LARGE RAINBOW MATCHINGS IN LARGE GRAPHS

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Abstract. A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. The *color degree* of a vertex v is the number of different colors on edges incident to v. We show that if n is large enough (namely, $n \geq 4.25k^2$), then each n-vertex graph G with minimum color degree at least k contains a rainbow matching of size at least k.

1. Introduction

We consider edge-colored *simple* graphs. A subgraph H of such graph G is *monochromatic* if every edge of H is colored with the same color, and *rainbow* if no two edges of H have the same color. In the literature, a rainbow subgraph is also called totally multicolored, polychromatic, and heterochromatic.

In anti-Ramsey theory, for given n and a graph H, the objective is to find the largest integer k such that there is a coloring of K_n using exactly k colors that contains no rainbow copy of H. The anti-Ramsey numbers and their relation to the Turán numbers were first discussed by Erdős, Simonovits, and Sós [4]. Solutions to the anti-Ramsey problem are known for trees [9], matchings [6], and complete graphs [15], [1] (see [7] for a more complete survey). Rödl and Tuza proved there exist graphs G with arbitrarily large girth such that every proper edge coloring of G contains a rainbow cycle [14]. Erdős and Tuza asked for which graphs G there is a d such that there is a rainbow copy of G in any edge-coloring of K_n with exactly $|E(G)|$ colors such that for every vertex $v \in V(K_n)$ and every color α, v is the center of a monochromatic star with d edges and color α. They found positive results for trees, forests, C_4, and K_3 and found negative results for several infinite families of graphs [5].

For $v \in V(G)$ and a coloring ϕ on $E(G)$, $\hat{d}(v)$ is the number of distinct colors on the edges incident to v. This is called the *color degree of* v. The smallest color degree of all vertices in G is the *minimum color degree of* G, or $\hat{\delta}(G,\phi)$. The largest color degree is $\hat{\Delta}(G,\phi)$.

Local anti-Ramsey theory seeks to find the maximum k such that there exists a coloring ϕ of K_n that contains no rainbow copy of H and $\hat{\delta}(K_n,\phi) \geq k$.

The topic of rainbow matchings has been well studied, along with a more general topic of rainbow subgraphs (see [10] for a survey). Let $r(G,\phi)$ be the size of a largest rainbow matching in a graph G with edge coloring ϕ. In 2008, Wang and Li [17] showed that $r(G,\phi) \geq$

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$\left\lceil \frac{5\hat{\delta}(G,\phi)-3}{12} \right\rceil$ for every graph G and conjectured that if $\hat{\delta}(G,\phi) \geq k \geq 4$ then $r(G,\phi) \geq \left\lceil \frac{k}{2} \right\rceil$. The conjecture is known to be tight for properly colored complete graphs. LeSaulnier *et al.* [13] proved that $r(G,\phi) \geq \left\lfloor \frac{k}{2} \right\rfloor$ for general graphs, and gave several conditions sufficient for a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$. In [11], the conjecture was proved in full. The only known extremal examples for the bound have at most $k+2$ vertices.

Wang [16] proved that every properly edge-colored graph (G,ϕ) with $\delta(G,\phi) = k$ and $|V(G)| \geq 1.6k$ has a rainbow matching of size at least $3k/5$ and that every such triangle-free graph has a rainbow matching of size at least $\lfloor 2k/3 \rfloor$. He also asked if there is a function, $f(k)$, such that for every graph G and proper edge coloring ϕ of G with $\hat{\delta}(G,\phi) \geq k$ and $|V(G)| \geq f(k)$, we have $r(G,\phi) \geq k$. The bound on $r(G,\phi)$ is sharp for any properly k-edge-colored k-regular graph.

Diemunsch et al. [2] answered the question in the positive and proved that $f(k) \leq 6.5k$. Shortly thereafter, Lo [12] improved the bound to $f(k) \leq 4.5k$, and finally Diemunsch et al. [3] combined the two manuscripts and improved the bound to $f(k) \leq \frac{98}{23}k$. The largest matching in a graph with n vertices contains at most $n/2$ edges, so $f(k) \geq 2k$. By considering the relationship of Latin squares to edge-colored $K_{n,n}$, the lower bound can be improved to $f(k) \geq 2k+1$ for even k. This is the best known lower bound on the number of vertices required for both the properly edge-colored and general cases.

