A Note on Cycle Spectra of Line Graphs

FLORIAN PFENDER
Universität Rostock
Institut für Mathematik
D-18051 Rostock, Germany
Florian.Pfender@uni-rostock.de

Abstract

We show that line graphs $G = L(H)$ with $\sigma_2(G) \geq 7$ contain cycles of all lengths k, $2\,\mathrm{rad}(H)+1 \leq k \leq c(G)$. This implies that every line graph of such a graph with $2\,\mathrm{rad}(H) \geq \Delta(H)$ is subpancyclic, improving a recent result of Xiong and Li. The bound on $\sigma_2(G)$ is best possible.

1 Introduction

All graphs considered here are simple. For all terms not defined here we refer the reader to [1]. We denote the neighborhood of a vertex set $X \subseteq V(G)$ in a graph G by $N_G(X)$ or $N(X)$. The degree of a vertex in $v \in V(G)$ is $d_G(v) = d(v) = |N_G(v)|$. The maximum degree of G is $\Delta(G)$, the minimum degree $\delta(G)$. Let $\sigma_2(G) := \min\{d(x) + d(y) \mid x, y \in V(G) \wedge xy \notin E(G)\}$. The number of vertices in G is denoted by $|G|$, the number of edges by $\|G\|$. The cycle with k edges is called C^k, and every cycle is given a direction. For a cycle C and two vertices $v, w \in V(C)$, vCw denotes the $v-w$ path following C in the direction of C, v^+ and v^- are the successor and the predecessor of v on C. For a tree T and two vertices $v, w \in V(C)$, vTw denotes the $v-w$ path following T.

The distance between two vertices $v, w \in G$ is $d_G(v, w) = d(v, w)$. The diameter of a graph G is $\mathrm{diam}(G) = \max_{v,w} d(v, w)$, and the radius is $\mathrm{rad}(G) = \min_v \max_w d(v, w)$. A subgraph $H \subseteq G$ is distance preserving if $d_H(v, w) = d_G(v, w)$ for all $v, w \in V(H)$. A shortening path of a subgraph H is a $v-w$ path P such that $V(H) \cap V(P) = \{v, w\}$ and $d_P(v, w) < d_H(v, w)$, i.e., a witness to the fact that H is not distance preserving.

We write $L(G)$ for the line graph of G. The complete bipartite graph $K_{1,3}$ is called a claw, and a graph is said to be claw-free if it does not contain a claw as an induced subgraph. All line graphs are claw-free.

A graph G is subpancyclic if it contains cycles of all lengths $3 \leq k \leq c(G)$, where $c(G)$ is the circumference of G, i.e. the length of the longest cycle in G.

Gould and Pfender [2] showed the following lemma about claw-free graphs.

Lemma 1. *Let G be a claw-free graph with $\sigma_2(G) \geq 9$. Suppose, for some $m > 3$, G has an m-cycle C, but no $(m-1)$-cycle. Then C is distance preserving.*

This yields as an immediate consequence the following corollary.

Corollary 2. *Let G be a claw-free graph with $\sigma_2(G) \geq 9$ and circumference $c(G)$. Then for every k with $2\,\mathrm{diam}(G) + 1 \leq k \leq c(G)$, G contains C^k.*

For line graphs, we strengthen Lemma 1 as follows.

Lemma 3. *Let G be a line graph with $\sigma_2(G) \geq 7$. Suppose, for some $m > 3$, G has an m-cycle C, but no $(m-1)$-cycle. Then C is distance preserving.*

Xiong and Li [3] prove the following theorem.

Theorem 4. *Let H be a graph and $G = L(H)$ its line graph with $\delta(G) \geq 6$, and $\operatorname{rad}(H) \leq \frac{\Delta(H)}{2}$. Then G is subpancyclic.*

We will prove the following.

Theorem 5. *Let H be a graph and $G = L(H)$ its line graph with $\sigma_2(G) \geq 7$. Then G contains cycles of all lengths k, $2\operatorname{rad}(H) + 1 \leq k \leq c(G)$.*

Since $G = L(H)$ trivially contains cycles of all lengths $3 \leq k \leq \Delta(H)$, we can improve Theorem 4.

Corollary 6. *Let H be a graph and $G = L(H)$ its line graph with $\sigma_2(G) \geq 7$, and $\operatorname{rad}(H) \leq \frac{\Delta(H)}{2}$. Then G is subpancyclic.*

Corollary 7. *Let H be a graph and $G = L(H)$ its line graph with $\delta(G) \geq 4$, and $\operatorname{rad}(H) \leq \frac{\Delta(H)}{2}$. Then G is subpancyclic.*

2 Proof of Lemma 3

The lemma can be proved very similarly to Lemma 1. Here is a sketch of the proof.

