THE 1-2-3 CONJECTURE FOR HYPERGRAPHS

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Abstract. A weighting of the edges of a hypergraph is called vertex-coloring if the weighted degrees of the vertices yield a proper coloring of the graph, i.e., every edge contains at least two vertices with different weighted degrees. In this paper we show that such a weighting is possible from the weight set $\{1, 2, \ldots, r+1\}$ for all linear hypergraphs with maximum edge size $r \geq 4$ and not containing isolated edges. The number $r+1$ is best possible for this statement.

Further, the weight set $\{1, 2, 3, 4, 5\}$ is sufficient for all hypergraphs with maximum edge size 3, as well as $\{1, 2, \ldots, 5r-5\}$ for all hypergraphs with maximum edge size $r \geq 4$, up to some trivial exceptions.

1. Introduction and Notation

Regular graphs have been studied in a lot of contexts, and have many properties not shared by other graphs. One may ask what is on the other side of the spectrum, and look for graphs which are as irregular as possible. But what is irregular? It is an easy observation that every graph with at least two vertices contains a pair of vertices of equal degree, so one can not hope for graphs which are *totally irregular* in the sense that all vertices have pairwise different degrees. This changes if one considers multigraphs. In fact, by multiplying some edges, one can make every graph totally irregular, as long as the original graph does not contain an isolated edge or two isolated vertices. This observation led to the definition of the *irregularity strength* of a graph in [3], the minimum maximum multiplicity one has to use on a given graph.

Later, Karoński, Łuczak and Thomason [6] asked a similar question inspired by this concept. What if we do not require *all* vertices to have pairwise different degrees, but only require this difference for *adjacent* vertices? In other words, we want to require that the degrees yield

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a proper vertex coloring. This question led to the so called 1-2-3-Conjecture, stated here in the obviously analogous form using edge weights instead of multiplicities.

Conjecture 1. *For every graph G without isolated edges, there is a weighting $\rho : E(G) \to \{1,2,3\}$, such that the induced vertex weights $\rho(v) := \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

The 1-2-3-Conjecture is known to be true for several classes of graphs, the best known result for general graphs is by the authors of the current article [5].

Theorem 2. *For every graph G without isolated edges, there is a weighting $\rho : E(G) \to \{1,2,3,4,5\}$, such that the induced vertex weights $\rho(v) := \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

Shortly thereafter, a total version of the 1-2-3-Conjecture, adaptly called the 1-2-Conjecture, was formulated by Przybyło and Wozniak [7].

Conjecture 3. *For every graph G, there is a weighting $\rho : E(G) \cup V(G) \to \{1,2\}$, such that the induced total vertex weights $w(v) := \rho(v) + \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

Kalkowski in [4] came close to settling this conjecture.

Theorem 4. *For every graph G, there are weightings $\rho : E(G) \to \{1,2,3\}$ and $\rho' : V(G) \to \{1,2\}$ such that the induced total vertex weights $w(v) := \rho'(v) + \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

One natural and promising approach for both conjectures is the use of Alon's Combinatorial Nullstellensatz (see [1]). In its most straightforward application, it would prove list versions of the conjectures if successful, leading to the following stronger conjectures, first stated by Bartnicky, Grytczuk and Niwczyk, and by Przybyło and Wozniak, and Wong and Zhu, respectively.

Conjecture 5. [2] *For every graph G without isolated edges, and for every assignment of lists of size 3 to the edges of G, there exists a weighting $\rho : E(G) \to \mathbb{R}$ from the lists, such that the induced vertex weights $\rho(v) := \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

Conjecture 6. [8],[10] *For every graph G, and for every assignment of lists of size 2 to the vertices and edges of G, there exists a weighting $\rho : V(G) \cup E(G) \to \mathbb{R}$ from the lists, such that the induced total vertex weights $w(v) := \rho(v) + \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

Conjecture 5 is open even if we allow larger lists of some fixed size k. For Conjecture 6, the best result due to Zhu and Wong in [11] generalizes Theorem 4.

Theorem 7. *For every graph G, and for every assignment of lists of size 2 to the vertices and of size 3 to the edges of G, there exists a weighting $\rho : V(G) \cup E(G) \to \mathbb{R}$ from the lists, such that the induced total vertex weights $w(v) := \rho(v) + \sum_{u \in N(v)} \rho(uv)$ properly color $V(G)$.*

All these questions also make sense for hypergraphs. Note that it is easy to construct totally irregular hypergraphs, so the irregularity strength of a hypergraph may actually be 1 in certain cases. In this manuscript, we want to first consider Conjecture 1 for hypergraphs. In Section 4, we will turn to Conjecture 6, and extend Theorem 7 to hypergraphs.

To start, we have to decide what we mean by a proper vertex coloring of a hypergraph as there are differing notions. We will consider the weakest notion and call a hypergraph properly colored if it does not contain a monochromatic edge, i.e. an edge containing only vertices from one color class.