In this note we prove an analogous result for arbitrary edge colorings of graphs.

Theorem 1. *Let G be an n-vertex graph and ϕ be an edge-coloring of G with $n > 4.25\hat{\delta}^2(G,\phi)$. Then (G,ϕ) contains a rainbow matching with at least $\hat{\delta}(G,\phi)$ edges.*

Our result gives a significantly weaker bound on the order of G than the bounds in [3] but for a significantly wider class of edge-colorings.

Several ideas in the proof came from Diemunsch et al.'s paper [2]. The full proof is presented in the next section.

2. Proof of the Theorem

Let (G,ϕ) be a counter-example to our theorem with the fewest edges in G. For brevity, let $k := \hat{\delta}(G,\phi)$. Since (G,ϕ) is a counter-example, $n := |V(G)| > 4.25k^2$. The theorem is trivial for $k=1$, and it is easy to see that if $\hat{\delta}(G) = 2$ and (G,ϕ) does not have a rainbow matching of size 2, then $|V(G)| \leq 4$. Therefore $k \geq 3$.

Claim 1. *Each color class in (G,ϕ) forms a star forest.*

Proof. Suppose that the edges of color α do not form a star forest. Then there exists an edge uv of color α such that an edge ux and an edge vy also are colored with α (possibly, $x=y$). Then the graph $G' = G - uv$ has fewer edges than G, but $\hat{\delta}(G',\phi) = k$. By the minimality of G, $r(G',\phi) \geq k$. But then $r(G,\phi) \geq k$, a contradiction. □

We will denote the set of maximal monochromatic stars of size at least 2 by $\mathcal{S}$. Let $E_0 \subseteq E(G)$ be the set of edges not incident to another edge of the same color, i.e. the maximal monochromatic stars of size 1.

Claim 2. *For every edge $v_1v_2 \in E(G)$, there is an $i \in \{1,2\}$, such that $\hat{d}(v_i) = k$ and v_1v_2 is the only edge of its color at v_i.*

Proof. Otherwise, we can delete the edge and consider the smaller graph. □

Claim 3. *All leaves $v \in V(G)$ of stars in $\mathcal{S}$ have $\hat{d}(v) = k$.*

Proof. This follows immediately from Claim 2. □

For the sake of exposition, we will now direct all edges of our graph G. With an abuse of notation, we will still call the resulting directed graph G. In every star in $\mathcal{S}$, we will direct the edges away from the center. All edges in E_0 will be directed in a way such that the sequence of color outdegrees in G, $\hat{d}_0^+ \geq \hat{d}_1^+ \geq \ldots \geq \hat{d}_n^+$ is lexicographically maximized. Note that by Claim 1,

(I) the set of edges towards v forms a rainbow star, and so $d^-(v) \leq \hat{d}(v)$.

Let C be the set of vertices with non-zero outdegree and $L := V \setminus C$. Let $\mathcal{S}^* \subseteq \mathcal{S}$ be the set of maximal monochromatic stars with at least two vertices in L, and let $E_0^* \subseteq E_0 \cup \mathcal{S}$ be the set of maximal monochromatic stars with exactly one vertex in L. For a color α, let $E_H[\alpha]$ be the set of edges colored α in a graph H. If there is no confusion, we will denote it by $E[\alpha]$.

Claim 4. *For every $v \in V(G)$ with $\hat{d}(v) \geq k+1$, $d^-(v) = 0$. In particular, $d^-(v) \leq k$ for every $v \in V(G)$. Moreover, for all $w \in L$, $d(w) = k$.*

Proof. Suppose that $\hat{d}(v) \geq k+1$, and let $w_i v$ be the edges directed towards v. By Claim 2 and (I), $\hat{d}(w_i) = k$ and $w_i v \in E_0$ for all i. Then $d^+(w_i) \leq \hat{d}(w_i) = k$. Reversing all edges $w_i v$ would increase the color outdegree of v with a final value larger than k while decreasing the color outdegree of each w_i, which was at most k. Hence the sequence of color outdegrees would lexicographically increase, a contradiction to the choice of the orientation of G.

