We substitute the elements of the block (8) which are determined by the correspondence (11) in the array (10) and subtract each, except the first, row from the preceding row:

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

The columns (12), taken as blocks, form a (23, 5, 10)-difference family in \( Z_{23} \) (\( \lambda = 10 \) is the smallest admissible value). Repeating the above-described procedure for the second block of the family (9), as the result we find a \( T_5(23, 5, 4) \)-difference family. Considering all possible cyclic permutations of elements in its blocks and adjoining five zero columns and one zero row, we obtain a (23, 6, 5)-difference matrix in \( Z_{23} \).

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SHARP A PRIORI ERROR ESTIMATES OF THE RAYLEIGH-RITZ METHOD
WITHOUT ASSUMPTIONS OF FIXED SIGN OR COMPACTNESS

A. V. Knyazev

I. We prove sharp a priori estimates of error inequalities type in the Rayleigh-Ritz method for a partial eigenvalue problem in a Hilbert space with a bounded self-adjoint operator. Each of the obtained inequalities is sharp and, consequently, cannot be improved without change of terminology, i.e., has a final character. Compactness, nondegeneracy, or the fixed sign property of the operator are not required.

Under these assumptions, only the asymptotic error estimates are known [1]. If the operator is assumed to be compact, nondegenerate, and positive, then estimates of the type of the inequalities of errors of approximations to the first eigenvalues and to the eigensubspace, corresponding to the maximum eigenvalue* exist in the literature [2-4]. Finally,*Vainikko has obtained in [10] asymptotically sharp error estimates of the inequalities type for the higher eigenvalues and vectors of a compact operator of fixed sign. In comparison with the present note, these estimates the magnitude of the norm of the error of approximation of the operator is additionally used.

for the finite-dimensional case, estimates of the type of inequalities of the errors of approximations to the eigensubspaces corresponding to some first eigenvalues have been published [5-9].

The results of the present note coincide asymptotically with those of [1]. The estimates for the eigenvalues in the above-indicated particular cases are analogous to those of [2, 3] and coincide with those of [4, 8, 9]. The estimate for the first eigensubspace coincides with that of [4, 5]. But the estimates for the higher eigensubspaces are new even for the finite-dimensional variant and improve [7-9]. Detailed computations, proofs of the sharpness of the obtained estimates, and their applications to the analysis of the conformal method of finite elements can be found in [11].

II. Let \( H \) be a real separable Hilbert or Euclidean space with the scalar product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \), \( \| u \|_H = \langle u, u \rangle^{1/2} \). Let there be defined on \( H \) the bounded symmetric bilinear form \( m(u, v) = m(v, u) \), \( \lambda_{\text{min}}(u) \leq m(u) \leq \lambda_{\text{max}}(u) \), where \( m(u) = m(u, u), \forall u, v \in H \). Let us consider the following eigenvalue problem: To find \( u \in H \setminus 0 \) and \( \lambda \in \mathbb{R} \) from

\[
m(u, v) = \lambda m(u, v), \quad \forall v \in H.
\]

The problem (1) is equivalent to the eigenvalue problem for the bounded self-adjoint operator \( L^1 M \) in \( H \) that is defined by the equality \( m(u, v) = l((L^1 M) u, v) \). Since, in general, the operator \( L^1 M \) is not assumed to be compact, we stipulate that the semiclosed interval \( (\lambda_{\text{min}}, \lambda_{\text{max}}] \) contains only the point spectrum \( \lambda_{\text{max}} = \lambda_1 > \ldots > \lambda_p, p < \infty \). Let \( k_1 < \infty \) denote the multiplicity of each \( \lambda_j \), and the corresponding eigensubspace be denoted by \( U_j \), \( \dim U_j = k_j (j = 1, \ldots, p) \). Let us set \( U = U_1 + \ldots + U_p \) and \( k = \dim U = k_1 + \ldots + k_p \). The partial eigenvalue problem consists in the determination of \( \lambda_j \) and \( U_j \), \( j = 1, \ldots, p \).

