ITERATIVE METHODS IN LINEAR ALGEBRA
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SESSION 1: COUPLED INNER OUTER ITERATION METHODS

ITERATIVE SOLUTION OF PDE WITH STRONGLY VARYING COEFFICIENTS: ALGEBRAIC VERSION

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Abstract. We start with an algebraic saddle point problem typical for finite approximation of a div-form PDE. The idea is to rewrite this problem with the help of special orthoprojectors. The new problem can be solved by applying any of standard iterative methods for non-negative definite symmetric systems. This approach gives a possibility to solve efficiently algebraic problems corresponding to PDE's with strongly varying coefficients, for example, with infinitely large coefficients in subdomains.

Key words. Saddle point problems, diffusion equation, strongly varying coefficients.

1. Introduction. In [1, 2], a new approach to the iterative solution of PDE's in div-form with strongly varying coefficients was presented. Taking the diffusion equation

$$div(\ k\ grad\ u - f) = 0 \ \ , u \in \overset{\circ}{W}{}_2^1\ (\Omega)$$

and letting $v = k\ grad\ u$, one gets the system

$$div(v - f) = 0 \ \ , \ k^{-1}v - grad\ u = 0.$$

The main idea in [1, 2] is to eliminate u here:

$$P(v - f) = 0,\ P^{\perp}k^{-1}v = 0 \tag{1.1}$$

with the help of the orthoprojectors

$$P \equiv grad\ \Delta^{-1}div:\ \mathbb{L}_2(\Omega) \rightarrow \mathbb{L}_2(\Omega),\ P^{\perp} = I - P,$$

and, with v known from (1.1), to compute $u = \Delta^{-1}div\ k^{-1}v$.

The advantage of this approach is the possibility to study the case where the coefficient function k^{-1} vanishes in subdomains of Ω.

Results from [1, 2] were generalized in [3] to elliptic PDE's systems and linear elasticity equations. An alternative approach for treating large jumps in the coefficient function was suggested in [4]. We investigate here the algebraic version of [1, 2].

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2. The reformulation of a saddle point problem. Consider the following saddle point point problem

$$\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ B^* f \end{pmatrix} \tag{2.1}$$

under the assumptions

$$A: \mathbb{H} \to \mathbb{H},\ A = A^* \geq 0,\ B: \mathbb{G} \to \mathbb{H},\ Ker(B) = \{0\} \tag{2.2}$$

in Euclidean spaces $\mathbb{H}, \mathbb{G}$,where $f \in \mathbb{H}$ is known and the pair $v \in \mathbb{H}, u \in \mathbb{G}$ is to be found out.

If $A > 0$, then $v = -A^{-1}Bu$ and (2.1) is equivalent to the div-form equation

$$B^*(A^{-1}Bu + f) = 0.$$

The motivation for considering (2.1) instead of this standard equation is the convenience to deal with degenerate A, but not with infinitely large A^{-1}.

THEOREM 2.1. *Under the assumption (2.2) on B, the operator*

$$P \equiv B(B^*B)^{-1}B^* : \mathbb{H} \to \mathbb{H}$$

is an orthoprojector and $\mathbb{P} \equiv Im(P) = Im(B) \subseteq \mathbb{H}$. *There exists a left inverse operator*

$$B^{-1} \equiv (B^*B)^{-1}B^* \mid_{\mathbb{P}}: \mathbb{P} \to \mathbb{G}$$

with $B^{-1}B = I$, $BB^{-1} = P$.

Proof. The assumption (2.2) on B leads to

$$B^*B = (B^*B)^* > O : \mathbb{G} \to \mathbb{G},$$

therefore the operator $(B^*B)^{-1}$ exists and $P = P^*$. The projector definition $P = P^2$ holds trivially and so are $B^{-1}B = I, BB^{-1} = P$. □

THEOREM 2.2. *Under the assumption (2.2), the problem (2.1) is equivalent to the system*

$$P^{\perp}Av = 0,\ P(v - f) = 0,\ P^{\perp} \equiv I - P \tag{2.3}$$

with unknown $v \in \mathbb{H}$ *only and*

$$u = -(B^*B)^{-1}B^*Av, \tag{2.4}$$

where v is a solution of (2.3).

