i} \quad (2.2)$$

$$\max_{i \in \omega_0} \{ \varepsilon_{L,i}, \varepsilon_{M,i} \} = \varepsilon$$

Let $Q = Q_{\omega_0}$ and $\overline{Q} = \overline{Q}_{\omega_0}$ be, respectively, the linear spans of $u_i$ and $\tilde{u}_i$, $i \in \omega_0$, $\dim Q = m, \overline{Q} = Q^\perp$ and $\overline{Q} = \overline{Q}^\perp$ are orthogonal in $H_M$ complements to $Q$ and $\overline{Q}$, $P$ and $\overline{P}$ are orthogonal in $H_M$ projectors onto $Q$ and $\overline{Q}$, $P^\perp = E - P$, $\overline{P}^\perp = E - \overline{P}$, and $E$ is the identity operator. The proximity of the subspaces $Q$ and $\overline{Q}$ will be defined from (2.2) and with the use of the openings $\Theta_M = \Theta_M(Q,\overline{Q})$ and $\Theta_L = \Theta_L(Q,\overline{Q})$ of these subspaces into $H_M$ and $H_L$, respectively. In this case (see [1]) we have

For $u \neq 0$ the foll

Let

$$\lambda_p^\perp = \min_{H_p \subseteq Q^\perp} \|u\|_p$$

where $H_p$ is a $p$-dim the $p$-th, in increments with $i \in \omega_0$ from th this case that will be

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Lemma 2.1. Let and

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\[ \Theta_M = \max_{n \in \tilde{Q}, \|u\|_M = 1} \|u - Pu\|_M = \|P - \overline{P}\|_M. \]  
(2.3)

For \( u \neq 0 \) the following notation is used:

\[ \mu(u) = \mu = \frac{(Lu, u)}{(Mu, u)}. \]  
(2.4)

Let

\[ \lambda^1_p = \min_{H_p \subseteq \tilde{Q}^+} \max_{u \in H_p} \mu(u), \quad \bar{\lambda}^1_p = \min_{H_p \subseteq \tilde{Q}^+} \max_{u \in H_p} \mu(u), \quad p = 1, \ldots, N - m \]  
(2.5)

where \( H_p \) is a \( p \)-dimensional subspace of \( H \). It is easily seen that \( \lambda^1_p \) coincides with the \( p \)-th, in increasing order, eigenvalue \( \lambda_i \) that has remained after eliminating all \( \lambda_i \) with \( i \in \omega_0 \) from the set \( \lambda_1, \ldots, \lambda_N \); if \( \lambda_i > \lambda^1_p \) for all \( i \in \omega_0 \), then \( \lambda^1_p = \lambda_p \) and it is this case that will be most important in what follows.

Let \( H^n_{p_0} \) be a prescribed \( p \)-dimensional subspace with the basis \( a^n_1, \ldots, a^n_p \). Consider the following algebraic eigenvalue problem:

\[ L^n_p \tilde{X} = \lambda \tilde{M}^n_p \tilde{X} \]  
(2.6)

where

\[ \tilde{X} = (x_1, \ldots, x_p)^T, \quad L^n_p = [(La^n_i, a^n_j)], \quad 1 \leq i, j \leq p \]
\[ \tilde{M}^n_p = [(Ma^n_i, a^n_j)], \quad 1 \leq i, j \leq p. \]

If \( a^n_1, \ldots, a^n_p \) is the orthonormal in \( H^n_M \) basis of \( H^n_p \), then \( \tilde{M}^n_p \) becomes an identity matrix and problem (2.6) is simplified. Let \( \lambda^1_1 \leq \lambda^1_2 \leq \ldots \lambda^1_p \) be its eigenvalues, with their eigenvectors \( \tilde{X}^{(1)} = (x^{(1)}_1, \ldots, x^{(1)}_p)^T \) forming an orthonormal system in accordance with the inner product \( \langle \tilde{M}^n_p \tilde{X}, \tilde{Y} \rangle = \tilde{Y}^T \tilde{M}^n_p \tilde{X} \). Then the vectors

\[ v_i = \sum_{j=1}^p x^{(1)}_j a^n_j, \quad i = 1, \ldots, p \]  
(2.7)

form an orthonormal in \( H^n_M \) basis of \( H^n_p \) and

\[ \mu(v_i) = \lambda^1_i = \min_{H_p \subseteq H^n_p} \max_{u \in H_i} \mu(u), \quad \lambda^1_p = \beta(H^n_p). \]  
(2.8)

Besides, the orthogonality of \( \tilde{X}^{(1)}, \ldots, \tilde{X}^{(p)} \) in accordance with the inner product \( \langle \tilde{L}^n_p \tilde{X}, \tilde{Y} \rangle \) entails the orthogonality of the vectors \( v_1, \ldots, v_p \) in \( L^n_\infty \) [see (2.7)].

**Lemma 2.1.** Let conditions (1.3) and (2.2) be satisfied, \( H^n_p \subseteq \tilde{Q}^\perp \), \( 0 < \gamma < 4 \beta^{-1}_1 \), and

\[ H^n_{p+1} = R^n_p H^n_p \]  
(2.9)

be an image of \( H^n_p \) in mapping \( R^n_p = E - \gamma \tilde{P}^{-1} B^{-1}(L - \beta^n M) \). Then \( H^n_{p+1} \) is a \( p \)-dimensional subspace of \( \tilde{Q}^\perp \).
Proof. Let $u \in H^n_p$ and $\|u\|_M = 1$, $\beta^n = \beta$ [see (1.2)]. Then $u = \bar{P}|u|$ and

$$
(R_p|u, u)_M = 1 - \gamma(B^{-1}Lu, Mu) + \gamma \beta^2 \|Mu\|_B^{-2}.
$$

$$
\geq 1 - \gamma\|B^{-1/2}L^{1/2}\| \|L^{1/2}u\| \|Mu\|_B^{-1} + \gamma \beta^2 \|Mu\|_B^{-2}.
$$

