A NEW UPPER BOUND FOR THE IRREGULARITY STRENGTH OF GRAPHS

MACIEJ KALKOWSKI, MICHAŁ KAROŃSKI, AND FLORIAN PFENDER

Abstract. A weighting of the edges of a graph is called irregular if the weighted degrees of the vertices are all different. In this note we show that such a weighting is possible from the weight set $\{1, 2, \ldots, 6\lceil \frac{n}{\delta} \rceil\}$ for all graphs not containing a component with exactly 2 vertices or two isolated vertices.

1. Introduction

All graphs in this note are finite and simple. For notation not defined here we refer the reader to [4].

For some $k \in \mathbb{N}$, let $\omega : E(G) \to \{1, 2, \ldots, k\}$ be an integer weighting of the edges of a graph G. This weighting is called irregular if the weighted degrees $d_\omega(v) = \sum_{u \in N(v)} \omega(uv)$ of the vertices are all different. It is easy to see that for every graph G which has at most one isolated vertex and no component isomorphic to K^2, there exists an irregular weighting for some smallest k, the *irregularity strength* $s(G)$ of G. If G contains a K^2 or multiple isolated vertices, we set $s(G) = \infty$.

The irregularity strength was introduced in [2] by Chartrand *et al.* . For all graphs with $n := |G| > 3$ and $s(G) < \infty$, Nierhoff [8] showed the tight bound $s(G) \leq n-1$, extending a result by Aigner and Triesch [1]. Faudree and Lehel considered regular graphs in [5]. They showed that if G is d-regular ($d \geq 2$), then $\left\lceil \frac{n+d-1}{d} \right\rceil \leq s(G) \leq \lceil \frac{n}{2} \rceil + 9$, and they conjectured that $s(G) \leq \left\lceil \frac{n}{d} \right\rceil + c$ for some constant c.

A first bound involving the minimum degree δ was given by Frieze *et al.* in [6] where they showed that $s(G) \leq 60\lceil \frac{n}{\delta} \rceil$, for graphs with maximum degree $\Delta \leq n^{1/2}$. For graphs with high minimum degree, Cuckler and Lazebnik showed that $s(G) \leq 48\lceil \frac{n}{\delta} \rceil + 6$ in [3]. Finally, Przybyło showed in [10] that $s(G) \leq 112\frac{n}{\delta} + 28$ for general graphs and in [9] that $s(G) \leq 16\frac{n}{d} + 6$ for d-regular graphs.

In this note we give a construction improving the bounds stated in the previous paragraph. We use ideas similar to the ones used in [7].

1991 *Mathematics Subject Classification.* 05C78, (05C15).

Key words and phrases. irregular graph labelings.

Theorem 1. *Let δ be the minimum degree of G and $n = |G|$. If $s(G) < \infty$, then $s(G) \leq 6\lceil \frac{n}{\delta} \rceil$.*

Considering the sharpness of these results, no graphs classes with $s(G) > \lceil \frac{n}{\delta} \rceil + c$ are known to us, similarly to the case of regular graphs mentioned above.

2. Proof

Since $s(G) \leq n - 1$, there is nothing to prove for $\delta \leq 6$, so we may assume that $\delta \geq 7$. Order the vertices $v_1, v_2, \ldots, v_n$ such that for $1 \leq i < k \leq j \leq n$, whenever v_i and v_j belong to the same component of G,

- v_k also belongs to that component of G, and
- v_i has a neighbor v_ℓ with $\ell > i$.

Going through the vertices in order, we will assign two weights ω_1 and ω_2 to each edge v_iv_j (where $i < j$), and $\omega(v_iv_j) = \omega_1(v_iv_j) + \omega_2(v_iv_j)$. The first weight $\omega_1(v_iv_j) \in \{1, 2, \ldots, 2\lceil \frac{n}{\delta} \rceil\}$ is assigned when we process v_i, the second weight $\omega_2(v_iv_j) \in \{0, 2\lceil \frac{n}{\delta} \rceil, 4\lceil \frac{n}{\delta} \rceil\}$ is initially set to $2\lceil \frac{n}{\delta} \rceil$ and finalized when we process v_j.