By the definition of L, if $w \in L$, then $d^+(w) = 0$. So in this case by the previous paragraph, $k \leq \hat{d}(w) \leq d^-(w) \leq k$, which proves the second statement. □

Claim 5. *No color class in (G, ϕ) has more than $2k-2$ components.*

Proof. Otherwise, remove the edges of a color class α with at least $2k-1$ components, and use induction to find a rainbow matching with $k-1$ edges in the remaining graph. This matching can be incident to at most $2k-2$ of the components of α, so there is at least one component of α not incident to the matching, and we can pick any edge in this component to extend the matching to a rainbow matching on k edges. □

We consider three cases. If $n > 4.25k^2$, then at least one of the three cases will apply. The first two cases will use greedy algorithms.

Case 1. $|\mathcal{S}^*| + \frac{1}{2}|E_0^*| \geq 2.5k^2$.

For every $S \in \mathcal{S}^*$, assign a weight of $w_1(e) = 1/|S \cap L|$ to each of the edges of S incident to L. Assign a weight of $w_1(e) = 1/2$ to every edge $e \in E_0^*$. Edges in $G[C]$ receive zero weight. Let $G_0 \subset G$ be the subgraph of edges with positive weight. For every set of edges $E' \subseteq E(G)$, let $w_1(E')$ be the sum of the weights of the edges in E'. For every vertex, let $w_1(v) = \sum_{a \in N^+(v)} w_1(va) + \sum_{b \in N^-(v)} w_1(bv)$. Note that G_0 is bipartite with partite sets C and L and that $w_1(e) \leq 1/2$ for every edge $e \in E(G)$. Furthermore,

$$\frac{1}{2} \sum_{v \in V(G)} w_1(v) = \sum_{e \in E(G)} w_1(e) = |\mathcal{S}^*| + \frac{1}{2}|E_0^*| \geq 2.5k^2.$$

Claim 6. *For every $v \in V(G)$, $w_1(v) \leq 2(k-1)$.*

Proof. Suppose (G, ϕ) has a vertex v with $w_1(v) > 2(k-1)$. Let $G' = G - v$. Then $\hat{\delta}(G', \phi) \geq k-1$ and $|V(G')| = n-1 > 4.25(k-1)^2$. By the minimality of (G, ϕ), the colored graph (G', ϕ) has a rainbow matching M of size $k-1$. At most $k-1$ of the stars v is incident to have colors appearing in M; each of them contributes a weight of at most 1 to $w_1(v)$. As $w_1(v) > 2(k-1)$, there are at least $2k-1$ edges incident to v with colors not appearing in M. At least one of these edges is not incident to M. Thus (G, ϕ) has a rainbow matching of size k, a contradiction. □

We propose an algorithm that will find a rainbow matching of size at least k. For $i = 1, 2, \ldots$, at Step i:

(0) If G_{i-1} has no edges or $i-1 = k$, then stop.
(1) If a vertex v of maximum weight has $w_1(v) > 2(k-i)$ in G_{i-1}, then set $G_i = G_{i-1} - v$ and go to Step $i+1$.
(2) If the largest color class $E[\alpha]$ of G_{i-1} has at least $2(k-i)+1$ components, then set $G_i = G_{i-1} - E[\alpha]$ and go to Step $i+1$.
(3) If $\max w_1(v) \leq 2(k-i)$ for all $v \in V(G_{i-1})$ and every color class has at most $2(k-i)$ components, then set $G_i = G_{i-1} - x - y - E[\phi(xy)]$ for some edge $xy \in E(G_{i-1})$.

We will refer to these as options (1), (2), and (3) for Step i. We call the difference in the total weight of the remaining edges between G_{i-1} and G_i the *weight of Step i* or $W_1(i)$. When both options (1) and (2) are possible, we will pick option (1). Option (3) is only used when neither of options (1) and (2) are possible.