In view of (1.3) it is easy to verify that $\|B^{-1/2}L^{1/2}\| \leq \delta_1^{1/2}$. In addition, as $u \in H^n_p$, then $\|L^{1/2}u\| \leq \delta^{1/2}$. Therefore,

$$(R_p|u, u)_M \geq 1 - \gamma \delta_1^{1/2} \delta^{1/2} + \gamma \beta^2$$

where $\delta = \|Mu\|_B^{-1}$. Hence,

$$(R_p|u, u)_M \geq 1 - \frac{1}{4} \gamma \delta_1^2 > K_0 > 0, \quad \|R_p|u\|_M > K_0$$

which proves the lemma. $\square$

Note that if $\omega_0$ is empty, then the operator $R_p = E - \gamma B^{-1}(L - \beta M)$ is self-adjoint in $H^n_p$ and $(R_p|u, u)_B > \gamma \|u\|^2$ for $0 < \gamma < \delta_1^{-1}$ and for any $u \in H^n_p$. If $u \in H^n_p$, then $(R_p|u, u)_p \geq \|u\|^2$ for any $\gamma > 0$. To find the basis $a_0, \ldots, a_p$ of the subspace $H^n_p$ it is sufficient to solve the system of equations:

$$
Bz_i = -\gamma(L\bar{z}_i - \beta M\bar{z}_i), \quad i = 1, \ldots, p
$$

and to set $z_i = v_i - \bar{P}_{v_i}$. Now we can again construct the problem of the type of (2.6) for $H^n_{p+1}$ and find an orthonormal in $H^n_p$ basis of $H^n_{p+1}$ consisting of the vectors $v^n_{1, \ldots, p}$ and $v^n_{1, \ldots, p}$ [see (2.7)].

Now we proceed to a study of the convergence of the quantity $\beta = \beta(H^n_p)$ from (1.2) to $\lambda^2_p$ with $H^n_p$ determined by iterative method (2.9) at $n = 0, 1, \ldots, p$.

Let $\dim Q_{\omega_0} = m$ and $\lambda_1^2 < \lambda_2^2 < \ldots < \lambda_K^2$ [see (2.5)]. Let $\kappa_i^2$ be an eigenvalue of $\lambda_i^2$ and strictly greater than $\lambda_i^2$.

**Lemma 2.2** (see [1]). Let the conditions of Lemma 2.1 be satisfied, $u \in \bar{O}^n$, $\|u\|_M = 1$, $\lambda = \lambda_i^2 < \mu(u) = \mu < \kappa_i^2 = \nu$, $\beta > 0$, $r_p = Lu - \beta Mu$. Then

$$
\|r_p\|_i^2 \geq \frac{\beta^2}{\mu} \left( \frac{(\mu - \lambda)(\nu - \mu)}{\nu} + 1 - \frac{\mu^2}{\beta} \right) \frac{\beta^2}{\nu} \sum_{\omega_0} e_{\omega_0}^2 (y - \lambda_i^2)(\lambda_i^2 - \lambda)\lambda_i^2
$$

where $\omega_0$ is a subset of the indices $i$ from $\omega_0$ such that $\lambda < \lambda_i < \nu$.

**Lemma 2.3.** Let the conditions of Lemma 2.1 be satisfied, $0 < \gamma < \delta_1^{-1}$, $u = u^n \in H^n_p$, $\|u\|_M = 1$, $\mu(u) = \mu^n = \mu$, $\beta^n = \beta$, $\delta = \delta_1/\delta_1$, $u^n + 1 = R_p|u^n|$. Then, for sufficiently small $\epsilon$ there exist such numbers $y = y(\delta, \delta_1^2) > 0$, $K_2 > 0$ that the following inequality holds:

$$
\beta^n - \mu^n + 1 \geq (\mu(u) - \mu) - (\bar{P}|r_p, u^n|_p) - K_2^2
$$

where $t(u) = y(\delta, \delta_1^2) > 0$.

**Proof.** Let $w = \bar{P}|B^{-1}r_p$. Then

$$
\mu^n - \mu^{n+1} = (2\gamma w, r_p) - \gamma^2 \|w\|^2 + \gamma^2 \mu \|w\|^2 + 2\gamma(\beta - \mu)(u,w)_M \|u^{n+1}\|_M^2.
$$

Let us also let $\beta > \mu$, from (2.14).
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Let us represent the term \((v,r_p) = (\bar{B} \perp B^{-1} r_p, r_p)\) in the form:

\[
(v,r_p) = \| r_p^B \|^2 - (PB^{-1} r_p, r_p) + ((\bar{B} \perp P \perp) B^{-1} r_p, r_p)
\]

and find a lower bound, by analogy with the estimate for \((v,r)\) given in [1]. Then

\[
\| (PB^{-1} r_p, r_p) \| \leq \| PM^{-1} r_p \| B^{-1} r_p \leq 2K_o \| r_p \| B^{-1} - \alpha_1 \| r_p \|^2_{B^{-1}} + \alpha_1^{-2} (K_o \| \delta \|^2)
\]

where

\[
2K_o \geq \lambda_1^{-1/2} \delta^{1/2} \left( \sum_{i \in \omega_0} (\lambda_i - \beta)^2 \right)^{1/2}, \quad \alpha_1 \in (0,1).
\]

Besides,

\[
\| (\bar{B} - P \perp) B^{-1} r_p, r_p) \| \leq \| P - \bar{B} \| B^{-1} - \delta^{1/2} \omega \| r_p \|^2_{B^{-1}}
\]

where

\[
\omega \geq \lambda_1^{-1/2} \left( m^{1/2} + \sum_{i \in \omega_0} \lambda_i^{1/2} \right).
\]