Next, we have to classify all hypergraphs which do not allow a vertex coloring edge weighting at all. What is the analogon of an isolated edge in the graph case? We will call a set of vertices of any cardinality *twin set* if the vertices in the set are contained in the exact same set of edges. With this notion, it is easy to verify that the only obstacle is an edge consisting of a twin set. In the absence of such edges, a vertex coloring edge weighting with integer weights is always possible. So we will ask for such graphs, what is the minimum maximum edge weight we have to use?

Going from graphs to hypergraphs, one discovers several important classes of hypergraphs invisible in the graph case, we will consider three special classes. A hypergraph is called *k-uniform* if all its edges have size k. If any two edges in a hypergraph intersect in at most one vertex, we call the hypergraph *linear* (this property is also called *simple* in other places). Note that graphs are exactly the 2-uniform linear hypergraphs. A hypergraph is called *bipartite* if it allows a proper 2-coloring. In general, for the ease of notation we allow multiple edges in our hypergraphs.

Starting with a hypergraph H with vertex set $V(H)$ and edge set $E(H)$ and a vertex $v \in V(H)$, we define the hypergraph $H - v$ (the deletion of v) as the hypergraph with

$$V(H-v) = V(H) \setminus \{v\},$$
$$E(H-v) = \{e \setminus \{v\} : e \in E(H)\}.$$

In other words, we delete v from every edge, and we keep the resulting edges.

On the other hand, for $X \subseteq V(H)$, we may consider the *induced* hypergraph $H[X]$ with

$$V(H[X]) = X,$$
$$E(H[X]) = \{e \in E(H) : e \subseteq X\}.$$

This time, we only allow edges completely contained in the smaller vertex set.

In the next section we provide some hypergraphs giving lower bounds for a number replacing the 3 in the 1-2-3-Conjecture. In particular, we show that the statement of the 1-2-3-Conjecture can not be true for general hypergraphs. In fact, it would fail even for linear bipartite hypergraphs.

In the third section, we present the main results of the paper—upper bounds for the question. We will get a bound for linear hypergraphs depending linearly on the size r of the largest edge, which matches our lower bound as long as $r \geq 4$. For non-linear hypergraphs our bound is weaker, but still linear.

2. Lower Bounds

Let F be any hypergraph with vertex set $V(F)$ and edge set $E(F)$. From this, we create another hypergraph H as follows. Let $V(H)$ consist of the vertex-edge incidences in F, i.e., pairs (v, e) where $v \in V(F)$, $e \in E(F)$ and $v \in e$. Let

$$E_1(H) = \bigcup_{v \in V(F)} \{(v, e) \in V(H) : e \in E(F)\},$$
$$E_2(H) = \bigcup_{e \in E(F)} \{(v, e) \in V(H) : v \in V(F)\},$$
$$E(H) = E_1(H) \cup E_2(H).$$

With this construction, H is linear and 2-regular, the largest edge in $E_1(H)$ has size equal to the maximum degree in F, and the largest edge in $E_2(H)$ has size equal to the largest edge in F. As H does not contain any odd cycles, H is bipartite: start with any vertex $(v, e) \in V(H)$, and partition the remaining vertices in the same component by the parity of their distance to (v, e). Further, if F has chromatic number

$\chi(F)$, then $V(H)$ can not be colored properly by the induced vertex weights from a weighting $\rho : E(H) \to \{1, 2, \dots, \chi(F) - 1\}$. We can see this as follows. Suppose there was such a weighting. For $h_1 = \{(v, e_1), (v, e_2), \dots, (v, e_r)\} \in E_1(H)$, we write shorter $\rho(v) = \rho(h_1)$, and similarly for $h_2 = \{(v_1, e), (v_2, e), \dots, (v_s, e)\} \in E_2(H)$, we write shorter $\rho(e) = \rho(h_2)$. Then the induced vertex weight of a vertex is

$$\rho((v, e)) = \rho(v) + \rho(e).$$

Thus, the edge h_2 from above is monochromatic if and only if all the $\rho(v_i)$ are the same. If ρ induces a proper coloring on the vertices of H, then ρ has to be a proper coloring on the vertices of F, a contradiction.

This construction gives us several lower bounds. If we start with a complete graph on $r + 1$ vertices, we obtain a hypergraph with maximum edge size r, which needs a weight set of at least $\{1, 2, \dots, r + 1\}$ on the edges to properly color the vertices. Starting with any other r-regular graph with chromatic number r, we obtain a hypergraph with maximum edge size r, which needs a weight set of at least $\{1, 2, \dots, r\}$ on the edges to properly color the vertices.

If we start with the Fano plane (or any other 3-regular 3-uniform non-bipartite hypergraph), we obtain a 2-regular 3-uniform hypergraph, which needs a weight set of at least $\{1, 2, 3\}$ on the edges to properly color the vertices.