Let G_r be the last graph created by the algorithm, i.e., $r = k$ or G_r has no edges. We will first show by reversed induction on i that

$$G_i \text{ has a rainbow matching of size at least } r - i. \tag{II}$$

This trivially holds for $i = r$. Suppose (II) holds for some i, and M_i is a rainbow matching of size $r-i$ in G_i. If we used Option (1) in Step i, then there is some edge $e \in E(G_{i-1})$ incident with v that is not incident with M_i and whose color does not appear on the edges of M_i, similarly to the proof of Claim 6. If we used Option (2) in Step i, then there is some component of $E_{G_{i-1}}[\alpha]$ that is not incident with M_i, and we let e be an edge of that component. If we used Option (3) in Step i, then let $e = xy$. In each scenario, $M_i + e$ is a rainbow matching of size $r-i+1$ in G_{i-1}. This proves the induction step and thus (II). So, if $r = k$, then we are done.

Assume $r < k$. Then the algorithm stopped because $E(G_{r+1}) = \emptyset$. This means that

$$\sum_{i=1}^{r} W_1(i) = \sum_{e \in E(G)} w_1(e) \geq 2.5k^2. \tag{1}$$

We will show that this is not the case. Suppose that at Step i, we perform Option (3). By the bipartite nature of G_0, we may assume that $y \in L$. By Claim 4, $w_1(y) - w_1(xy) \leq \frac{k-1}{2}$. Because Options (1) and (2) were not performed at Step i, $w_1(x) + w_1(E_{G_{i-1}}[\phi(xy)]) \leq 4(k-i)$. Therefore the weight of Step i is at most $\frac{k-1}{2} + 4(k-i) < 4.5k - 4i$.

By Claims 5 and 6, Option (3) is performed at Step 1. If $W_1 < 4.5k - 4i$ for all i, then $\sum_{i=1}^{r} W_1(i) < \sum_{i=1}^{r} 4.5k - 4i = 4.5kr - 2r(r+1) \leq 2.5k(k-1)$, a contradiction to (1). Let i be the first time that $W_1(i) \geq 4.5k - 4i$, and $j < i$ be the last time Option (3) is performed prior to i. By the choice of i, $W_1(a) < 4.5k - 4a$ when $a \leq j$. Because Option (1) and (2) were not chosen at Step j, $W_1(i') \leq 2(k-j)$ for each Step i' such that $i' > j$ and Option (1) or (2) is used. Note that by choice of i and j, this bound applies for all steps between $j+1$ and i. Furthermore, by the choice of i, $2(k-j) > 4k - 4i' - 1$ for $i' > i$. It follows that $W_1(b) \leq 2(k-j)$ for each $b > j$, and so

$$\sum_{a=1}^{r} W_1(a) \leq \sum_{a=1}^{j}(4.5k - 4a) + 2(k-j)(r-j) \leq 4.5kj - 2j(j+1) + 2(k-j)(k-1-j)$$

$$= k(0.5j + 2k - 2) < 2.5k^2,$$

a contradiction to (1).

Case 2. $|C| \geq 1.75k^2$.

We will use a different weighting: For every vertex $v \in C$ and outgoing edge vw, if $vw \in E_0$, we let $w_2(vw) = 1/\hat{d}^+(v)$, where $\hat{d}^+(v)$ is the color outdegree of v, and if vw is in a star $S \in \mathcal{S}$, then we let $w_2(vw) = 1/(\hat{d}^+(v)\|S\|)$. For a vertex $v \in V(G)$, let $w^+(v)$ and $w^-(v)$ denote the accumulated weights of the outgoing and incoming edges, respectively, and $w_2(v) = w^+(v) + w^-(v)$. By definition, $w^+(v) = 1$ for each $v \in C$. Then

$$\sum_{e \in E(G)} w_2(e) = \sum_{v \in V(G)} w^-(v) = \sum_{v \in V(G)} w^+(v) = |C| \geq 1.75k^2.$$

Claim 7. *Let uv be a directed edge in G and e an edge incident to u that is not uv. Then $w_2(e) \leq 1/2$.*

