MATH 4320, ADVANCED CALCULUS II
FINAL EXAM
DUE MONDAY, MAY 10, 2010

1. Using the power series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad -1 < x \leq 1,$$

prove that

$$\int_0^x \frac{du}{1+u} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad -1 < x \leq 1,$$

justifying all the important steps with theorems from the text book.

2. Let (X, d) and $(X, \hat{d})$ be metric spaces with $d \equiv \hat{d}$. (Recall, this means that

$$\forall x \in X, \forall \epsilon > 0, \ \exists \delta > 0, \text{ such that } \hat{N}_\delta(x) \subset N_\epsilon(x),$$

and vice-versa.) Suppose $A \subset X$ is open in (X, d). Prove that A is also open in $(X, \hat{d})$.

3. Let

$$f(x, y) = e^{x+2y-3xy}.$$

   (a) Find the first and second order approximations for $f(x, y)$ at $(0, 0)$.

   (b) Find and classify the critical points of $f(x, y)$.

   (c) Prove that

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - (1 + x + 2y + \frac{1}{2}(x - 2y)^2)}{x^2 + y^2}$$

   does not exist. (Bonus: "Fix" the problem so the limit exists and equals $\frac{1}{2}$.)