On the other hand, this construction cannot give us non-trivial examples for r-unform hypergraphs with $r \geq 4$. Thomassen shows in [9] that all r-uniform r-regular hypergraphs are bipartite for $r \geq 4$, leaving open the possibility of a vertex coloring edge weighting from the set $\{1, 2\}$.

3. Upper Bounds

For $r \geq 4$ and linear hypergraphs, we show that the bound of $r + 1$ we got in the last section is in fact best possible. Notice that for linear hypergraphs without multiple edges, edges consisting of a twin set are exactly isolated edges and edges of size at most 1.

Theorem 8. *For every linear hypergraph H with edges of order at most $r \geq 2$, and no edge consisting of a twin set, there is a weighting $\rho : E(H) \to \{1, 2, \dots, \max\{5, r + 1\}\}$, such that the induced vertex weights $\rho(v) := \sum_{e \ni v} \rho(e)$ properly color $V(H)$.*

Proof. We prove the statement by induction on $n = |V(H)|$. In fact, we will prove a slightly stronger statement to make the induction work. The statement is stronger because we can pick ρ' to be constant:

For every linear hypergraph H with all edges of order between 2 and $r \geq 2$, and without isolated edges, and for every weighting of the vertices $\rho' : V(H) \to \mathbb{N}$, there is a weighting $\rho : E(H) \to \{1, 2, \ldots, \max\{5, r+1\}\}$, such that the induced vertex weights $\rho(v) := \rho'(v) + \sum_{e \ni v} \rho(e)$ properly color $V(H)$.

The statement is easy for $n = 3$, so assume that $n \geq 4$. We may assume that every vertex lies in an edge of size 2. Otherwise, pick a vertex v which is in no edge of size 2, such that the degree of v is minimal. Then the hypergraph $H - v$ is linear and contains no isolated edges: if the removal of v had created an isolated edge, then another vertex in that edge would have been chosen instead of v by the minimality of the degree of v. Any edge weighting inducing a proper coloring on $H - v$ then induces a proper coloring on H as well.

The main idea of the proof is as follows. We order the vertices in a specific ordering and then only consider an associated graph on the same vertex set, and all edges consisting of the first two vertices in each edge of H. Then we proceed very similar to the proof in [5] to weight these edges respecting the order of the vertices, guaranteeing that in the end, the first two vertices of each edge in H have different weighted degrees. Minor short comings we can fix in the end. The main difficulty in this approach lies in the fact that in the proof for graphs we must pick a vertex ordering with specific properties, but as the associated graph of the hypergraph depends on the ordering of the vertices, we can not change this vertex ordering after restricting our view to the associated graph. To circumvent this problem, we very carefully pick the ordering such that the resulting associated graph already has properties very close to what we need to make the graph process work without reordering.

In this spirit, for any ordering π of the vertices, define E_2 to be the set of edges of size 2, and let E_π be the set of pairs of vertices appearing first and second in π in an edge of size at least 3 in H. Now let us find a suitable ordering π.

If H contains a vertex incident to at least two edges in E_2, make such a vertex the last vertex v_n, and skip forward to next paragraph of this construction. If H contains no such vertex, then E_2 is a perfect matching. As H is linear, every edge of size at least 3 will share at most one vertex with each edge in this matching. Find a hyperedge h of minimal size $3 \leq t \leq \frac{n}{2}$, and choose π such that $h = \{v_n, v_{n-2}, \ldots, v_{n-2t+2}\}$, and $v_{n-2i}v_{n-2i-1} \in E_2$ for $0 \leq i \leq t-1$.

Then, successively for $i \geq 1$ or $i \geq 2t$, respectively, let v_{n-i} be a vertex not in $\{v_{n-i+1}, \ldots, v_n\}$ with an edge in $E_2 \cup E_\pi$ into $\{v_{n-i+1}, \ldots, v_n\}$, as long as such a vertex exists. Note that at this point, π is determined sufficiently to decide if there is an edge in E_π into $\{v_{n-i+1}, \ldots, v_n\}$. If we arrive at v_1 this way, π is determined. If the process stops before, say after assigning i labels, delete the previously ordered vertices $\{v_{n-i+1}, \ldots, v_n\}$ from H to form a subhypergraph H' on $n' = n - i$ vertices. Clearly, H' contains no edges of size 1. As H contains no twin set of size 2, H' contains no twin set of size 2 either, and therefore no isolated edges. By induction, we can find a vertex coloring edge weighting on H'. Similarly, let H'' be the connected hypergraph induced on $\{v_{n-i+1}, \ldots, v_n\}$. Add the weights of edges intersecting both $V(H')$ and $V(H'')$ which we computed in the weighting of H' to the respective vertex weights in H'', and use induction to weight the edges in H'', finishing the proof. Thus, we may assume in the following that π is completely determined, and that the graph $G = G_\pi$ with edge set $E_2 \cup E_\pi$ is connected up to possibly a few isolated edges in the end of the ordering; in the case that E_2 is a perfect matching, the edges $v_{n-2i}v_{n-2i-1}$ are isolated for $0 \leq i \leq t-3$. The component of G with more than one edge is ordered in a way that every vertex but the last vertex in the component has a neighbor later in the order, and the last vertex in the component has degree at least 2.