Therefore,

\[
(v,r_p) \geq \| r_p \|^2_{B^{-1}} (1 - \omega \delta^{-1/2} \omega - \alpha_1 - \alpha_1^{-1} K_o \delta^{2}).
\]

(2.13)

For \(-\| w \|^2_{L}\), similar to [1], we find

\[
-\| w \|^2_{L} \geq -(1 + \omega \delta^{-1/2} \omega - \alpha_1) - \omega^{-1} K_o \delta^{2}
\]

which, combined with (2.13), yields the inequality:

\[
2\gamma (v,r_p) - \gamma^2 \| w \|^2_{L} \geq \gamma (\gamma, c, a_1) \| r_p \|^2_{B^{-1}} - K_1 \delta^{2}
\]

where

\[
K_1 = 2\gamma K_o a_1^{-1}, \quad \gamma (\gamma, c, a_1) = 2\gamma (1 - \omega \delta^{-1/2} \omega - \alpha_1) - \gamma^2 \omega_1 (1 + \omega \delta^{-1/2} \omega) > 0
\]

for \(\gamma < 2\delta^{-1}\) and small \(\omega \) and \(a_1\). Hence,

\[
\mu^n - \mu^{n+1} \leq \frac{\gamma \| r_p \|^2_{B^{-1}} + \gamma^2 \mu \| w \|^2_{L} + 2\gamma (\beta - \mu) (a, w)_{M}}{1 - 2\gamma (a, w)_{M} + \gamma^2 \| w \|^2_{M}} - K_1 \delta^{2}
\]

\[
\geq \frac{\gamma \| r_p \|^2_{B^{-1}} + \gamma^2 \mu \| w \|^2_{L} + \beta - \mu}{1 - 2\gamma (a, w)_{M} + \gamma^2 \| w \|^2_{M}} + \mu - \beta - K_1 \delta^{2}.
\]

(2.12)

Let us also take into consideration that, in view of Lemma 2.1,

\[
\| u^{n+1} \|_{M} = \| R_{\mu} u^{n+1} \|_{M} > K_0 > 0.
\]

Therefore, setting \(K_1 = K_1 / K_0\) we obtain

\[
\beta - \mu^{n+1} + K_1 \delta^{2} \geq \frac{\gamma \| r_p \|^2_{B^{-1}} + \gamma^2 \mu \| w \|^2_{L} + \beta - \mu}{1 - 2\gamma (a, w)_{M} + \gamma^2 \| w \|^2_{M}}.
\]

(2.14)

Let us increase the denominator in the right-hand side of (2.14) by replacing the term \(-2\gamma (a, w)_{M}\) with its upper bound \(a \| u \|^2_{M} + a^{-1} \| w \|^2_{M}, a > 0\). Then, allowing for \(\beta > \mu\), from (2.14) we have
\[ \beta - \mu^{n+1} - K_2 \varepsilon^2 > \min \left\{ \frac{t(u)}{1 + a}, \frac{\beta}{1 + a - 1} \right\}. \]

We may assume that \( r_p \neq 0 \) and take \( a = \frac{t(u)}{\beta} \). In view of this, we obtain lower bound maximal in \( a \) for \( \beta - \mu^{n+1} \), which leads to (2.12). \( \square \)

**Theorem 2.1.** Let \( \omega_0, \bar{Q}, \Theta, \) and \( p > 1 \) be such that we have \( \lambda_i > \lambda_p \) for any \( i \in \omega_0 \) and \( \Theta_L(\omega, \bar{Q}) < \varepsilon, \Theta_M(\bar{Q}) < \varepsilon \). Assume that we have an initial \( p \)-dimensional subspace \( H_p^{L} \leq \bar{Q} \) with \( \beta(H_p^{L}) = \beta^0 < \beta^1 < \gamma_p = \gamma \), where \( \gamma_p \) coincides with the eigenvalue \( \lambda_p \), \( \nu \in \omega_0 \), closest to \( \lambda_p = \lambda \) and strictly greater than \( \lambda \). Let inequalities (1.3) and \( 0 < \gamma < 2\delta^{-1} \) be true. Let \( \varepsilon \) be sufficiently small and \( \gamma \delta_0 \leq 1 \) (see Lemma 2.3). Assume that there have been defined the constants \( K_\omega, K_1, \) and \( K_2 \) from Lemma 2.3, the same for all \( \beta \) belonging to \( [\lambda, \bar{\beta}] \), and

\[ K_3 = \frac{\bar{\beta}}{\lambda_p} \sum_{i \in \omega_0} \frac{(\lambda_i - \lambda)(\nu - \lambda_i)}{\lambda_i}, \quad K_4 = K_3 \bar{\beta} + K_2(1 + K_2 \varepsilon^2) \]

Then for iterative method (2.9) the following estimate is valid:

\[ \beta^n + 1 - \lambda \leq \rho(\beta^n)(\beta - \lambda) + K_4 \varepsilon^2 \]

and the method enables one to obtain \( \beta \in [\lambda_p, \lambda] + \varepsilon^2(K_4 + K_2) \) in \( s = [\log_2(\varepsilon^2 K_4(\beta - \lambda)^{-1})] + 1 \) iterations for a certain value of \( r \leq s \).