Remark. The analogous problem for the minimum eigenvalues (1), if their existence is assumed, reduces to the preceding one on replacing \( m(\cdot, \cdot) \) by \(-m(\cdot, \cdot)\). In this manner, all the results of the present note are carried over to the case of computation of the minimum eigenvalues.

Let there be given a certain subspace \( U \subseteq H \), \( \dim U = k \). Let us construct the projection of the problem (1) on \( U \): To find \( \bar{u} \in U \setminus 0 \) and \( \bar{\lambda} \in \mathbb{R} \) from

\[
m(\bar{u}, v) = \bar{\lambda} m(\bar{u}, v), \quad \forall v \in U.
\]

The problem (2), constructed by the projection method, is equivalent to the problem of the Rayleigh–Ritz method: To find \( \bar{u} \in U \setminus 0 \) as a fixed point in \( U \) of the Rayleigh ratio \( \lambda(\cdot) = m(\cdot) / \| \cdot \| \) and to take \( \bar{\lambda} = \lambda(\bar{u}) \). This follows, as usual, from the symmetry of the problem (1).

We number all the \( k \) eigenvalues (2) in decreasing order: \( \lambda_{j,1} \geq \lambda_{j,2} \geq \ldots \geq \lambda_{j,k_j} \), and analogously the linearly independent eigenvectors corresponding to them. Let us set \( \tilde{\lambda}_j = \lambda_{j,k_j} \) and \( U_j = \text{span} (\tilde{\bar{u}}_{j,1}, \ldots, \tilde{\bar{u}}_{j,k_j}) \) and call the numbers \( \tilde{\lambda}_j \) and the subspaces \( U_j \) the approximations by the Rayleigh–Ritz method in the subspace \( U \) to the eigenvalues \( \lambda_j \) and the eigensubspaces \( U_j \).

For comparing the closeness of a subspace \( V \subset H \) to a subspace \( W \subset H \), \( \dim V \leq \dim W < \infty \) we will use the quantity

\[
\Theta(V; W) = \max_{v \in V \setminus 0} \frac{\text{dist} (v, W)}{\| v \|_H}.
\]

If \( \dim V = \dim W \), then (3) coincides with the definition of the angle between subspaces.

**Theorem 1.**

\[
0 \leq \lambda_j - \tilde{\lambda}_j \leq (\lambda_j - \lambda_{\text{min}}) \Theta^2(U; U) (j = 1, \ldots, p).
\]

**Theorem 2.**

\[
\Theta^2 (U_j; U_i) \leq (\tilde{\lambda}_i - \tilde{\lambda}_j) / (\lambda_i - \lambda_j).
\]

If \( \lambda_{j-1} > \lambda_j \) and \( \tilde{\lambda}_j > \tilde{\lambda}_{j-1} \) for a certain \( j \in \{2, \ldots, p\} \), then

\[
\Theta^2 (U_j; U_i) \leq (\tilde{\lambda}_{j-1} - \tilde{\lambda}_j)^{-1} (\lambda_{j-1} - \lambda_i)^{-1} (\tilde{\lambda}_{j-1} - \lambda_{j-1}) (\lambda_j - \tilde{\lambda}_j).
\]

Each of the estimates (4)-(6) is sharp.
III. Let us outline the scheme of application of the inequalities (4)-(6) to the a priori analysis of the rate of convergence of the conformal method of finite elements. Let $H_0$ and $H_1$ be real separable Hilbert spaces with the norms $\| \cdot \|_0$ and $\| \cdot \|_1$, such that in unit (in $H_1$) sphere from $H_0 \cap H_1 \neq \emptyset$ is approximated in $H_0$ by a family of finite-dimensional subspaces $H^h \subset H_0$, $h \to 0$ being a parameter, i.e.,
\[
\text{dist}_0(u; H^h) \leq \epsilon_r h \| u \|_1,
\forall u \in H_0 \cap H_1, \quad \epsilon_r (h) \to 0.
\] (7)

