Math 3191
Applied Linear Algebra

Lecture 20: Discrete Dynamical Systems

Stephen Billups

University of Colorado at Denver
Sec. 5.4: Eigenvectors and Linear Transformations (cont.)

Review:

Last time, we looked at how to represent a linear transformation $T : V \rightarrow W$ with a matrix $A$, in the sense that

$$w = T(v) \iff [w]_C = A[v]_B.$$ 

Note that the matrix $A$ depends on the bases $B$ and $C$. If you choose different bases, you will get a different matrix.

Tonight, we look at linear operators:

- A linear transformation $T : V \rightarrow V$ that maps a vector space $V$ into itself, is called an operator.

- When we represent an operator by a matrix, we use the same basis for the domain and codomain.
Example

Consider the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$ 

What is the matrix representing this transformation with respect to the standard basis for $\mathbb{R}^2$?
A different basis

What happens if we use the basis $B = \{b_1, b_2\} = \{(1, 1), (-1, 1)\}$ instead?

$$T(b_1) = \begin{bmatrix} \_ \\ \_ \end{bmatrix}, \quad T(b_2) = \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$

$$[T(b_1)]_B = \begin{bmatrix} \_ \\ \_ \end{bmatrix}, \quad [T(b_2)]_B = \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$

Matrix representation relative to the basis $B$:

$$B = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix}.$$
Another Look at Similarity

In the previous two slides, we saw that the linear transformation could be represented by two different matrices (relative to two different bases).

We will show that two square matrices $A$ and $B$ represent the same linear transformation (with respect to two different bases), if and only if they are similar. That is, there exists some invertible matrix $P$ such that

$$A = PBP^{-1}.$$ 

In our example, if we let $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then

$$PBP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A.$$
Theorem 8

Suppose $A = PBP^{-1}$. If $B$ is a basis for $\mathbb{R}^n$ formed from the columns of $P$, then $B$ is the $B$-matrix for the operator $T$ defined by $T(x) = Ax$.

Proof: See page 331.

In other words: If $A$ and $B$ are similar, then they both represent the same linear transformation.
Pictures
Finding a Basis that Yields a Diagonal Matrix Representation

Wouldn’t it be great if we could choose a basis for $V$ so that the linear operator $T : V \rightarrow V$ has a **diagonal** matrix representation?

This can’t always be done. But if it can, here is how to do it:

1. Choose any basis (the standard basis is easiest).
2. Find the matrix $A$ that represents $T$ with respect to that basis.
3. Try to diagonalize $A$ (using method of Section 5.3). If $A$ is diagonalizable, then

$$A = PDP^{-1}$$

for some diagonal matrix $D$ and some invertible matrix $P$.
4. The columns of $P$ are the desired basis $B$, and $D$ is the corresponding $B$-matrix representation of $T$. 
Example

Let \( A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \). Find a basis \( B \) for \( \mathbb{R}^2 \) such that \([T]_B\) is diagonal.
If there is a nonzero vector \( \mathbf{v} \in V \) and a scalar \( \lambda \) satisfying

\[ T(\mathbf{v}) = \lambda \mathbf{v}, \]

then

- \( \lambda \) is an eigenvalue of \( T \).
- \( \mathbf{v} \) is an eigenvector of \( T \) associated with \( \lambda \).

Key Fact: Every matrix representation of a linear operator has exactly the same eigenvalues as the operator has.
Discrete Dynamical Systems

- Eigenvalues and eigenvectors are a key to understanding the long-term behavior of a dynamical system described by a difference equation

\[ x_{k+1} = Ax_k. \]

- In such a system, the vector \( x_k \) represents the state of the system at time \( k \).

- The difference equation describes how that state will change at the next time point \( k + 1 \).
Example: Predator–Prey System

Let $O_k$ represent the number of owls in a region at time $k$.

Let $R_k$ represent the number of rats (in thousands) at time $k$.

$$x_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}.$$  

**Changes in Owl population:**

- Owls need rats for food.
- If there are no rats, half the owls will starve to death each time period.
- If rats are abundant, many more owls will survive. And with new owls being born, the owl population could actually increase.

$$O_{k+1} = (.5)O_k + (.4)R_k.$$
Changes in Rat population:

- Rats reproduce rapidly, but are killed by owls.
- The more owls there are, the more rats are killed.

\[ R_{k+1} = -pO_k + (1.1)R_k \]

where \( p \) is a positive parameter (called the predation parameter) to be specified.

Determine the long term behavior of this system when the predation parameter is \( p = 0.104 \).
Solution:

- The two equations $O_{k+1} = (.5)O_k + (.4)R_k$ and $R_{k+1} = -.104O_k + (1.1)R_k$
  are equivalent to the difference equation:

  $$x_{k+1} = Ax_k, \text{ where } A = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}$$

- The eigenvalues of this matrix are $\lambda_1 = 1.02$ and $\lambda_2 = .58$, with corresponding eigenvectors

  $$v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \text{ and } v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$ 

- Let $x_0$ represent the starting state. This can be rewritten as $x_0 = c_1v_1 + c_2v_2$. 

- Then,

  $$x_k = c_1 A^k v_1 + c_2 A^k v_2 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$$

  $$= c_1 (1.02)^k v_1 + c_2 (.58)^k v_2.$$ 

- For large $k$, the second term approaches zero, so we get

  $$x_k = c_1 (1.02)^k v_1.$$
Graphical Descriptions of Solutions

When the state is described by a vector in $\mathbb{R}^2$, we can examine the evolution of the system graphically.

We will look at several types of steady-state solutions:
- Attractors.
- Repellors.
- Saddle Points.
Attractors and Repellors

- If both eigenvalues are less than 1, then $x_k \to 0$ as $k \to 0$. In this case, the origin is an attractor.

- If both eigenvalues are bigger than 0, the origin is a repellor.
Saddle Points

If one eigenvalue is less than 1, and one is bigger than 1, the origin is a saddle point.