**Proof.** Substitute \( u^n \in H_p^n \) in (2.12) so that \( \mu(u^{n+1}) = \beta(H_p^{n+1}) = \beta^{n+1} \). Then, taking into account the monotonicity in \( t \) of the function \( t(1 + ut)^{-1} \) with \( a > 0, t > 0 \), from (2.12) we obtain

\[ \beta^n - \beta^{n+1} \geq \varphi(1 + \beta^{-1} \varphi)^{-1} - K_2 \varepsilon^2 \]

where

\[ \rho(\beta) = \frac{1 - \gamma \delta_0(1 - \gamma^{-1})}{1 + \gamma \delta_0(\beta - \lambda - 1)(1 - \beta^{-1})} \]

\[ \varphi = \max_{\lambda < \beta < \beta^0} \rho(\beta) < 1, \quad K_2 > K_4(1 - q_p)^{-1}. \]

Then for iterative method (2.9) the following estimate is valid:

\[ \beta^n + 1 - \lambda \leq \rho(\beta^n)(\beta^n - \lambda) + K_4 \varepsilon^2 \]

and the method enables one to obtain \( \beta \in [\lambda_p, \lambda] + \varepsilon^2(K_4 + K_2) \) in \( s = [\log_2(\varepsilon^2 K_4(\beta - \lambda)^{-1})] + 1 \) iterations for a certain value of \( r \leq s \).

3. DETERMINATION

**Lemma 3.1.** Let

\[ \lambda_p > \lambda \]
and
\[ t(\mu) \geq \bar{\gamma} \delta_0 [\beta^2(\lambda + \nu - \mu)(\lambda \nu)^{-1} + \mu - 2\beta] + \beta - \mu - K_3 \varepsilon^2 = g_p(\mu) - K_2 \varepsilon^2 \]
for \( \mu > \lambda \) due to the assumption made about \( \omega_0 \), the form of \( \omega_{0,0} \) [see (2.11)], and the validity of inequalities (2.2). As
\[ \bar{\gamma} \delta_0 [1 - \beta^2(\lambda \nu)^{-1}] < 1, \quad g_k(\lambda_{k+1}) = g_{k+1}(\lambda_{k+1}) \]
then
\[ \min_{k \leq p} \min_{\mu \in [\lambda_k, \min(\nu, \beta)]} g_k(\mu) = g_p(\beta) \]
and
\[ t(\mu) \geq \bar{\gamma} \delta_0 \beta(1 - \beta/\nu)(\beta/\lambda - 1) - K_3 \varepsilon^2 \]
which, together with (2.16), leads to inequality (2.15).

It is easy to verify that \( \beta^n \geq \lambda^n \geq \lambda_p \). Besides, from (2.15) it follows that
\[ \beta^n - \lambda < q_p^2(\beta^0 - \lambda) + K_3 \varepsilon^2 \]
and, hence, the last s-th iteration will have the accuracy required. In fact, such an accuracy could have been reached previously, since there could be a violation in the monotonicity of decrease of \( \beta^n \), which could happen only for \( \beta^r \in [\lambda_p, \lambda + K_3 \varepsilon^2] \) [see (2.15)]. It is easily seen that for small \( \varepsilon \) the best value of \( \gamma \) is \( \gamma = \delta_1^{-1} \), since with this value we obtain the least values of \( \rho(\beta) \) and \( g_p \).

In the particular case, when \( s = 0 \) (or when \( \omega_0 \) is empty), the investigation of the method is significantly simplified. Then it can be proved [3] that for \( \gamma = \delta_1^{-1} \), the inequalities \( \beta^n - \lambda_p < \sigma(n)(\beta^0 - \lambda_p) \) hold, where
\[ \sigma(n) = \prod_{i=0}^{n-1} \rho(\beta^i) \leq q_p^n \]
\[ \rho(\beta) = \frac{1 - \delta(1 - \beta/\nu)}{1 + \delta(\beta/\lambda - 1)(1 - \beta/\nu)}, \quad \delta = \delta_0 \delta_1, \quad q_p = \max(\rho(\beta^0), \rho(\lambda_p)) \]

3. DETERMINATION OF \( U_{\lambda_p} \)

Lemma 3.1. Let \( H_{\nu}^* \subseteq \bar{Q}^+ \) be a subspace such that, in view of Theorem 2.1, the following estimate is valid:
\[ 0 < \beta^r - \lambda_p < K_6 \varepsilon^2 \]
and \( \lambda_i > \beta^r \) for any \( i \in \omega_0 \). Let \( p' \) be a dimension of a direct sum of the eigen subspaces \( U_{\lambda_{p'}} \) of problem (2.1), provided \( \lambda_i < \lambda_p \). Then, in \( H_{p'}^* \) there exists a \((p - p')\)-dimensional subspace \( U' \) such that
\[ \Theta_m(U'_{\nu}U_{\lambda_p}) \leq (\beta^r - \lambda_p)(\nu_{p'} - \lambda_p)^{-1} m \varepsilon^2)^{1/2} < K_7 \varepsilon \]
where \( K_7 = [K_6(\nu_{p'} - \lambda_p)^{-1} + m)^{1/2} > 1 \).
Proof. Take $\bar{U} = H_p \cap L(u_{p+1}, \ldots, u_N)$. Then, for any $u \in \bar{U}$ [see (2.2)] with $\|u\|_M = 1$, we have

$$u = \sum_{i=1}^{N} c_i u_i, \quad \sum_{i=p+1}^{N} c_i^2 (\lambda_i - \mu) = 0$$

where $\mu = \mu(u) \in [\lambda_p, \beta']$. Therefore,

$$\sum_{i \in \omega_2} (\lambda_i - \mu)c_i^2 = \sum_{i=p+1}^{p} c_i^2 (\mu - \lambda_p) + \sum_{i \in \omega_0} (\mu - \lambda_i)c_i^2$$

where $\omega_2 = \{p+1, \ldots, N\} \setminus \omega_0$. Take into account that $\lambda_i - \mu \geq \gamma_p - \beta'$ for $i \in \omega_2$ and that $\mu - \lambda_i < 0$ for $i \in \omega_0$. Then

$$\sum_{i \in \omega_2} c_i^2 \leq (\gamma_p - \beta')^{-1} (\beta' - \lambda_p)$$

and

$$b_{\Phi}^2(\mu; U_p) = \sum_{i \in \omega_0 \cup \omega_2} c_i^2 \leq (\beta' - \lambda_p)(\gamma_p - \beta')^{-1} + me^2 \tag{3.3}$$

since

$$\sum_{i \in \omega_0} c_i^2 \leq me^2.$$ 

Relation (3.3), together with (2.3) and (3.1), gives (3.2). \(\square\)

For $H_p$ consider the numbers $\lambda_1' < \ldots < \lambda_p'$ [see (2.6)], $\lambda_1' > \lambda_1$, $\lambda_p < \lambda_p'+1 < \ldots < \lambda_p' = \beta'$.