Proof. First equation in (2.1) is equivalent to the pair

$$P^{\perp}Av = 0,\ PAv + Bu = 0,$$

in which the second one is equivalent to $B^{-1}PAv + u = 0$ and, as a consequence of Theorem 2.1, to (2.4). Similarly, the second equations in (2.1) and (2.3) are equivalent to each other. □

THEOREM 2.3. *Let* $v \in \mathbb{H}$ *be a solution of (2.3) for* $f \in \mathbb{H}$. *Then v is a unique solution of (2.3) iff*

$$Ker(A) \cap \mathbb{P}^{\perp} = \{0\}, \tag{2.5}$$

otherwise all the solutions for given $f \in \mathbb{H}$ *form the affine space*

$$v + (Ker(A) \cap \mathbb{P}^{\perp}),\ v \text{ is a solution of } (2.3)\ .$$

Proof. Solutions of (2.3) for $f = 0$ form the subspace $Ker(P^{\perp}A) \cap \mathbb{P}^{\perp}$. Therefore, solutions of (2.3) for $f \in \mathbb{H}$ are the points of an affine space $v + (Ker(P^{\perp}A) \cap \mathbb{P}^{\perp})$, *v is a solution of (2.3)*. The last step is the verification of

$$Ker(P^{\perp}A) \cap \mathbb{P}^{\perp} = Ker(A) \cap \mathbb{P}^{\perp}. \tag{2.6}$$

From $Ker(P^{\perp}A) \supseteq Ker(A)$ it follows that (2.6)holds with "$\supseteq$" instead of the desired equality "$=$". We now prove the inclusion "$\subseteq$" in (2.6). Let $v \in \mathbb{P}^{\perp}$, $P^{\perp}Av = 0$, then

$$0 = (P^{\perp}Av, v) = (Av, v) = \|\ A^{\frac{1}{2}}v\ \|^2,$$

so $A^{\frac{1}{2}}v = 0$ and $v \in Ker(A)$. □

COROLLARY 2.1. *For a given* $f \in \mathbb{H}$ *a solution pair* $\{v, u\}$ *of (2.3) is not necessarily unique, but its part u is.*

Proof. By Theorem 2.3, the difference between two solutions of (2.3) $v_1 - v_2 \in Ker(A)$, so the vector u does not depend of this difference by formula (2.4). □

The computation of u from (2.4) is numerically stable in the sense of the following

PROPOSITION 2.1. *If* $\{v^*, u^*\}$ *is a solution of the system (2.1) and a vector* $u \in \mathbb{G}$ *is found from (2.4) by a given* $v \in \mathbb{H}$, *then*

$$\|\ B(u^* - u)\ \| \leq \|\ A\ \|\ \|\ v^* - v\ \|\ .$$

Proof. From (2.4) it follows that $B(u^* - u) = -PA(v^* - v)$. □

We conclude this section with a summary comment: to find the solution $u \in \mathbb{G}$ of (2.1), it is sufficient to compute a solution $v \in \mathbb{H}$ of (2.3).

3. Iterative solution of a Ritz equation for a non-negative definite case. Denote $q \equiv P^{\perp}v$, $p \equiv Pf$ and write (2.3) in the following form

$$P^{\perp}A(q + p) = 0,\ q \in \mathbb{P}^{\perp},\ p \in \mathbb{P}. \tag{3.1}$$

One can see that (3.1) is the equation of the Ritz procedure for solving a linear system with non-negative definite coefficient matrix A and with slightly nonstandard right-hand side.

To solve (3.1), choose for our analysis the simplest one-step stationary iterative method (Richardson method). Other well-known methods (multi-step, variational) are available as well.

The iterative scheme is

$$q^{n+1} = q^n - \tau P^{\perp}A(q^n + p),\ n = 0, 1, \ldots \tag{3.2}$$

THEOREM 3.1. *Let* $\mathbb{N} \equiv Im(P^{\perp}A)$. *Assume*

$$(Av, v) \geq \kappa\ (v, v),\ \kappa > 0,\ v \in \mathbb{N}. \tag{3.3}$$

Then, there exists a unique solution of (3.1) $q \in \mathbb{N}$, *the stability inequality*

$$\| q \| \leq \frac{1}{\kappa} \| A \| \| p \| \tag{3.4}$$

holds, the iterative procedure (3.2) with the initial guess

$$q^0 \in \mathbb{N} \tag{3.5}$$

(for example, $q^0 = 0$*) converges for* $\tau = 1/ \| A \|$ *and*

$$\| q^n - q \| \leq (1 - \kappa / \| A \|)^n \| q^0 - q \| \tag{3.6}$$

Proof. Let

$$L \equiv P^{\perp} A |_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}, \tag{3.7}$$

then (3.3) and (2.2) lead to

$$0 < \kappa I \leq L = L^* \leq \| A \| I. \tag{3.8}$$

Write (3.1) as an equation

$$Lq = r \in \mathbb{N}, \tag{3.9}$$

with $r \equiv -P^{\perp} Ap$. Due to (3.8) this equation has a unique solution $q \in \mathbb{N}$ and the following stability inequality holds

$$\| q \| \leq \| L^{-1} \| \| r \|,$$

from which (3.4) follows.