Now we weight very similarly to the proof in [5], we repeat large parts of this proof here so that this article is self contained. When we assign a weight to an edge in G, we are assigning it at the same time to the edge in H that corresponds to the edge in G.

Let $G[\{v_1, \ldots, v_s\}]$ be the first component of G, where $s = n$ or $s = n - 2t + 4$. We start by assigning the provisional weight $\rho(e) = 3$ to every edge and adjust it at most twice while going through all vertices in order—once when we are considering the first vertex in the edge, and once when we consider the second vertex. To every vertex v_i with $i < s$, we will assign a set of two colors $W(v_i) = \{w(v_i), w(v_i) + 2\}$ with $w(v_i) \in \{0, 1\} \mod 4$, so that for every edge $v_jv_i \in E(G)$ with $1 \leq j < i$, we have $W(v_j) \cap W(v_i) = \emptyset$, and we will guarantee that $\rho(v_i) = \sum_{v_i \in e \in E(H)} \rho(h) \in W(v_i)$. Finally we will adjust the weights of the edges incident to v_s in G to make sure that $\rho(v_s)$ is different from $\rho(v_i)$ for all $v_i \in N_G(v_s)$.

To this end, let $\rho(v_1) = \rho'(v_1) + 3d_G(v_1)$, and pick the set $W(v_1) = \{w(v_1), w(v_1) + 2\}$ so that $\rho(v_1) \in W(v_1)$ and $w(v_1) \in \{0, 1\} \mod 4$. Let $2 \leq k \leq n - 1$ and assume that we have picked $W(v_i)$ for all $i < k$ and

- $\rho(v_i) \in W(v_i)$ for $i < k$,
- $\rho(v_k v_j) = 3$ for all edges in G with $j > k$, and
- if $\rho(v_i v_k) \neq 3$ for some edge in G with $i < k$, then $\rho(v_i v_k) = 2$ and $\rho(v_i) = w(v_i)$ or $\rho(v_i v_k) = 4$ and $\rho(v_i) = w(v_i) + 2$.

If $v_i v_k \in E(G)$ for some $i < k$ we can either add or subtract 2 to $\rho(v_i v_k)$ keeping $\rho(v_i) \in W(v_i)$. If v_k has d such neighbors, this gives us a total of $d + 1$ choices (all of the same parity) for $\rho(v_k)$. In addition to this we will allow to alter the weight $\rho(v_k v_j)$ by 1, where $j > k$ is smallest such that $v_k v_j \in E(G)$. This way, $\rho(v_k)$ can take all values in an interval $[a, a + 2d + 2]$. We want to adjust the weights and assign $w(v_k)$ so that

(1) $\rho(v_i) \in W(v_i)$ for $1 \leq i \leq k$,
(2) $w(v_i) \neq w(v_k)$ for $v_i v_k \in E(G)$ with $i < k$, and
(3) either $\rho(v_k) = w(v_k)$ and $\rho(v_k v_j) \in \{2, 3\}$ or $\rho(v_k) = w(v_k) + 2$ and $\rho(v_k v_j) \in \{3, 4\}$.

Condition (2) can block at most $2d$ values in $[a, a+2d+2]$, and condition (3) can block only the values a and $a+2d+2$ (for all other values $\rho(v_k)$ with $\rho(v_k v_j) \neq 3$, we have the choice between $\rho(v_k v_j) = 2$ and $\rho(v_k v_j) = 4$). At least one value remains open for $\rho(v_k)$.

This way, we can assign the sets $W(v_k)$ step by step for all $k \leq s-1$ without conflict. Note that the first time $\rho(v_k)$ may get changed by an adjustment of an edge $v_k v_i$ for $i > k$ is when $i = j$, so we don't run into problems with edges weighted 2 or 4.

As the final step, we have to find an open weight for v_s. This time, we don't have an extra edge $v_s v_j$ to work with, but we don't have to worry about later vertices. If $v_i v_s \in E$ for some $i < s$ we can again either add or subtract 2 to $\rho(v_i v_s)$ keeping $\rho(v_i) \in W(v_i)$. These possible adjustments give a total of $d_G(v_s) + 1 \geq 3$ options (all of the same parity) for $\rho(v_s)$. Hence if the smallest such option a has $a \in \{2, 3\}$ mod 4, then picking the lower possible weight on each edge incident to v_s gives a proper coloring of the vertices. If $a \in \{0, 1\}$ mod 4 and there is a $v_i \in N(v_s)$ with $w(v_i) \neq a$, then picking the higher weight on $v_i v_s$ and the lower weight on all other edges gives $\rho(v_s) = a + 2$ in a proper coloring. Finally, if $a \in \{0, 1\}$ mod 4 and $w(v_i) = a$ for all $v_i \in N(v_s)$, picking the higher weight on at least two edges gives a proper coloring.