**Theorem 3.1.** Provided (3.1) is true, for $\lambda_p' + 1$ the following estimate holds for sufficiently small $\epsilon$:

$$\lambda_p' + 1 \leq \beta' [1 + \phi_\beta / \beta']^{-1} + K_2 \epsilon^2 \tag{3.4}$$

where

$$\phi_\beta = \bar{\rho} \delta_0 \beta' \left(1 - \beta'/\gamma_p \right)(\beta'/\lambda_p - 1) - K_2 \epsilon^2 > 0. \tag{3.5}$$

**Proof.** Let

$$T^r = H_p \cap L(u_1, \ldots, u_{p+1}, \ldots, u_N), \quad T^{r+1} = R_p T^r$$

$$\dim T^r = \dim T^{r+1} = p'$$

and $u \in T^r$, with $\|u\|_M = 1$, be such that

$$\mu(R_p u) = \mu^{r+1} = \beta(T^{r+1}) \geq \lambda_p' \geq \lambda_p.$$

Then, in view of (2.12),

$$\lambda_p' + 1 \leq \mu^{r+1} \leq \beta' + K_2 \epsilon^2 - t(u) [1 + t(u)/\beta']^{-1}$$

$$\leq \beta' + K_2 \epsilon^2 - \bar{\rho}_1 [1 + \bar{\rho}_1 / \beta']^{-1} = \beta' [1 + \bar{\rho}_1 / \beta']^{-1} + K_2 \epsilon^2 \tag{3.6}$$

where $\bar{\rho}_1 > 0$ is the low for $\bar{\rho}_1$. If we drop the (2.5) then, as $T^r \subseteq T'$ reasoning, as used in
As a result, we obtain the required sufficiently small $\epsilon$ we

Therefore, $\lambda_{p+1}', \ldots, \lambda_p'$
group that is a finite $d$ from the results of the

**Theorem 3.2.** Let the

Let problem (2.6) consider orthonormalized in ac eigenvalues be put in orthonormal in $H_M$ b

Then

$$\Theta_M (U)$$

**Proof.** Let us estin
Both these $(p - p')$-dimensional Euclidean product $(\bar{Y}, \bar{X})_M$, whe then

Since

$$\sum_{i=1}^{N} c_i^2 \leq me^2.$$
where \( \bar{\varphi}_1 \geq 0 \) is the lower bound for \( \tau(u), u \in T' \), \( \|u\|_M = 1 \). Let us find the expression for \( \bar{\varphi}_1 \). If we drop the values \( \lambda^r_i \) with \( i = p' + 1, \ldots, p \), from the set \( \lambda^r_1, \ldots, \lambda^r_{N-r} \) [see (2.5)] then, as \( T' \leq \bar{Q}^r \cap L^r(u_{p+1}^r, \ldots, u_p^r) \), inequality (2.11) is valid and the same reasoning, as used in the proof of Theorem 2.1, applies to the lower bound of \( \tau(u) \). As a result, we obtain: \( \tau(u) \geq \bar{\varphi}_1 = \varphi_1 \). Combining the lower bound with (3.6) we obtain the required inequality (3.4). From inequality (3.4) it follows that for sufficiently small \( \varepsilon \) we have

\[
\lambda^r_p \leq \lambda^r_{p+1} \leq \ldots \leq \lambda^r_0 = \beta^r < \lambda^r_p + K_g \varepsilon^2.
\]

Therefore, \( \lambda^r_{p+1}, \ldots, \lambda^r_0 \), with an accuracy of \( K_g \varepsilon^2 \) can be combined into a separate group that is a finite distance apart from \( \lambda^r_p \). Thus, the number \( p' \) can be determined from the results of the calculations.

**Theorem 3.2.** Let the conditions of Lemma 3.1 be satisfied and \( \lambda^r_p < \lambda^r_0 \) [see (3.6)]. Let problem (2.6) correspond to \( H^r_p \) and \( \bar{X}^r(0), i = 1, \ldots, p \), be the eigenvectors of (2.6) orthonormalized in accordance with the inner product \( (\bar{Y}, \bar{X})_{H^r_p} = \bar{Y}^T \bar{X} \). Let these eigenvectors be put in correspondence with the vectors \( v^r_i = v_i' \) [see (2.7)] forming an orthonormal in \( H^q_M \) basis of \( H^r_p \). Let \( U^r \) be a linear span of the vectors \( v_{p+1}^r, \ldots, v_p^r \). Then

\[
\Theta_M(U^r; U^r_p) = K_q \varepsilon + (\beta^r - \lambda^r_p)^{1/2}(\beta^r - \lambda^r_p)^{-1/2} = K_q \varepsilon. \tag{3.7}
\]

**Proof.** Let us estimate \( \Theta_M(\bar{U}^r; U^r) \), where \( \bar{U}^r \) has been defined in Lemma 3.1. Both these \((p - p')\)-dimensional subspaces belong to \( H^r_p \). To estimate \( \rho_M(z; U^r) \), where \( x \in \bar{U}^r \) and \( \|z\|_M = 1 \), we perform an isometric in \( H^r_M \) transition from \( H^r_p \) to the \( p \)-dimensional Euclidean space \( H^r_M \), consisting of \( \bar{X} \equiv (x_1, \ldots, x_p)^T \) with the inner product \((\bar{Y}, \bar{X})_{\bar{M}^r} \), where \( \bar{M} = \bar{M}^r \) [see (2.6)]. In this case, if

\[
z = \sum_{i=1}^{p} z_i v_i^r
\]

then

\[
\bar{z} = \sum_{i=1}^{p} z_i \bar{X}^r(0), \quad \rho^2_M(z; U^r) = \sum_{i=1}^{p} z^2_i.
\]