A typical example: $H_0$ is a subspace of $W_2^1(\Omega)$, $H_1 \equiv W_2^2(\Omega)$, $\Omega \subset \mathbb{R}^2$, and use of piecewise-linear elements gives $\epsilon_r (h) = K h^{-4}$. We will assume that the spaces $H$ and $H_0$ coincide elementwise, $\sigma_0 \| \cdot \|_0 \leq \| \cdot \|_1 \leq \sigma_1 \| \cdot \|_0$, $\sigma_0 > 0$, and $U \subset H_0 \cap H_1$. Then, for $\dim H^h \geq k$ from (7) we have
\[
\Theta (U; H^h) \leq \sigma_0^2 \sigma_1 \epsilon_r (h),
\]
\[
\epsilon_r \equiv \max_{u \in U, \vartheta} \| u \|_1 / \| u \|_0.
\] (8)

We take a subspace $U \subset H^h$ such that $\Theta (U; U) = \Theta (U; H^h)$. Now the error estimate for the approximations to the eigenvalues in $U$ follows from (4) and (8). Then the same estimates for approximations of $\lambda^h_j$ in $H^h$ follow from the variational principle [12] and $U \subset H^h$. If $U$ and $\epsilon_r$ are replaced by $U_i + \ldots + U_j$ and $\epsilon_r$, respectively in (8), then the same arguments lead to somewhat better (on account of $\epsilon_r (h) < \epsilon_r (h)$) estimate:
\[
0 \leq \lambda_j - \lambda^h_j \leq (\lambda_j - \lambda_{\min}) \sigma_0^2 \sigma_1^2 \epsilon_r (h) \quad (j = 1, \ldots, p).
\]
The error estimates in $H$ for approximations to eigensubspaces obviously follow from (5), (6), and the indicated estimates for eigenvalues, and the conditions $\lambda_{j+1} > \lambda_j$ and $\lambda_j > \lambda_{j+1}$, lead to the condition for "smallness" of $\epsilon_r (h)$. Using the estimates for the angles in $H$, we easily deduce the estimates in $H_0$ also when necessary [10].

IV. Proof of Theorem 1. The right-hand inequality in (4) for $\Theta (U; U) = 1$ and the left-hand inequality in (4) follow directly from the variational principle [12] -- a generalization of the Courant--Fisher theorem. We prove the right-hand inequality in (4) for $\Theta < 1$, following the methodology of [3, 8, 9].

Let $P$ be the orthogonal (in $H$) projection on $U$. Then $P$ maps $U$ onto $U$ in a one-to-one manner and $P$ is orthogonal also in the scalar product $(m - \lambda) (u, v) \equiv (m (u, v) - \lambda (u, v), \lambda_{\min} > \lambda$ is a parameter. We take $\tilde{u} \in U \setminus 0$. We have $P \tilde{u} \subset U \setminus 0$ and
\[
\lambda (P \tilde{u}) - \lambda = (m - \lambda) (P \tilde{u}, \tilde{u}) \lesssim (m - \lambda) (\tilde{u}, \tilde{u}) (1 - \Theta)^{-1} = (1 - \Theta)^{-1} (\lambda (\tilde{u}) - \lambda)
\]
It follows from this inequality and the Courant--Fisher theorem in $U$ and in $U$ that $\lambda_j - \lambda \leq (1 - \Theta)^{-1} (\lambda_j - \lambda)$ $(j = 1, \ldots, p)$. Both the sides of the last inequality continuously depend on the parameter $\lambda < \lambda_{\min}$. Consequently, it is valid for $\lambda = \lambda_{\min}$ also, which leads to (4).