If $s = n$, then this finalizes the weighting of the edges in this component. Note that in this case, we have only used edge weights from the set $\{1, 2, \ldots, 5\}$, and we have not used that H is linear in this part of the proof.

If $s < n$, we have to make sure that in addition to $G[\{v_1, \ldots, v_s\}]$, the following isolated edges are also colored properly by the weighting. By our construction, there are $t-2$ of them. For this, consider the edge $h = \{v_n, v_{n-2}, \ldots, v_{n-2t+2}\}$ again. We will now change the weight of h to make the coloring proper on the t edges from E_2 intersecting h. Note that h is colored properly no matter to which value we change its weight, as this affects all its vertices the same way, and $\rho(v_{n-2t+2}) \neq \rho(v_{n-2t+4})$ from the previous argument. As we have $r+1 \geq t+1$ choices for the weight of h, this can easily be done. For the case $t = 3$, we have at least 5 choices for h, so on top of a proper coloring of the edges just mentioned, we can also make sure that the weight of v_{s-2} and v_{s-1} is different. Note that this may result in improper colorings of some edges in E_π, which will be corrected next.

As E_2 is a matching, and H is linear, and since t was chosen minimal, every edge other than h in H of size greater than 2, yielding an edge in G incident to v_s or v_{s-2} must contain at least $t-2$ of the $t-1$ vertices in $\{v_{s+1}, v_{s+3}, \ldots, v_{n-1}\}$. Therefore, if $t = 3$, there are at most 3 edges in E_π incident to $\{v_{s-2}, v_s\}$ other than $v_{s-2}v_s$, coming from edges in H containing $\{v_{s-2}, v_{s+1}\}$, $\{v_{s-2}, v_{s+3}\}$ and $\{v_s, v_{s+3}\}$, respecively. Each such edge must contain at least one of these three pairs of vertices, and no two such edges can contain the same pair by linearity.

If $t = 4$, there are at most 2 edges in E_π incident to $\{v_{s-2}, v_s\}$ other than $v_{s-2}v_s$, one of them containing v_{s-2} and the other containing v_s, as any two such edges in H share a vertex in $\{v_{s+1}, v_{s+3}, v_{s+5}\}$. If $t \geq 5$, there is at most one edge in E_π incident to $\{v_{s-2}, v_s\}$ other than $v_{s-2}v_s$, as any two such edges in H would share two vertices in $\{v_{s+1}, v_{s+3}, v_{s+5}, v_{s+7}\}$. By changing the weights of $v_{s+1}v_{s+2}, \ldots v_{n-1}v_n$, we can make every such edge proper in H. Note that only in the case $t = 3$ there can be such an edge in H with no vertex in $\{v_{s+1}, \ldots v_n\}$, and this edge contains v_{s-2} and v_{s-1}, for which we ensured different weights when we chose the final weight for h. This finishes the proof. □

For $r = 3$, we can get rid of the linearity condition, by being more careful in the last step of the proof.

Theorem 9. *For every hypergraph H with all edges of order at most 3, and no edge consisting of a twin set, there is a weighting $\rho : E(H) \to \{1, 2, \ldots, 5\}$, such that the induced vertex weights $\rho(v) := \sum_{e \ni v} \rho(e)$ properly color $V(H)$.*

Proof. The proof is the same as for Theorem 8, only the case where E_2 is a matching is treated differently. So let us assume that E_2 is a matching.

Let $E_1(H) \subseteq E(H)$ be the set of edges in H which contain an edge of E_2, and let $E_0(H) = E(H) \setminus E_1(H)$.

If $E_0(H) = \emptyset$, then the proper weighting is easy. Changing the weight of an edge in $E_1(H)$ does not impact the properness of the coloring of the contained edge in $E(G)$. As every edge in $E(G)$ has an intersection of exactly one vertex with at least one 3-edge, we can greedily make all edges in E_2 properly colored by changing the weights of the 3-edges only. Since all 3-edges contain 2-edges, this in turn makes the coloring of every 3-edge proper.

So there exists a 3-edge $e \in E_0(H)$. Order the vertices such that $e = \{v_{n-4}, v_{n-2}, v_n\}$, and such that $v_{n-5}v_{n-4}, v_{n-3}v_{n-2}, v_{n-1}, v_n \in E_2$. Let d_1 be the number of edges in $E_0(H)$ containing both v_{n-2} and v_{n-1}, and let d_2 be the number of edges in $E_0(H)$ containing both v_{n-2} and v_n. We may assume that e and the ordering of the vertices was chosen such that the sum $d_1 + d_2$ is maximized, and $d_2 \geq d_1$.

Now continue the reordering process as before until there are no more vertices with forward edges into the current component. Again, if this process stops before we reach v_1, we can use induction to complete the weighting, so we may assume that we can complete π this way on the first try.