Since

\[
\sum_{i=1}^{p} z^2_i (\lambda^r_i - \mu) = 0, \quad \mu = \mu(z) = \lambda^r_p, \quad \lambda^r_i \geq \lambda^r
\]

[see (2.4) and (2.8)], we have

\[
\sum_{i=1}^{p} z^2_i (\mu - \lambda^r) = \sum_{i=p+1}^{p'} z^2_i (\lambda^r - \mu)
\]

and

\[
(\mu - \lambda^r_p) \sum_{i=1}^{p} z^2_i \leq (\beta^r - \mu) \left( 1 - \sum_{i=1}^{p} z^2_i \right).
\]
Hence,
\[ \rho_M^2(z;U') < (\beta' - \mu)(\beta' - \lambda_p')^{-1} < \rho_M(U';U')^{-1} \]
and
\[ \Theta_M(\tilde{U}';U') < (\beta' - \lambda_p')^{1/2}(\beta' - \lambda_p')^{-1/2}. \] (3.8)
As the opening of the subspaces possesses the properties of a metric, then inequality (3.7) follows from (3.2) and (3.8). Now it is easy to estimate \( \Theta_L(U';U_p) \), since for any \( u \in U_p \) with \( \|u\|_M = 1 \) and for any \( v \) with \( \|v\|_M = 1 \), we have the equality:
\[ \|v - u\|^2_L = \mu(v) - \lambda_p + \lambda_p\|u - v\|^2_M. \]
Therefore
\[ \Theta_L(U';U_p) = (\beta' - \lambda_p + \lambda_p\Theta_M^2(U';U_p))^{1/2} < K_p. \] (3.9)

Having found \( U_p \) with an accuracy of \( \Theta(\varepsilon) \) [see (3.7) and (3.8)], we are in a position to change the structure of \( \omega_0 \) and \( Q_{p_0} \) and to turn to the determination of \( \lambda_{p'} \) and \( U_{p'} \). By virtue of (3.4), we may take the linear span of the vectors \( v_{1'}, \ldots, v_{p'} \) belonging to \( H' \) as an initial iteration \( H_{p'}^0 \). If \( \lambda_{p'} \) is sufficiently far away from \( \lambda_{p'+1} \), orthogonalization can turn out to be unnecessary. In case \( \lambda_{p'} \) differs from \( \lambda_{p'+1} \) only slightly, the use of orthogonalization with respect to \( U' \) can be an essential factor.

Inserting the indices \( p'+1, \ldots, p \) into \( \omega_0 \) with possible elimination of some higher indices enables one to obtain the convergence rate estimates of such iterative methods no worse than the given estimates for the case of finding \( \lambda_p \) and \( U_p \) considered above. Thus, it provides a possibility of determining \( \lambda_i \) and \( U_{p_i} \) with the accuracy required for all \( i < p \), given, for example, an initial iteration \( H_p^0 \) with \( \beta(H_p^0) < \lambda_{p+1} \).

4. NUMERICAL EXPERIMENTS

As previously mentioned, the most important applications of modified gradient methods are connected with the solution of grid eigenvalue problems that approximate, for example, the corresponding ones with elliptic operators. In order to carry out a larger number of numerical experiments on the study of the effect of various conditions and parameters on the convergence, while preserving reasonable computational costs, the calculations were performed only for model problems. These were the simplest difference schemes on a uniform grid with the step \( h = \pi/(N+1) \) for the problems of the type:
\[ -(a(x)u'(x))' = \lambda q(x)u(x), \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0. \]
The problems were of the form:
\[ -h^{-2}[q_{i+1/2}(u_{i+1} - u_i) - q_{i-1/2}(u_i - u_{i-1})] = \lambda q u_i, \quad i = 1, \ldots, N \] (4.1)
and, on eliminating \( U_0 \) and \( U_{N+1} \), they were reduced to standard algebraic problems (1.1). The operator \( L \) with \( a \equiv 1 \) defined the operator \( B \). For the examples considered, we too determined from the

For this reason, the main attention of \( \varepsilon \) by tw and the group me approximations wer approximation was performed the iterat:
\[ \hat{u}^{n+1} = \quad \]
with \( u^0 = u_{1'} \) until \( v \)

Then at the second we constructed a ne

with \( \hat{u}_1 \) coinciding with \( k_2 \) iterations:
\[ \hat{u}^{n+1} = \quad \]
with orthogonalizat of (4.3) for a numb

In the group we and we performed i

Then the vector orthogonal in \( H_M \) respect to \( \tilde{U}_2 \) and:

was applied. With form of (4.3) to be the number \( k_1 + k_2 \).