By induction from (3.5) deduce $q^n \in \mathbb{N}, n = 0, 1, ...$, so we find the iterative procedure (3.2) has the form

$$q^{n+1} = q^n - \tau(Lq^n - r), n = 0, 1, ... \tag{3.10}$$

Making now use of (3.8) yields (3.6). □

REMARK 3.1. *The inequality (3.3) is always fulfilled for some* $\kappa > 0$. *If the problem (2.1) is a discrete approximation of the PDE mentioned in the introduction, then* κ *does usually not depend on the mesh size parameter, see [3].*

Finally, consider a special family of non-degenerate operators $A \equiv A_\omega$, for which the problem (2.1) can be viewed as a finite dimensional approximation to a PDE with strongly varying coefficients. The condition (3.3) is possibly not uniformly fulfilled for all $\omega > O$; nevertheless, some modifications in the Theorem 3.1 allow to cover this case too, as we see in

THEOREM 3.2. *Let* $\mathbb{H} = \mathbb{D} \oplus \mathbb{D}^{\perp}$ *for some subspace* $\mathbb{D}$ *and*

$$A = A_D + \omega D^{\perp},\ Im(A_D) = \mathbb{D},\ A_D = A_D^* \geq 0,\ 0 < \omega \leq \| A_D \|, \tag{3.11}$$

where $D^{\perp}$ *is the orthoprojector on* $\mathbb{D}^{\perp}$. *Let* $\mathbb{N}_D \equiv P^{\perp} \mathbb{D}$. *Assume that*

$$(A_D v, v) \geq \kappa_D (v, v),\ \kappa_D > 0,\ v \in \mathbb{N}_D. \tag{3.12}$$

Then there exists a unique solution of (3.1) $q \in \mathbb{N}_D$, *the following stability inequality*

$$\| q \| \leq \frac{1}{\kappa_D} \| A_D \| \, \| p \|,$$

holds, the iterative procedure (3.2) with the initial guess $q^0 \in \mathbb{N}_D$ *(for example,* $q^0 = 0$*) converges for* $\tau = 1/ \| A_D \|$ *and*

$$\| q^n - q \| \leq (1 - \kappa_D / \| A_D \|)^n \, \| q^0 - q \| .$$

Proof. The subspace $\mathbb{N}_D$ is invariant for the operator $P^\perp A$, because $P^\perp A_D \mathbb{N}_D \subseteq \mathbb{N}_D$ and $P^\perp D^\perp \mathbb{N}_D = P^\perp D^\perp P^\perp \mathbb{D} = P^\perp D^\perp P \mathbb{D} = P^\perp D P \mathbb{D} \subseteq \mathbb{N}_D$, so we can denote (cf. with (3.7)):

$$L \equiv P^\perp A \,|_{\mathbb{N}_D} : \mathbb{N}_D \to \mathbb{N}_D.$$

From (3.11), (3.12), it follows that

$$0 < \kappa_D I \leq L = L^* \leq \| A_D \| \, I.$$

Equation (3.1) may be written as $Lq = r \equiv -P^\perp Ap = -P^\perp A_D p - \omega P^\perp D^\perp p \in \mathbb{N}_D$, because $p \in P$ and $P^\perp D^\perp \mathbb{P} = P^\perp D \mathbb{P} \subseteq \mathbb{N}_D$. Further arguments are similar to the corresponding ones in the proof of the Theorem 3.2 so we drop them. □

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REFERENCES

[1] N. Bakhvalov and A. Knyazev, *A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations*, Sov. Math. Doklady, 41 (1991), Translated from Dokl. Akad. Nauk SSSR, 312 (1990), pp. 783–786. In Russian.

[2] ——, *A new iterative algorithm for solving the fictitious fluxes method problems for elliptic equations*, in Proc. EQUADIFF 7, Praha, 1989.

[3] ——, *Methods of effective computation of homogenized properties for the composites with a periodic structure wich consists of essentially different components*, in Comp. Processes Systems N 8, NAUKA, Moscow, 1990, pp. 52–94. In Russian.

[4] ——, *An effective computation of homogenized properties for the composites with a periodic structure which consists of essentially different components*, Sov. Math. Doklady. Translated from Dokl. Akad. Nauk SSSR, 313 (1990), pp. 777-781. In Russian.