Let H and G be as in the statement of the lemma, and let C be an m-cycle in G. Suppose first that C has a shortening path of length at most two. Pick four vertices $s_1,\ t_1,\ s_2,\ t_2$ such that there are shortening paths P_i of length at most two between s_i and t_i, $s_i^+ \notin s_{2-i}Ct_{2-1}$ and the s_iCt_i are minimal according to these conditions. Let K_i be the set of vertices on $s_i^+Ct_i^-$ which are not incident to a chord of C. By symmetry, we may assume that either $|K_1| < |K_2|$ (in which case note that all but at most two vertices in K_2 have degree at least 4), or $|K_1| = |K_2|$ and $\min_{v \in K_2} d(v) \geq 4$. Let $C' = t_1Cs_1P_1t_1$. Then $|C'| \leq m-1$. Now we can extend C' one vertex at a time by inserting the vertices of $V(s_1^+Ct_1^-) \setminus K_1$. Then, we can insert all neighbors outside C' of vertices in K_2. Note that every such neighbor has at most two adjacent vertices on $s_2^+Ct_2^-$, so $|N(K_2) \setminus C| \geq \frac{1}{2}\sum_{v \in K_2}(d(v) - 2) \geq |K_1| - 1$. Thus, we can insert vertices until we have a C^{m-1}.

On the other hand, if there is no shortening path of length at most 2, we can construct from C and a shortening path P a cycle C' with $|C'| \leq m-1$ and $|C' \cap C| \geq \frac{m}{2}$, which we can again extend one by one through vertices in $N(C' \cap C) \setminus V(P)$ until we have a C^{m-1}. This contradiction shows that there is no shortening path of C in G, and thus C is distance preserving. □

3 Proof of Theorem 5

For the sake of contradiction, suppose that H and $G = L(H)$ are graphs as in the statement, and suppose that for some $m > 2\operatorname{rad}(H) + 1$, G contains a C^m but no C^{m-1}. The cases that $m \in \{4, 5\}$ (and thus $\operatorname{rad}(H) = 1$) are easy to rule out, so we may assume that $m \geq 6$. By Lemma 3, this cycle is distance preserving, so its line graph original in H is an induced cycle C on m vertices which is distance preserving as well.

Since G contains no C^{m-1}, we know that G contains no induced C^k for $\frac{2}{3}(m-1) \leq k \leq m-1$, as each such cycle could easily be extended to a C^{m-1}. Thus, H contains no C^k with $\frac{2}{3}(m-1) \leq k \leq m-1$ (the line graph operation bijectively maps cycles in H to induced cycles of the same length in G).

Let S be the graph obtained from H through a single subdivision of every edge. Then $2\operatorname{rad}(H) \leq \operatorname{rad}(S) \leq 2\operatorname{rad}(H) + 1$ and S contains no C^k with $\frac{4}{3}(m-1) \leq k \leq 2m-2$. Note that all cycles in S have even length. Let Z be the $2m$-cycle in S obtained from C. Choose $z \in V(S)$ such that $\max_{v \in V(Z)} d(z,v)$ is minimal, and therefore at most $\mathrm{rad}(S)$. Let T be a minimal tree in S such that $d_{Z \cup T}(z,v) = d(z,v)$ for all $v \in V(Z)$. Since Z is distance preserving, T intersects Z exactly in the leaves of T. Let $\{v_1,\ldots,v_\ell\} = V(T) \cap V(Z)$ be the leaves of T in the order they appear on Z. For ease of notation, let $v_{\ell+1} = v_1$. Let $P_i = v_i T z$.

Now consider the cycles $Z_{i,j} = v_i Z v_j T v_i$. We have $\|Z_{i,i+1}\| \le 2\,\mathrm{rad}(S) \le 2m-2$, since otherwise there would be a vertex v on $v_i Z v_{i+1}$ with $d_{Z \cup T}(z,v) > \mathrm{rad}(S)$. Therefore, we get $\|Z_{i,i+1}\| < \frac{4}{3}(m-1)$, as S contains no C^k with $\frac{4}{3}(m-1) \le k \le 2m-2$. As Z is distance preserving, this implies that $\|v_i Z v_{i+1}\| \le \frac{1}{2}\|Z_{i,i+1}\| < \frac{2}{3}(m-1)$.

Let us pick $i,j \in \{1,\ldots \ell\}$ such that

1. there is a vertex $u \in v_i Z v_j$, such that $d(u,z) = \max_{v \in V(C)} d(z,v)$,
2. $\|v_i Z v_j\| < \frac{2}{3}(m-1)$,
3. $\|Z_{i,j}\| < \frac{4}{3}(m-1)$, and
4. $\|v_i Z v_j\|$ is maximal under these conditions.