For \( N = 10 \) and on eigenvalues and method of orthog span of the grid fu

Let us give some
Finding lower eigenvalues

(3.8)

ric, then inequality \( (U^p; U'_p) \), since for the equality:

\[
\begin{align*}
\delta_0 &= \min_{x \in [0,1]} a(x), \quad \delta_1 = 1.
\end{align*}
\]

For this reason, the iterative parameter \( \gamma \) was always taken equal to \( \gamma = \delta_1^{-1} = 1 \). The main attention was paid to the problem of determining \( \lambda_1 \) and \( \lambda_2 \) with an accuracy of \( \varepsilon \) by two methods: the conventional modified gradient method (method 1) and the group method with \( p = 2 \) (method 2). With both methods, the initial approximations were determined by the same subspace \( H_2^0 \). In method 1 the initial approximation was usually \( u_1^0 \) with \( \| u_1^0 \|_M = 1 \) and \( \mu(u_1^0) = \min_{u \in H_2^0} \mu(u) \). Then we performed the iterations:

\[
\begin{align*}
\bar{u}^{n+1} &= u^n - B^{-1} (Lu^n - \mu(u^n)Mu^n), \quad u^{n+1} = \bar{u}^{n+1} - \bar{u}^{n+1} \| \bar{u}^{n+1} \|_M^{-1} \| \bar{u}^{n+1} \|_M^{-1}
\end{align*}
\]

(4.2)

with \( u^0 = u_1^0 \) until we obtained the inequality:

\[
\mu(u_1^0) - \mu(u_1^{k+1}) < \varepsilon.
\]

(4.3)

Then at the second stage of the method, for \( u_2^0 \) with \( \| u_2^0 \|_M = 1 \) and \( \mu(u_2^0) = \beta(H_2^0) \), we constructed a new approximation:

\[
u^0 = \bar{u}_1^0 - \bar{u}_2^0 \| \bar{u}_1^0 \|_M^{-1}
\]

(4.4)

with \( \bar{u}_1^0 \) coinciding with previously determined \( u_1^{k+1} \). After this, we performed next \( k_2 \) iterations:

\[
\begin{align*}
\bar{u}^{n+1} &= u^n - B^{-1} (Lu^n - \mu(u^n)Mu^n), \quad u^{n+1} = \bar{u}^{n+1} - \bar{u}^{n+1} \| \bar{u}^{n+1} \|_M^{-1} \| \bar{u}^{n+1} \|_M^{-1}
\end{align*}
\]

with orthogonalization with respect to \( \bar{u}_1^0 \) until we obtained an inequality of the type of (4.3) for a number \( k_2 \). The value \( u_2^{k+1} \) obtained was taken equal to \( \bar{u}_2^0 \).

In the group method 2, (without orthogonalization) the set \( \omega_0 \) was taken empty and we performed iterations until we obtained the inequality:

\[
\beta(H_2^0) - \beta(H_2^{k+1}) < \varepsilon
\]

Then the vector \( \bar{u}_2 \in H_2^{k+1} \) with \( \| \bar{u}_2 \|_M = 1 \) and \( \mu(\bar{u}_2) = \beta(H_2^{k+1}) \) yielded an orthogonal in \( H_2 \) complement \( \bar{Q}_1^2 \). Then method (4.2) with orthogonalization with respect to \( \bar{u}_2 \) and with the initial iteration \( u^0 \in H_2^{k+1} \) and

\[
\mu(u_2) = \min_{u \in H_2^{k+1}} \mu(u)
\]

was applied. With this method, \( r_2 \) iterations were required for an inequality of the form of (4.3) to be obtained. In this case the computational costs can be estimated by the number \( k_1 + k_2 \) for method 1 and \( 2k_1 + r_2 \) for method 2.

For \( N = 10 \) and \( N = 20 \) the results were compared with the available information on eigenvalues and eigenvectors obtained by using the conventional program of the method of orthogonal rotations. The subspace \( H_2^{k+1} \) usually coincided with the linear span of the grid functions \( \sin x \) and \( \sin 2x \).

Let us give some conclusions and illustrations.
(1) The choice of an initial approximation that does not obey the conditions of the type of \( \beta^0 < \lambda_3 \) as a rule did not prevent us from obtaining the convergence to the required values \( \lambda_1 \) and \( \lambda_2 \), when using both method 1 and method 2. For some highest eigenvalues \( \lambda_i \) we did sometimes observe the convergence slowing-down. The iterative process in the vicinity of \( \lambda_3 \) came to a halt only when the initial iteration virtually coincided with the linear span of certain eigenfunctions. For example, Fig. 1 demonstrates the behaviour of \( \mu(\mu^n) \) at the first stage of method 1 (determination of \( \lambda_1 \), while Fig. 2 shows the behaviour of \( \beta^n \) at the first stage of method 2 (determination of \( \lambda_2 \)) for problem (4.1) with \( N = 10, a(x) = 10^{-3} \) for \( 0 \leq x \leq 1 \) and \( a(x) = 1 \) for \( 1 \leq x \leq \pi \), \( \delta_0 = 10^{-3}, \ v = 10^{-4}, \ \text{and} \ \lambda_1 = 0.0072, \ \lambda_2 = 0.0245, \ \lambda_3 = 0.0418, \ \lambda_4 = 0.5391, \ \lambda_5 = 4.686, \ \beta(H_2^0) = 2.91, \ r_1 = 81, \ r_2 = 3, \ k_1 = 27, \ k_2 = 69. \)

From Fig. 2 it is seen that the convergence becomes slower in the vicinity of the point \( \lambda_4 \). In Figs. 1, 2 for simplicity, instead of functions defined on the integers, continuous functions are depicted.

![Figure 1](image1)

![Figure 2](image2)

The most aggravating and 2 with a special ch 
\( u_2^0 = u_4 + u_5 \), where \( u_1 \) 
\( a(x) = 10^{-2} \) on the \( \delta_0 = 10^{-2}, \ v = 10^{-4}, \lambda_5 = 4.7133 \). An exam 
conditions is provided i

Method 1, stopped 
(\( \mu^{100} = 0.0890 \)). Meth 
approximation to \( \lambda_3 \) i 
were required to obtain 

(2) Almost in all th 
and \( \mu^n \) happened at i 
iterations accounted fo 

(3) While applying 
monotonically. Howev 
good approximations 
required a significant 
than at the first stage. 

