MATH 4310 Introduction to Real Analysis I
Midterm Exam March 11, 2021

You can use any books, notes, or online information to help you with the problems on this exam, but you are required to work alone. You can use theorems proved in our textbook in your proofs, but not results proved elsewhere. (Make sure it is clear which result you are using.) Please hand in solutions that are neat and easy to follow. When you finish the exam, create a single pdf file of your work and email it directly to eric.culver@ucdenver.edu. (Eric Culver is the grader for this course.) The exam is due before midnight MST on Friday, March 12, 2021.

1. (20 points) Provide an example of the following, or prove it is impossible.
   (a) A sequence (a_n) where $\forall k,\ a_k > \liminf a_n$
   (b) A sequence (a_n) where $\forall k,\ a_k > \limsup a_n$
   (c) A sequence (a_n) where $\limsup a_n = \liminf a_n$
   (d) A sequence (a_n) where $\limsup a_n < \liminf a_n$

2. (20 points) Let (b_n) be a bounded sequence, and suppose $a_n \to a$.
   (a) Prove that $b_n(a_n - a) \to 0$.
   (b) Let $c_n = a_n b_n$. Does c_n converge? (Prove or find a counterexample.)
   (c) Prove (c_n) has a convergent subsequence.

3. (20 points) Let $f(x)$ and $g(x)$ be continuous at x_0, and suppose $f(x_0) > g(x_0)$. Prove $\exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow f(x) > g(x)$.

4. (20 points) Suppose $f(x)$ is continuous on $[0, 1]$ with $f(0) > 0$ and $f(1) = 0$. Prove $\exists x_0 \in (0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for $x \in [0, x_0)$. Is the statement still true if $f(x)$ is not continuous? (Prove or find a counterexample.)

5. (20 points) Prove or find a counterexample:
   (a) If $f(x)$ and $g(x)$ are uniformly continuous on (a, b) then $f(x)g(x)$ is uniformly continuous on (a, b).
   (b) If $f(x)$ and $g(x)$ are uniformly continuous on R then $f(x)g(x)$ is uniformly continuous on R.

6. (Bonus 10 points) Consider the following series:
   (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$
   (b) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \cdots$

   Prove that both series converge. Do they converge to the same limit?