Math 3000
Injective, Surjective, and Bijective Functions

Define:

• A function

• An injective (one-to-one) function

• A surjective (onto) function

• A bijective (one-to-one and onto) function

A few words about notation:

To define a specific function one must define the domain, the codomain, and the rule of correspondence. In other words,

\[ f : A \to B \text{ defined by } f : x \mapsto f(x) \]

is the full definition of the function \( f \).

For example, the function that maps a real number to its square is defined as

\[ f : \mathbb{R} \to \mathbb{R} \text{ defined by } f : x \mapsto x^2. \]

Most people are more familiar with the following notation for this same function

\[ f : \mathbb{R} \to \mathbb{R} \text{ defined by } f(x) = x^2. \]

These definitions are equivalent!

”Line Tests”:

The “vertical line test” is a (simplistic) tool used to determine if a relation \( f : \mathbb{R} \to \mathbb{R} \) is function. The “horizontal line test” is a (simplistic) tool used to determine if a function \( f : \mathbb{R} \to \mathbb{R} \) is injective.

Examples:

• An example of a relation that is not a function

\[ f : \{0, 1, 2\} \to \{0, 1, 2, 3\} \text{ defined by } f = \{(0, 1), (1, 1), (0, 2), (2, 3)\} \]

• An example of an injective function

\[ f : \mathbb{R} \to \mathbb{R} \text{ defined by } f : x \mapsto x(x - 1)(x + 2) \]

• An example of a surjective function

\[ f : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\} \text{ defined by } f(x) = |x| \]

• An example of a bijective function

\[ f : \mathbb{R} \to \mathbb{R} \text{ defined by } f : x \mapsto x^3 \]
A few for you to try:
First decide if each relation is a function. Then decide if each function is injective, surjective, bijective, or none of these.

1. \( f : \mathbb{N} \to \mathbb{N} \) defined by \( f : n \mapsto n^2 \)

2. \( g : \mathbb{Z} \to \{0, 1\} \) defined by \( g(n) = n \mod 2 \)

3. Let \( I : L \to \{ x \in \mathbb{R} : x \geq 0 \} \) where \( L \) is the set of functions whose square is integrable on \((0, 1)\). Define
   \[
   I(f) = \int_{0}^{1} (f(x))^2 \, dx
   \]

4. \( h : \mathbb{R} \to \mathbb{R} \) defined by \( h : x \mapsto (2x + 1) \)

5. \( E : \mathbb{R} \to \mathbb{R} \) defined by \( E : x \mapsto e^x \)

6. \( J : \mathbb{R} \to \mathbb{R}^+ \) defined by \( J(x) = e^x \)

7. \( k : [-2\pi, 2\pi] \to \mathbb{R} \) defined by \( k(x) = \sin(x) \)

8. \( f : [-1, 1] \to [-1, 1] \) defined by \( f = \{(x, y) \in [-1, 1] \times [-1, 1] : x^2 + y^2 = 1\} \)

9. \( C : [-1, 1] \to [0, 1] \) defined by \( C = \{(x, y) \in [-1, 1] \times [0, 1] : x^2 + y^2 = 1\} \)

10. Let \( A = \{1, 2, 3, 4\} \). Define \( v : \mathcal{P}(A) \to \mathbb{N} \) by \( v : S \mapsto |S| \).

11. Let \( A = [0, 1] \subset \mathbb{R} \). Define \( \chi_A : \mathbb{R} \to \{0, 1\} \) \( F \) such that
   \[
   \chi_A(x) = \begin{cases} 
   1, & x \in A \\
   0, & x \notin A
   \end{cases}
   \]

12. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T : \vec{x} \mapsto A\vec{x} \) where
   \[
   A = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
   \]
   (If you haven’t had any linear algebra then don’t worry about this problem)

13. (True / False) Exactly one of problems 1 - 12 is not a function. (Hint: The answer is “true”) \(^1\)

14. **Proof**: Prove that \( h \) defined in problem 4 is a bijection.

\(^1\)(fi),(fs),(fs),(fb),(fi),(fb),(fn),(nf),(fs),(fn),(fs),(fb),(t)