Section 25

(9) show $\sum x^n$ converges unif on $[-a,a]$, $a<1$

Pf: let $M_k = a^k$

$\therefore \ |g_k(x)| \equiv |x^k| \le M_k, \quad \forall x \in [-a,a]$

$\therefore$ Since $\sum M_k = \frac{1}{1-a} < \infty$,

The Weierstrass M-test $\Longrightarrow$

$\sum x^k$ converges unif on $[-a,a]$ ✓

But convergence is <u>not</u> uniform on $(-1,1)$

Pf: $\sum x^k = \frac{1}{1-x}$ on $(-1,1)$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

$$\therefore \left| \frac{1}{1-x} - \sum_{k=0}^{n} x^k \right| = \left| \frac{x^{n+1}}{1-x} \right|$$

let $x_n = \left(\frac{1}{2}\right)^{\frac{1}{n+1}}$

$$\therefore \left| \frac{x^{n+1}}{1-x} \right| = \left| \frac{1/2}{1-x_n} \right| > \frac{1}{2}$$

$\therefore f_n \to f$ not unif on $(-1,1)$

let $f(x) = \frac{1}{1-x}$

$f_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

$\therefore$ find (x_n) s/t

$|f_n(x_n) - f(x_n)| > \varepsilon$

let $\varepsilon = \frac{1}{2}$

(10) (a) Show $\sum \frac{x^n}{1+x^n}$ converges on $[0,1)$

Pf: $\frac{x^n}{1+x^n} < x^n$ on $[0,1)$

$\therefore$ since $\sum x^n$ converges on $[0,1)$,

so does $\sum \frac{x^n}{1+x^n}$ by Comparison Test.

(b) $\left|\frac{x^n}{1+x^n}\right| < a^n$ for $x \in [0,a]$ if $0 \le a < 1$

$\therefore$ since $\sum a^n$ converges,

~~so does~~ $\sum \frac{x^n}{1+x^n}$ converges unif on $[0,a]$ by Weierstrass M-test.

(c) Show convergence <u>not</u> unif on $[0,1)$.

Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{1+x^k}$, $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{1+x^k}$, $x \in [0,1)$

$$\therefore \left|f(x) - f_n(x)\right| = \sum_{k=n+1}^{\infty} \frac{x^k}{1+x^k} > \sum_{k=n+1}^{\infty} \frac{x^k}{2}$$

$$= \frac{1}{2}\,\frac{x^{n+1}}{1-x} > \frac{x^{n+1}}{2}$$

$\therefore$ let $x_n = \left(\frac{1}{2}\right)^{\frac{1}{n+1}}$

$\therefore \forall n$, $\left|f(x_n) - f_n(x_n)\right| > \frac{1}{4}$ $\therefore$ $f_n \to f$ <u>not</u> unif on $[0,1)$.