(4) The effect of th 
and 2 was not profou 
the number of iteratio 
\( \lambda_1 = 0.29, \ \lambda_2 = 1.19, \ \\
k_2 = 26 \) iterations (\( \mu \) 
iterations for \( \varepsilon = 10^{-7} 
method 2 required res 

(5) The calculation 
and 1000. An increase 

(6) In all the abo 
determination of \( \lambda_2 \) 
instance, in the exam 
is this fact that was a 
Recall that method 
Presumably, the met 
and \( \lambda_2 (u_1 \ and \ u_2) \) 
would be reasonable

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>Stage</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>
Finding lower eigenvalues

The most aggravated situation (we did this intentionally) occurred for methods 1 and 2 with a special choice of \( H_2^0 \) coinciding with the linear span of \( u_1^0 = u_2 + u_6 \) and \( u_2^0 = u_4 + u_5 \), where \( u_1 \) denote the corresponding eigenfunctions. In this case we had \( a(x) = 10^{-2} \) on the segment \([0, 1]\) and \( a(x) = 1 \) on the segment \((1, \pi)\), \( q = 1 \), \( \delta_0 = 10^{-2}, \ \epsilon = 10^{-4}, \ \lambda_1 = 0.06952, \ \lambda_2 = 0.2387, \ \lambda_3 = 0.4126, \ \lambda_4 = 0.5814, \) and \( \lambda_5 = 4.7333 \). An example of the dependence of \( \mu^n \) on \( n \) method 1 gave under these conditions is provided in Table 1.

Method 1, stopped after 100 iterations, gave a rather crude approximation to \( \lambda_1 \) (\( \mu^{100} = 0.0890 \)). Method 2 after 100 iterations at the first stage gave quite an accurate approximation to \( \lambda_3 \) (instead of \( \lambda_2 \)) while at the second stage as little as 16 iterations were required to obtain \( \mu^{16} = 0.0699 \) close to \( \lambda_1 \).

(2) Almost in all the experiments it was noted that the strongest decrease in \( \beta^n \) and \( \mu^n \) happened at several first iterations. Frequently, this change at the first \( 3 \sim 5 \) iterations accounted for as much as 90% of the total initial error.

(3) While applying method 2, we noticed that, in general, \( \lambda_i^n \) changed non-monotonically. However, the last iterations at the second stage gave as a rule very good approximations to \( u_i \). Therefore, the second stage of the method usually required a significantly smaller number of iterations (sometimes, 10 times smaller) than at the first stage.

(4) The effect of the value of \( \epsilon \) on the required number of iterations in methods 1 and 2 was not profound. A 100-fold decrease in \( \epsilon \) often called for mere doubling of the number of iterations. For example, in the problem with \( a(x) = 1 - x/\pi, \ q = 1, \) and \( \lambda_1 \approx 0.29, \ \lambda_2 \approx 1.19, \ \lambda_3 = 2.66, \ \lambda_4 = 7.25, \ \lambda_5 = 10.5, \) method 1 required \( k_1 = 17 \) and \( k_2 = 26 \) iterations \( (\mu^{0} = 0.5) \) for \( \epsilon = 10^{-3} \) and it required \( k_1 = 27 \) and \( k_2 = 32 \) iterations for \( \epsilon = 10^{-5} \) and at the worst initial value \( \mu^{0} = 3 \). In the same situation, method 2 required respectively \( r_1 = 26, \ r_2 = 3 \) and \( r_1 = 48, \ r_2 = 8 \).

(5) The calculations were performed with the values of \( \delta_1/\delta_0 \) varying between 2 and 1000. An increase in the value of \( \delta_1/\delta_0 \) was not profound either.

(6) In all the above examples method 2 was used in the situations when the determination of \( \lambda_2 \) was a more difficult problem than the determination of \( \lambda_1 \). For instance, in the example from item 4 we have \( \lambda_2/\lambda_1 = 4 \), but \( \lambda_3/\lambda_2 = 2 \). Probably, it is this fact that was a reason why method 2 usually had no advantages over method 1. Recall that method 2 is, first of all, aimed at the cases when \( \lambda_1 \approx \lambda_1 < \lambda_2 \). Presumably, the method may be of particular assistance when one has to separate \( \lambda_1 \) and \( \lambda_2 \) \( (u_1 \) and \( u_2) \) given a relatively good approximation to \( U_{\lambda_1} + U_{\lambda_2} \). Evidently, it would be reasonable to obtain such an approximation using method 1.

<table>
<thead>
<tr>
<th>Stage</th>
<th>( n )</th>
<th>0</th>
<th>4</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>70</th>
<th>90</th>
<th>100</th>
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<tbody>
<tr>
<td>1</td>
<td>( \mu^n )</td>
<td>6.35</td>
<td>0.36</td>
<td>0.297</td>
<td>0.173</td>
<td>0.13</td>
<td>0.107</td>
<td>0.0925</td>
<td>0.0890</td>
</tr>
<tr>
<td>2</td>
<td>( \mu^n )</td>
<td>2.65</td>
<td>0.573</td>
<td>0.572</td>
<td>0.562</td>
<td>0.55</td>
<td>0.53</td>
<td>0.482</td>
<td>0.4148</td>
</tr>
</tbody>
</table>
REFERENCES


Optimal method problems with...

A. D. JAVADIAN

Abstract – The paper deals with inhomogeneous elliptic problems allowing us to solve these problems is given.

1. POSING THE PROBLEM
Let us consider a smooth convex domain

where $L$ is an elliptic operator, $\partial_k = \partial / \partial x_k$, $k = 1, 2, 3$.

In the traditional Sobolev–Slobodetskii spaces $\mathcal{D}_0^s = \mathcal{D}(\Omega, L, f, 0)$ and $\mathcal{S} = H^0(\Omega)$, then problems $\mathcal{D}_0^s$ and Sobolev–Slobodetskii spaces.

When solving the problems $\mathcal{S}$ and $\mathcal{D}_0^s$ so that they are solvable with the optimal methods.

Let us elucidate the method of approximate solution.

Let us set

where we shall omit the details of these sets.