Without loss of generality we may assume that $1 \le i < j \le \ell$, and that $\|P_i \cap Z_{i,j}\| \le \|P_j \cap Z_{i,j}\|$. Consider $Z_{i,j+1}$. If $\|Z_{i,j+1}\| \le 2m-2$, then in fact again $\|Z_{i,j+1}\| < \frac{4}{3}(m-1)$, $\|v_i Z v_{j+1}\| < \frac{2}{3}(m-1)$, and we get a contradiction to the maximality of $\|v_i Z v_j\|$. Thus, $\|Z_{i,j+1}\| \ge 2m$.

If $Z_{i,j+1}$ contains no edges of $E(P_j) \setminus E(P_i)$, then

$$\|Z_{i,j+1}\| \le \|Z_{i,j}\| + \|Z_{j,j+1}\| - 2|E(P_j) \setminus E(P_i)| \le \|Z_{j,j+1}\| + \|Z_{i,j}\| - \tfrac{2}{4}\|Z_{i,j}\| < 2m-2,$$

a contradiction. Thus, $Z_{i,j+1}$ contains edges of $E(P_j) \setminus E(P_i)$.

Let $u_1 \in V(v_i Z v_j)$ such that $\|u_1 Z v_i P_i z\| = \|u_1 Z v_j P_j z\|$ and $u_2 \in V(v_j Z v_{j+1})$ such that $\|u_2 Z v_j P_j z\| = \|u_2 Z v_{j+1} P_{j+1} z\|$. Then $\|u_1 Z v_j P_j z\| \ge d(u,z) \ge \|u_2 Z v_j P_j z\|$, and therefore $\|u_1 Z v_j\| \ge \|u_2 Z v_j\|$. But now

$$\begin{aligned}
\|Z_{i,j+1}\| &= \|Z_{i,j}\| + \|Z_{j,j+1}\| - 2\|Z_{i,j} \cap Z_{j,j+1}\| \\
&= \|Z_{i,j}\| + \|Z_{j,j+1}\| - 2(\tfrac{1}{2}\|Z_{j,j+1}\| - \|u_2 Z v_j\|) \\
&= \|Z_{i,j}\| + 2\|u_2 Z v_j\| \\
&\le \|Z_{i,j}\| + 2\|u_1 Z v_j\| \le \|Z_{i,j}\| + \tfrac{2}{4}\|Z_{i,j}\| \quad < \quad 2m-2.
\end{aligned}$$

This contradiction concludes the proof of the Theorem. □

4 Sharpness

Consider the following graph H_1 (see Figure 4) with $G_1 = L(H_1)$ demonstrating that the condition $\sigma_2(G) \ge 7$ is best possible in Lemma 3 and Theorem 5. For $k \ge 3$, start with two copies of C^{2k} and identify them at one vertex. At every vertex at even distance from the vertex with degree 4, attach a star $K_{1,4}$ by identifying one of its leaves with the vertex, resulting in a graph H_1.

Then $G_1 = L(H_1)$ has minimum degree $\delta(G_1) = 3$ (and $\sigma_2(G_1) = 6$), contains a C^{4k} and no C^ℓ for $3k+2 \le \ell \le 4k-1$. But, the C^{4k} has chords and is thus not distance preserving, showing that the bound on σ_2 is best possible for Lemma 3. The radius of H_1 is $k+1 \le \mathrm{rad}(H) \le k+2$, concluding that the bound on σ_2 is best possible for Theorem 5 as well.

To see that the bound on the radius in Theorem 5 is best possible, start with a complete graph K^4, and subdivide the three edges incident to some vertex v k-times each for some $k \ge 2$. Add three pendant edges to every vertex of degree 2 to get a graph H_2, and let $G_2 = L(H_2)$. We have $c(G_2) = 8k+7$, $\delta(G_2) = 4$, $\mathrm{rad}(H_2) = k+1$, and G_2 contains no C^{2k+2}.

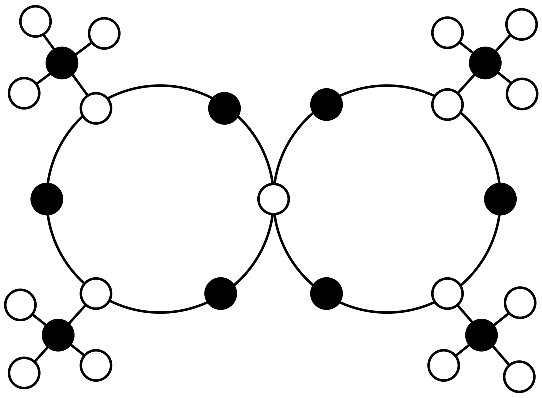

Figure 1: The graph H_1 for $k = 3$

References

[1] R. Diestel, Graph Theory, Springer-Verlag New York (1997).

[2] R. Gould and F. Pfender, Pancyclicity in Claw-free Graphs, Discrete Math. 256 (2002), 151–160.

[3] L. Xiong and M. Li, Radius and subpancyclity in line graphs, preprint, 13 pages.