Proof. The result is easy if e is in a monochromatic star with size at least 2, so assume $e \in E_0$. If e is directed away from u, then $\hat{d}^+(u) \geq 2$ and the claim follows. Suppose now that e is directed towards u, say $e = wu$, and $w_2(e) = 1$. Then $d^+(w) = 1$, and reversing e we obtain the orientation of G where the outdegree of w decreases from 1 to 0, and the outdegree of u increases from $d^+(u) \geq 1$ to $d^+(u) + 1$. The new orientation has a lexicographically larger outdegree sequence, which is a contradiction. □

Claim 8. *For every color α, we have $w_2(E[\alpha]) \leq 1.5(k-1)$.*

Proof. Otherwise, remove the edges of a color class $E[\alpha]$ with $w_2(E[\alpha]) > 1.5(k-1)$, and use induction to find a rainbow matching with $k-1$ edges in the remaining graph. For every directed edge $vw \in M$, v can be incident to a component of $E[\alpha]$ of weight at most $1/2$, and w can be incident to a component of $E[\alpha]$ of weight at most 1, so there is at least one component of $E[\alpha]$ not incident to the vertices of M, and we can pick any edge in this component to extend M to a rainbow matching of k edges. □

We will use the following greedy algorithm: Start from G, and at Step i, choose a color α with the minimum value $w_2(E[\alpha]) > 0$, and pick any edge $e_i \in E[\alpha]$ of that color, and put it in the matching M, and then delete all edges of G that are either incident to e_i or have the same color as e_i. Without loss of generality, we may assume that edge e_i has color i. If we can repeat the process k times, we have found our desired rainbow matching, so assume that we run out of edges after $r < k$ steps, and call the matching we receive M. Let $h \leq k-1$ be the first step after which only edges with colors present in M remain in G_h. Let β be a color not used in M such that the last edges in $E[\beta]$ were deleted at Step h. Such β exists, since G has at least k colors on its edges.

By Claim 7, one step can reduce the weight $w_2(E[\beta])$ by at most 1.5. It follows that $w_2(E[\beta])$ at Step $i \leq h$ is at most $1.5(h-i+1)$. As we always pick the color with the smallest weight, the color $i \leq h$ also had weight at most $1.5(h-i+1)$ when we deleted it in Step i. Every color $i > h$ which appears in M has weight at most $1.5(k-1)$ by Claim 8. Thus, the total weight of colors in M at the moment of their deletion is at most $1.5\sum_{i=1}^{h} i + 1.5(k-1)(k-1-h)$.

Claim 9. *For each vertex v, $w_2(v) \leq (k+1)/2$.*

Proof. Suppose there are two edges, e_1 and e_2, incident with v such that $w_2(e_1) = w_2(e_2) = 1$. By Claim 7, both edges are directed towards v and are in E_0. Consider the orientation of G where the directions of e_1 and e_2 have been reversed. Then the outdegree of v has been increased by 2, while the outdegree of two other vertices changed from 1 to 0. This creates a lexicographically larger outdegree sequence, a contradiction.

By Claim 4, if $\hat{d}(v) \geq k+1$, then $w_2(v) = 1$. If $\hat{d}(v) = k$, then by the above $w_2(v) \leq 1 + (k-1)/2$. $\square$

If an edge e has a color β not in M or has color $i \leq r$ but was deleted at Step j with $j < i$, then e is incident to the edges $\{e_1, \dots e_h\}$. By Claim 9, the total weight of such edges is at most $2h(k+1)/2$.

However, this is a contradiction because it implies

$$|C| \leq h(k+1) + \frac{3}{2}\sum_{i=1}^{h} i + \frac{3}{2}(k-1)(k-1-h) = \frac{3k^2}{2} - 3k + \frac{3}{2} + \frac{3h^2}{4} - \frac{hk}{2} + \frac{13h}{4} < 1.75k^2.$$

Case 3. $|L| > |\mathcal{S}^*| + 0.5|E_0^*|$.

We will introduce yet another weighting, now of vertices in L. For every star $S \in \mathcal{S}^*$, add a weight of $1/|L \cap V(S)|$ to every vertex in $L \cap V(S)$. For every edge $e \in E_0^*$, add a weight of $1/2$ to the vertex in $L \cap e$. For every $v \in L$, let $w_3(v)$ be the resulting weight of v.