If there is an edge $\{v_{n-2}, v_{n-1}, v_n\} \in E(H)$, we may delete this edge and pretend that we are in the same case, as this edge will be colored properly in the end due to the fact that $v_{n-1}v_n$ will be colored properly.

Consider first the case that $d_1 = 0$ and $d_2 = 1$. Notice that in this case, e is the only edge containing v_{n-4} and a vertex later in the order. Otherwise, we could change the order of the last three edges in E_2 such that $d_1 + d_2 \geq 2$, contradicting the maximimality of $d_1 + d_2$. Similarly, there can be at most one edge e' in $E_0(H)$ containing v_{n-3} and a vertex in $\{v_{n-1}, v_n\}$, and if e' exists, it contains v_{n-3} and v_{n-1}. We now run the weighting algorithm vertex by vertex until we have dealt with v_{n-5}. Next, we adjust the weight of e to make all of $v_{n-5}v_{n-4}$, $v_{n-3}v_{n-2}$ and $v_{n-1}v_n$ properly colored. Finally, we adjust the weight of $v_{n-1}v_n$ to make e and e' properly colored, finishing this case. Notice that adjusting the weights of e and $v_{n-1}v_n$ does not affect the properness of the edges considered up to v_{n-5}. Further, every edge in $E_1(H)$ not previously considered contains either $v_{n-3}v_{n-2}$ or $v_{n-1}v_n$ and is therefore properly colored now.

It remains to consider the case that $d_1+d_2 \geq 2$. We run the weighting algorithm until we reach v_{n-2}. Now there are 2 weight options for each of the $d_1 + d_2$ edges incident to v_{n-2} other than $v_{n-3}v_{n-2}$, where d_1 of these choices affect the weight of v_{n-1}, and d_2 of these choices affect the weight of v_n. If we pick the values of these edges such that in the end, $\rho(v_{n-2}) \notin \{\rho(v_{n-3}), \rho(v_{n-1}), \rho(v_n)\}$, and $\rho(v_{n-1}) \neq \rho(v_n)$, then the induced weighting properly colors H.

To this end, start with a weighting always using the smaller of the two options on the $d_1 + d_2$ edges. By switching up to two of these edges (one of them being e) to the higher option, we can change the pair $(\rho(v_{n-2}) - \rho(v_{n-3}), \rho(v_n) - \rho(v_{n-1}))$ by $\{(0,0),(2,2),(4,4)\}$ or $\{(0,0),(2,2),(2,-2),(4,0)\}$ depending on the choice of the second edge. Thus, we have a choice which will make both $\rho(v_{n-2}) \neq \rho(v_{n-3})$ and $\rho(v_n) \neq \rho(v_{n-1})$. Finally, we can adjust the weight of $v_{n-1}v_n$ to achieve $\rho(v_{n-2}) \neq \rho(v_{n-1})$ and $\rho(v_{n-2}) \neq \rho(v_n)$. □

For general (non-linear) hypergraphs, we have the following bound.

Theorem 10. *For every hypergraph H with all edges of order between 2 and r, and no edge consisting of a twin set, there is a weighting $\rho : E(H) \to \{1, 2, \ldots, 5r - 5\}$, such that the induced vertex weights $\rho(v) := \sum_{e \ni v} \rho(e)$ properly color $V(H)$.*

Proof. Again, the proof follows similar lines as the proof for Theorem 8. But since we have trouble with the isolated edges in G_π which may appear, we consider these edges first, and later make sure that these edges do not become monochromatic.

We start by finding a similar ordering π as above. In the case that two edges in E_2 intersect, we use the ordering from above. If we do not reach v_1 this way on the first try, we can again use induction to show the existence of the weighting in the theorem statement. If we do reach v_1, we will find a weighting $\rho : E(H) \to \{1, 2, 3, 4, 5\}$.

In the case that E_2 is a matching, we can not guarantee the existence of a hyperedge h of the form described above. Instead, we will use a hyperedge h which intersects some edge in E_2 in exactly one vertex, and, given that constraint, intersects the minimal number of edges in E_2 (say a total of t intersected edges). Now, as above, put the edges in E_2 intersecting h last in the ordering $v_{n-1}v_n, v_{n-3}v_{n-2}, \ldots, v_{n-2t+1}v_{n-2t+2}$ with $\{v_{n-2t+2}, v_{n-2t+4}, \ldots, v_n\} \subseteq h$ and $v_1, v_2, \ldots, v_{n-2t+1} \notin h$. Again, extend this component backwards as far as possible, guaranteeing that every vertex has a neighbor in G_π later in the ordering.

If this process stops before we reach v_1, we can not use the same inductive argument as before—the reasons will become clear very soon. Instead, we delete all previously ordered vertices and continue with the remaining hypergraph until all vertices are in order, resulting in more than one component with more than two vertices.

Now give every edge a preliminary weight of $2r-1$. Next, we change the weight of some edges in a way that all edges in E_2 intersecting h other than possibly $v_{n-2t+1}v_{n-2t+2}$ have two weights on their vertices which are different modulo $r-1$. In the final step we will only adjust weights of edges by multiples of $r-1$, and in this way these preprocessed edges can never become monochromatic.