Proof of Theorem 2. We will prove (5) and (6) separately, using the following common method. At first, we prove the estimate (5) ((6)) in the simplest particular case of a two-dimensional (three-dimensional) space $H$. Then for large dimension we construct a special two-dimensional (three-dimensional) subspace $H^h (H_0)$, the restriction of the problem (1) to this subspace is a two-dimensional (three-dimensional)-eigenvalue problem; we call it (1'). The estimate (5) ((6)), proved already for this particular case, is valid for the problem (1'); we denote this estimate by (5') ((6')). Finally, the construction of $H^h (H_0)$ is chosen such that (5) ((6)) follows from (5') ((6')). Here we use the fact that the right-hand side of (5) ((6)) depends monotonically on its parameters $\lambda$ and $\lambda$. The proof of (5). In order to avoid confusion, we equip all the symbols, concerning the two-dimensional case, with the affix 2; e.g., the estimate (5) is written down as
\[
\Theta (U_2, U_2^2) \approx (\lambda_2 - \lambda_1^2) / (\lambda_2^2 - \lambda_1^2).
\]
(5')

A trivial verification shows that even the equality is valid in (5').

Let us construct a two-dimensional subspace $H^2 \subset H$ such that the projection of the problem (1) on $H^2$ has the eigenvalues $\lambda_2 = \lambda_1$ and $\lambda_2^2 \leq \lambda_2$ and the eigensubspace $U_2^2$ and we can indicate a test subspace $U_2^2 \subset H^2$, such that $\Theta (U_1; U_2) = \Theta (U_1, U_2) \equiv \lambda_2$. If a similar subspace $H^2$ is constructed, then (5) follows obviously from the estimate (5').
We introduce the operator $P_1$ of orthogonal (in $H$) projection on $U_1$ and set $P_2 = E - P_1$. We take a unit vector $u_1^1 \in U_1 \setminus 0$ such that $\Theta (U_1; u_1) = \text{dist} (u_1^1, U_1)$ and define orthogonal vectors $u_1^2 = P_2 u_1^1$ and $u_2^1 = P_2 u_2$.

If $u_1^1 = 0$, then $\bar{u}_1^1 \in P_2 H$ and, by virtue of the variational principle, $\lambda_2 \geq \lambda (u_2^1) \geq \lambda_1$. In this case, the left-hand side of (5) is equal to 1 and the right-hand side is greater than or equal to 1.

If $u_2^1 = 0$, then $\bar{u}_2^1 \subset U_1$ and then the left-hand and the right-hand sides of (5) and equal to zero.

If $u_1^1, u_2^1 \neq 0$, then we set $H^2 = \text{span} \{ u_1^1, u_2^1 \}$. The eigenvectors of projection $u_1^1$ and $u_2^1$ correspond to the eigenvalues of projection $\lambda_1 = \lambda (u_1^1) = \lambda_1$ and $\lambda_2 = \lambda (u_2^1) = \lambda_2$. We take $\bar{u}_2^1$ as a test vector in $H^2$. Then we have

$\Theta (U_1; U_1) = \text{dist} (\bar{u}_1^1, u_1^1) = \text{dist} (\bar{u}_1^1, u_1^1) = \Theta (U_1; U_1)$,

$\lambda_2 = \lambda (u_2^1) \geq \lambda_1$.

The desired two-dimensional problem is constructed.

**Proof of (6).** It is easily verified that the following equation is valid for dim $H = 3$ (we assume $\lambda_1 > \lambda_2 > \lambda_3$ and dim $U = 2$):

$$\Theta^2 (U_1; U_2) = 1 - \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}.$$

(9)

The right-hand side of (9) increases monotonically with respect to $\lambda_1$. In order to eliminate $\lambda_1$, we let $\lambda_1 \to \infty$; then the right-hand side of (9) transforms into the right-hand side of (6). Using the affix 3, we rewrite the estimate (6) for dim $H = 3$, proved in this manner:

$$\Theta^2 (U_1; U_2)^2 \leq (\lambda_2 - \lambda_1)^1 (\lambda_3 - \lambda_2)^1 (\lambda_3 - \lambda_1)^1 (\lambda_2 - \lambda_3)^2.$$

(6')

