MATH 4310 Introduction to Real Analysis I
Final Exam May 6, 2021

You can use any books, notes, or online information to help you with the problems on this exam, but you are required to work alone. You can use theorems proved in our textbook in your proofs, but not results proved elsewhere. (Make sure it is clear which result you are using.) Please hand in solutions that are neat and easy to follow. When you finish the exam, create a single pdf file of your work and email it directly to eric.culver@ucdenver.edu. (Eric Culver is the grader for this course.) The exam is due before midnight MST on Friday, May 7, 2021.

1. Let $f(x)$ be twice differentiable on $[0,1]$. Suppose $f(0)=1$, $f(1)=3$, and $f(c)=0$ for some $c \in (0,1)$.
   (a) (10 points) Prove $f'(\alpha)=0$ for some $\alpha \in (0,1)$.
   (b) (10 points) Prove $f'(\beta)=1$ for some $\beta \in (0,1)$.
   (c) (10 points) Prove $f''(\gamma)>4$, for some $\gamma \in (0,1)$.
2. (10 points) Let $g(x)$ be bounded on $[a,b]$, and let $f_n(x)=\frac{n}{n+1}g(x)$. Prove that $f_n \to g$ uniformly on $[a,b]$.
3. (15 points) Let
$$f(x)=\frac{1}{e^x-1}-\frac{1}{x}.$$
Use Taylor's Theorem (31.3) on e^x to find $\lim_{x\to 0} f(x)$. Justify your steps. Hint: Look at Example 9 in section 30 so you know the right answer.
4. Let $(x_n) \subset [0,1]$ be a sequence of distinct real numbers.
   (a) (10 points) Give an example where (x_n) has exactly k subsequential limits, $k=1,2$.
   (b) (10 points) Is it possible that (x_n) has no subsequential limits? Explain.
   (c) (5 points) Is it possible that the set of subsequential limits of (x_n) is countably infinite? If it's possible, give an example. Note: $(x_n)=$ rational numbers in $[0,1]$ is not an example since the set of limit points is the whole interval, which is uncountable.

5. Let $(r_1, r_2, \ldots)$ be the rational numbers in $[0, 1]$ (the does not matter for this problem), and let

$$f(x) = \sum_{\{i \,:\, r_i \leq x\}} 2^{-i}.$$

(a) (10 points) At which $x \in [0, 1]$ (if any) is f continuous?

(b) (10 points) Prove that f is integrable on $[0, 1]$.

(c) (Bonus) Find $\int_0^1 f$ in terms of $(r_1, r_2, \ldots)$.

Hint: If there were only five rational numbers then f would look like this.