So for this, let E^* contain all edges in E_2 consisting of the last two vertices of a component of G, unless that component ends with a vertex with at least two different neighbors in G. As none of these edges contains a twin set of size at least 2, for every $f \in E^*$ there exists an edge $e_f \in E(H)$ containing exactly one of the two vertices of f. Every edge $e \in E(H)$ can have a unique intersection with at most $r-2$ edges $f \in E^*$: If e intersects r or $r-1$ different edges in E_2 uniquely, then e does not fully contain an edge in E_2, and so the first two vertices of e are not in E^*. Thus, we can greedily add the lowest suitable values from $\{0, \ldots, r-2\}$ to the weights of the edges e_f one-by-one, making the vertex weights in each edge $f \in E^*$ different modulo $r-1$.

We proceed very similarly as above. Instead of adjusting edge weights by $\{-2, -1, 0, 1, 2\}$, we adjust them by $\{-2(r-1), -(r-1), 0, (r-1), 2(r-1)\}$. Since we preprocessed the edges in E^*, these adjustments will never make te edges in E^* monochromatic. Again, by induction it is enough to look at one large component of G now, say the last.

We stop the algorithm after adjusting the edges around v_{n-2t+2}. Note that the biggest change required on h at this step is $r-1$, so h has weight at most $(2r-1)+(r-2)+(r-1) = 4r-4$. The edges in E^* are properly colored due to the preprocessing, all other edges other than h are properly colored due to the algorithm. To make h properly colored, we can adjust $v_{n-1}v_n$, which does not affect the properness of any other edge at this point.

After the algorithm runs, every edge has weight at least $(2r-1) - 2(r-1) = 1$ and at most $(2r-1)+(r-2)+2(r-1) = 5r-5$, proving the theorem. $\square$

4. Total List Colorings

In this section, we generalize Theorems 4 and 7 to hypergraphs.

Theorem 11. *For every hypergraph H without edges of size 0 or 1, there are weightings $\rho : E(G) \to \{1,2,3\}$ and $\rho' : V(G) \to \{1,2\}$ such that the induced total vertex weights $w(v) := \rho'(v) + \sum_{e \ni v} \rho(e)$ properly color $V(H)$.*

We will not present the proof here, as the statement is implied by the list version.

Theorem 12. *For every hypergraph H without edges of size 0 and 1, and for every assignment of lists of size 2 to the vertices and of size 3 to the edges of G, there exists a weighting $\rho : V(G) \cup E(G) \to \mathbb{R}$ from the lists, such that the induced total vertex weights $w(v) := \rho(v) + \sum_{e \ni v} \rho(e)$ properly color $V(G)$.*

Proof. The proof is an adaptation of the proof of Theorem 7 to hypergraphs, using some ideas from our previous proofs. Order the vertices in some order. For every edge $e \in E(H)$, let $u_e, v_e \in e$ be the first two vertices in the edge. We will show the stronger statement, that we can find a weighting such that $w(u_e) \neq w(v_e)$ for every edge $e \in E(H)$. We may assume that $\{u_e, v_e\} \neq \{u_{e'}, v_{e'}\}$ for any pair of edges $e, e' \in E(H)$. Otherwise, pick a weight from the list of e', and add it to the lists of all vertices in e'. If we can now find a coloring weighting in the Hypergraph $H - e'$ from the new lists, the corresponding weighting in H will also induce a proper coloring.

Now consider the polynomial

$$\phi(\rho) = \prod_{e \in E(H)} (w(u_e) - w(v_e)).$$

Note that non-zeros of ϕ correspond exactly to total weightings of H with $w(u_e) \neq w(v_e)$ for every edge $e \in E(H)$. Let $n = |V(H)|$, $m = |E(H)|$, and let A_H be an $m \times (m+n)$ matrix, where the rows are labeled with the elements of $E(H)$, the columns are labeled with the elements of $E(H) \cup V(H)$, and

$$A_H(e,z) = \begin{cases} 1, & \text{if } z \in V(H) \text{ and } z \in e, \\ 1, & \text{if } z \in E(H) \text{ and } u_e \in z, v_e \notin z, \\ -1, & \text{if } z \in E(H) \text{ and } v_e \in z, u_e \notin z, \\ 0, & \text{otherwise.} \end{cases}$$

Let B be an $m \times m$ matrix consisting of columns of A_H, such that every column corresponding to a vertex of H appears at most once, and every column corresponding to an edge of H appears at most twice. Then the permanent $\mathrm{per}(B)$ equals a coefficient of a maximum degree monomial in ϕ. By a standard application of Alon's Combinatorial Nullstellensatz, there exists a non-zero of ϕ (and thus a proper coloring by total vertex weights from the lists) from any given list assignment with vertex lists of size two and edge lists of size three, if we can find such a matrix B with $\mathrm{per}(B) \neq 0$.