For fixed $j \in [2, p]$ let us construct a three-dimensional subspace $H^3 \subset H$, such that the projection of the problem (1) on $H^3$ has the eigenvalues $\lambda_3 = \lambda_j$ and $\lambda_2 = \lambda_{j+1}$ and we can indicate a two-dimensional test subspace $U^3 \subset H^3$ that satisfies the conditions $\Theta (U_1; U_2) = \Theta (U_j; U_j)$, $\lambda_3 \geq \lambda_{j+1}$ and $\lambda_2 \geq \lambda_j$. It follows from the conditions $\lambda_{j+1} > \lambda_j \geq \lambda_{j+1}$ of Theorem 2, by construction, that $\lambda_j > \lambda_2 \geq \lambda_j$. Let us observe that under these conditions the right-hand side of (6') is monotonically nondecreasing with respect to $\lambda_3$ and non-increasing with respect to $\lambda_2$ and $\lambda_1$. Therefore, we can conclude that (6) follows from (6').

We introduce the operators $P_1$ and $P_2$ of orthogonal (in $H$) projection on $U_1 + \ldots + U_{j-1}$ and $U_j$, respectively, and set $P_3 = E - P_1 - P_2$.

We take a unit vector $u_2 \in U_j$ such that $\text{dist} (u_2, U_j) = \Theta (U_j; U_j)$. It follows from the conditions of the theorem and the variational principle (by contradiction) that $P_2 u_2 \neq 0$. We take the vector $\bar{u}_1 \in U_1 + \ldots + U_{j-1} \setminus 0$ arbitrary if $P_2 U_2 = 0$, and such that $P_2 \bar{u}_1 = P_2 u_2$ if $P_2 u_2 = 0$. Such a vector $\bar{u}_1$ exists in the latter case, since $\text{codim} \{ v \in H : P_2 v \in \text{span} (P_2 u_2) \} = k_1 + \ldots + k_{j-1} - 1$. We take observe that $\lambda_3 \geq \lambda_{j+1} \geq \lambda_j$ and $\lambda_2 \geq \lambda_j$. Let us set $\bar{u}_1 = (P_1 + P_2 + P_3) \bar{u}_1$, where $P_1$ is the orthogonal (in $H$) projection on the one-dimensional subspace $\text{span} (u_2)$; we have $\bar{u}_1 \neq 0$ since, in the contrary case, $\bar{u}_1 \in U_j$ and $\lambda (\bar{u}_1) = \lambda_j$, which contradicts the assumption $\lambda_{j+1} > \lambda_j$. We have $\lambda (\bar{u}_1) > \lambda_j$ and $m (\bar{u}_1, \bar{u}_2) = 0$.

Let us consider the case dim $\text{span} (\bar{u}_1, \bar{u}_2, u_2) = 2$. It is easily seen that then simply $\bar{u}_2 = u_2$ and, consequently, $\Theta (U_j; U_j) = 0$.

Further, we assume that dim $\text{span} (\bar{u}_1, \bar{u}_2, u_2) = 3$. By construction, $(P_1 + P_2) \text{span} (\bar{u}_1, \bar{u}_2, u_2) = \text{span} (P_1 \bar{u}_1, u_2)$. Therefore, the variational principle leads to

$$\min_{u \in \text{span} (\bar{u}_1, \bar{u}_2, u_2)} \lambda (u) \leq \lambda_{j+1}.$$

Let us now define the three-dimensional problem

$$H^3 = \text{span} (\bar{u}_1, \bar{u}_2, u_2), U^3 = \text{span} (\bar{u}_1, \bar{u}_2),$$

Summarizing what we have said above, we get.
The desired problem is constructed, which completes the proof of Theorem 2.

Let us observe that it is easy to improve the estimate (6) if we also use the term \( \lambda_1 \) in addition. However, this new and better estimate is substantially more tedious (see (6') and (9)).

The main aim of this note from the beginning has been the generalization of the results of [4], in which D'yakonov took the lead. The author is thankful to him.

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