Since $\sum_{v \in L} w_3(v) = |\mathcal{S}^*| + 0.5|E_0^*| < |L|$, there is a vertex $v \in L$ with $w_3(v) < 1$. Let $S_1, S_2, \dots S_k$ be the k maximal monochromatic stars incident to v ordered so that $|L \cap V(S_i)| \leq |L \cap V(S_j)|$ for $1 \leq i < j \leq k$ (where $S_1 \in E_0$ is allowed). Since $v \notin C$, all these stars have different centers. Now we greedily construct a rainbow matching M of size k, using one edge from each S_i as follows. Start from including into M the edge in S_1 containing v. Assume that for $\ell \geq 2$, we have picked a matching M containing one edge from each S_i for $1 \leq i \leq \ell - 1$. Since $w_3(v) < 1$, we know that $|L \cap V(S_\ell)| > \ell$ for $\ell \geq 2$. As every edge in M contains at most one vertex in L, we can extend the matching with an edge from the center of S_ℓ to an unused vertex in $L \cap V(S_\ell)$.

To finish the proof, let us check that at least one of the above cases holds. If Cases 1 and 2 do not hold, then $|C| < 1.75k^2$ and $|\mathcal{S}^*| + 0.5|E_0^*| < 2.5k^2$. Then, since $n > 4.25k^2$, $|L| > 4.25k^2 - 1.75k^2 = 2.5k^2$, and we have Case 3. $\square$

References

[1] H. Chen, X. Li, and J. Tu, Complete solution for the rainbow numbers of matchings. *Discrete Math.* 309 (2009), 3370–3380.

[2] J. Diemunsch, M. Ferrara, C. Moffatt, F. Pfender, and P. Wenger, Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs. *arXiv:1108.2521*

[3] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, and P. Wenger, Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs. *(Submitted)*

[4] P. Erdős, M. Simonovits, and V.T. Sós, Anti-Ramsey theorems. *Colloq. Math. Soc. Janos. Bolyai, Vol. 10, Infinite and Finite Sets, Keszthely (Hungary)* (1973), 657–665.

[5] P. Erdős and Zs. Tuza, Rainbow subgraphs in edge-colorings of complete graphs. *Quo vadis, graph theory?*, 81–88, *Ann. Discrete Math.*, 55, *North-Holland, Amsterdam* (1993).

[6] S. Fujita, A. Kaneko, I. Schiermeyer, and K. Suzuki, A rainbow k-matching in the complete graph with r colors. *Electron. J. Combin.* 16 (2009), #R51.

[7] S. Fujita, C.Magnant, and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs and Combin.* 26 (2010), 1-30.

[8] A. Gyarfas, J. Lehel, R. H. Schelp, and Zs. Tuza, Ramsey Numbers for Local Colorings. *Graphs and Combin.* 3 (1987), 267–277.

[9] T. Jiang and D. West, Edge-colorings of complete graphs that avoid polychromatic trees. *Discrete Math.* 274 (2004), 137–145.

[10] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs–a survey. *Graphs Combin.* 24 (2008), 237–263.

[11] A. Kostochka and M. Yancey, Large Rainbow Matchings in Edge-Coloured Graphs. to appear in *Combinatorics, Probability and Computing.*

[12] A. Lo, A note on rainbow matchings in properly edge-coloured graphs. *arXiv:1108.5273*

[13] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, Rainbow Matching in Edge-Colored Graphs. *Electron. J. Combin.* 17 (2010), Paper #N26.

[14] V. Rödl and Zs. Tuza, Rainbow subgraphs in properly edge-colored graphs. *Random Structures Algorithms* 3 (1992), 175 – 182.

[15] I. Schiermeyer, Rainbow numbers for matchings and complete graphs. *Discrete Math.* 286 (2004), 157 – 162.

[16] G. Wang, Rainbow matchings in properly edge colored graphs. *Electron. J. Combin.* 18 (2011), Paper #P162.

[17] G. Wang and H. Li, Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.* 15 (2008), Paper #R138.

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