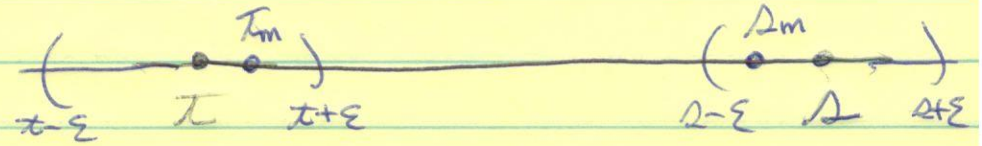
Notes on problem #1 and #4 from section 12

<u>Lemma</u> $(s_n), (t_n)$ bounded. $\forall n$, $s_n \le t_n$

Suppose $s_n \to s$ and $t_n \to t$. Then $s \le t$

<u>Pf</u> Suppose $t < s$



choose $\varepsilon < \frac{s-t}{2}$ $\therefore$ $s-t > 2\varepsilon$

choose N s/t $m > N \Rightarrow |s_m - s| < \varepsilon$, $|t_m - t| < \varepsilon$

Then, for $m > N$

$$s_m - t_m = \underbrace{s_m - s}_{>-\varepsilon} + \underbrace{s - t}_{>2\varepsilon} + \underbrace{t - t_m}_{>-\varepsilon} > 0 \quad \#$$

<u>def</u> for sequence (s_n) let

$$\underline{s_n} = \inf(s_n, s_{n+1}, \dots)$$
$$\overline{s_n} = \sup(s_n, s_{n+1}, \dots)$$

Then $\underline{s_n} \nearrow$ and $\overline{s_n} \searrow$ (make sure you understand why)

<u>def</u> $\liminf s_n = \lim \underline{s_n}$, $\limsup s_n = \lim \overline{s_n}$ — always exist, possibly $\pm\infty$.

Thm (problem 1 from section 12)

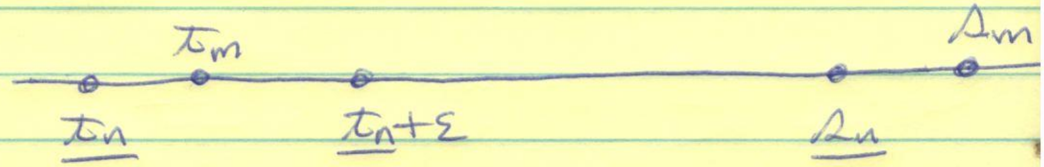
(s_n), (t_n) bounded

$\exists N_0$ s/t $n > N_0 \Longrightarrow s_n \le t_n$

Then (a) $\liminf s_n \le \liminf t_n$

(b) $\limsup s_n \le \limsup t_n$

Pf: (a) Let $n > N_0$ and suppose $\underline{t}_n < \underline{s}_n$



Let $\varepsilon < \underline{s}_n - \underline{t}_n$

Then $\exists m \ge n$ s/t $t_m < \underline{t}_n + \varepsilon$ (make sure yo understand wh

So, $t_m < \underline{t}_n + \varepsilon < \underline{s}_n \le s_m$ $\#$

since $\underline{s}_n$ ~~#~~ is lower b for $\{s_n, s_{n+1}, \dots s_m$

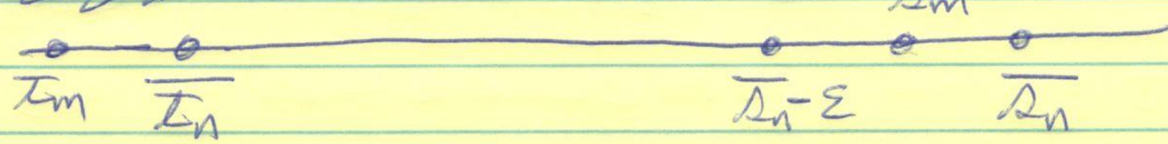
Thus, $\forall n > N_0$, $\underline{s}_n \le \underline{t}_n$,

which implies (by Thm above)

$$\underbrace{\lim \underline{s}_n}_{\lim\cancel{\sup}\inf s_n} \le \underbrace{\lim \underline{t}_n}_{\liminf t_n}$$

(b) likewise, if $\overline{t}_n < \overline{s}_n$ for some $n > N_0$

~~then~~



Then choose $\varepsilon < \frac{\overline{s}_n - \overline{t}_n}{2}$ and choose $m \geq n$

s/t $s_m > \overline{s}_n + \varepsilon$.

Thus,

$$s_m > \overline{s}_n - \varepsilon > \overline{t}_n \geq t_m \quad \#$$

since $\overline{t}_n$ is upper bound for $\{t_n, t_{n+1}, \dots, t_m, \dots\}$

$\therefore \ \forall n > N_0, \ \overline{s}_n \leq \overline{t}_n$,

which implies $\underbrace{\lim \overline{s}_n}_{\limsup s_n} \leq \underbrace{\lim \overline{t}_n}_{\limsup t_n}$

(4) $\limsup(s_n+t_n) \le \limsup(s_n) + \limsup(t_n)$

Pf: $\overline{s}_n = \sup(s_n, s_{n+1}, \dots) \ge s_n$

$\overline{t}_n = \sup(t_n, t_{n+1}, \dots) \ge t_n$

$\therefore\ s_n + t_n \le \overline{s}_n + \overline{t}_n$

make sure you understand why

Likewise

$$s_{n+1} + t_{n+1} \le \overline{s}_{n+1} + \overline{t}_{n+1} \le \overline{s}_n + \overline{t}_n$$

In general $m > n \Rightarrow s_m + t_m \le \overline{s}_n + \overline{t}_n$

$$\therefore\ \underbrace{\sup(s_n+t_n,\ s_{n+1}+t_{n+1}, \dots)}_{= \overline{s_n+t_n}} \le \overline{s}_n + \overline{t}_n$$

$$\therefore\ \underbrace{\lim \overline{s_n+t_n}}_{= \limsup(s_n+t_n)} \le \cancel{\overline{s}_n + \overline{t}_n}\ \underbrace{\lim(\overline{s}_n + \overline{t}_n)}_{= \lim \overline{s}_n + \lim \overline{t}_n}$$

$$= \limsup s_n + \limsup t_n$$

Likewise: $\liminf(s_n+t_n) \ge \liminf s_n + \liminf t_n$