We will find such a B by induction on n. For $n = 0$, the statement is trivial as by definition, the permanent of a 0×0 matrix is 1. For $n \geq 1$, let $u \in V(H)$ be the first vertex in the order, and consider the hypergraph H' induced by H on $V(H) \setminus \{u\}$, i.e., H' contains all edges of H which do not contain u. Let $k = \deg(u)$, then by induction there is an $(m-k) \times (m-k)$-matrix B' with $\operatorname{per}(B') \neq 0$ consisting of columns of $A_{H'}$, at most two columns equal to $A_{H'}(., e)$ for each edge, and at most one column equal to $A_{H'}(., v)$ for each vertex.

Now build an $m \times m$ matrix C from B' by the use of the corresponding columns of A_H, and the addition of k identical columns equal to $A_H(., u)$. These added columns all have k entries equal to 1, and the remaining entries 0, so $\operatorname{per}(C) = k! \operatorname{per}(B') \neq 0$. Now, whenever $u \in e \in E(H)$, and C contains $A_H(., v_e)$, replace that column by the column $A_H(., e)$. Note that $A_H(., e) = A_H(., u) - A_H(., v_e)$. By the multilinearity of the permanent, the difference of permanents of C and the matrix after the switch is equal to the permanent of a matrix containing $(k+1)$ copies of $A_H(., u)$, a singular matrix with permanent 0. Thus, we can make all these switches one-by-one, arriving at a matrix D without columns equal to $A_H(., v_e)$ and at most one column equal to $A_H(., e)$ for $u \in e \in E(H)$, and with $\operatorname{per}(D) = \operatorname{per}(C)$.

To now get B from D, we will replace the columns equal to $A_H(., u)$ appropriately one-by-one. For every $e \ni u$, replacing one column $A_H(., u)$ by one of $A_H(., v_e)$ and $A_H(., e)$ will result in a matrix with per $\neq 0$. If both the resulting matrices had permanent 0, then by the multilinearity of the permanent, the matrix before the replacement would have had permanent 0, a contradiction. After replacing all columns equal to $A_H(., u)$ this way, we arrive at B with the desired properties. □

5. Conclusion and Open Questions

Linearity helps us in the proof of Theorem 8, but we believe that this is just a technical problem, and we believe that the following is true for all hypergraphs.

Conjecture 13. *For every hypergraph H with all edges of order between 2 and r, and no edge consisting of a twin set, there is a weighting $\rho : E(H) \to \{1, 2, \ldots, r+1\}$, such that the induced vertex weights $\rho(v) := \sum_{e \ni v} \rho(e)$ properly color $V(H)$.*

The only class of hypergraphs we know achieving this bound is the one constructed above, stemming from the complete graphs K_{r+1}. Possibly, it is true that this is the unique example for $r \geq 3$, and in all other cases a set $\{1, 2, \ldots, r\}$ is sufficient.

Note that most of our examples on the lower bounds are highly non-uniform, they contain very small and very large edges. For r-uniform hypergraphs, there may be a constant upper bound instead, independent of r. But what is it? As mentioned, it may even be true that for $r \geq 4$, the set $\{1, 2\}$ is sufficient. For $r = 3$, we conjecture the same bound which is conjectured for $r = 2$:

Conjecture 14. *For every* 3*-uniform hypergraph* H *without an isolated edge, there is a weighting* $\rho : E(H) \to \{1, 2, 3\}$*, such that the induced vertex weights* $\rho(v) := \sum_{e \ni v} \rho(e)$ *properly color* $V(H)$*.*

References

1. N. Alon, Combinatorial Nullstellensatz, *Combin. Prob. Comput.* **8** (1999), 7-29.
2. T. Bartnicki, J. Grytzcuk, S. Niwczyk, Weight choosability of graphs, *J. Graph Theory* **60**, (2009), 242-256.
3. G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, and F. Saba, Irregular Networks, *Congressus Numerantium* **64** (1988), 187-192.
4. M. Kalkowski, A note on the 1,2-conjecture, in Ph.D. Thesis, 2009.
5. M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: Towards the 1-2-3 conjecture, *Journal of Combinatorial Theory (B)* **100** (2010), 347-349.
6. M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combinatorial Theory (B)* **91** (2004), 151- 157.
7. J. Przybyło, M. Woźniak, On a 1-2 conjecture, *Discrete Mathematics and THeoretical Computer Science* **12** (2010), 101-108.
8. J. Przybyło, M. Woźniak, Total weight choosability of graphs, *Electronic J. Combinatorics* **18** (2011).
9. C. Thomassen, the even cycle problem for directed graphs, *Journal of the AMS* **5** (1992), 218-229.
10. T. Wong, X. Zhu, Total weight choosability of graphs, *J. Graph Theory* **66** (2011), 198-212.
11. X. Zhu, T. Wong, Every graph is $(2, 3)$-choosable, preprint.

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