Let

$$\mathcal{W} := \left\{ \{a + 4b\lceil \tfrac{n}{\delta} \rceil, a + (4b+2)\lceil \tfrac{n}{\delta} \rceil\} \mid a, b \in \mathbb{Z}, 0 \leq a \leq 2\lceil \tfrac{n}{\delta} \rceil - 1 \right\}$$

be a set of disjoint pairs of integers covering $\mathbb{Z}$, and for a given ω and $1 \leq i \leq n$, let $W(v_i) \in \mathcal{W}$ be the unique pair containing $d_\omega(v_i)$.

Let X be the set of indices i, such that either v_i or v_{i+1} is the final vertex of a component. For $i \leq n$, assume that all vertices v_k with $k < i$ have been considered already.

If $i \notin X$, we want to adjust ω such that $W(v_i) \neq W(v_k)$ for all $k < i$. In the remainder of the construction, $W(v_i)$ will not change anymore. Reserving a pair of values for $d_\omega(v_i)$ like this gives us the freedom to later adjust $\omega_2(v_iv_j)$ for $j > i$ without creating a conflict.

To this end, we can freely choose $\omega_1(v_iv_j)$ for $i < j$ and choose $\omega_2(v_kv_i)$ for $k < i$ from one of the two values keeping $d_\omega(v_k)$ in $W(v_k)$. If v_i has $d^+ \geq 1$ neighbors v_j with $j > i$ and d^- neighbors v_k with $k < i$, this gives us

$$2\lceil \tfrac{n}{\delta} \rceil (d^+ + d^-) - d^+ \geq 2n - d^+ > 2i$$

consecutive options for $d_\omega(v_i)$. These options intersect more than i pairs of $\mathcal{W}$. At most $i - 1$ of these pairs can already be used as some $W(v_k)$ by a neighbor v_k of v_i with $k < i$, so we can find the desired pair $W(v_i)$, together with a preliminary weighting ω.

If $\{i, i+1\} \subseteq X$, note that $v_i v_{i+1}$ is an edge. We may choose $\omega_1(v_i v_{i+1})$ such that no three vertices v_j with $j \in X$ and $j \leq i+1$ have the same weight ω_1 as there are less than $\lceil \frac{n}{\delta} \rceil$ components and thus $|X| < 2\lceil \frac{n}{\delta} \rceil$.

Let $j \in \{i, i+1\} \subseteq X$. We want to adjust $\omega_2(v_k v_j)$ for edges with $k < i$ so that all weighted degrees $d_\omega(v_\ell)$ for $\ell \leq j$ are different. At this stage we allow that $W(v_j) = W(v_\ell)$ for one $\ell < j$, since both $d_\omega(v_j)$ and $d_\omega(v_\ell)$ are finalized in this step as they don't have neighbors v_s with $s > i+1$. There are at least $\delta - 1$ neighbors v_k of v_j with $k < i$. As we have picked all the $W(v_k)$ after finalizing $d_\omega(v_s)$ for all $s \in X$ with $s < i$, at most one of the pairs $W(v_k)$ may contain the weighted degrees of two vertices (namely, $d_\omega(v_k)$ and $d_\omega(v_i)$ if $j = i+1$). Thus, we may adjust $\omega_2(v_k v_j)$ on all these edges but possibly one, keeping $d_\omega(v_k) \in W(v_k)$.

This gives us $\delta - 1$ options for $d_\omega(v_j)$, an arithmetic progression with step size $2\lceil \frac{n}{\delta} \rceil$. These options completely contain at least $\frac{\delta-3}{2} \geq 2$ pairs in $\mathcal{W}$. At most one such pair may contain some $d_\omega(v_\ell)$ with $j > \ell \in X$ by our choice of $\omega_1(v_i v_{i+1})$, so there is a pair left which does not contain such a weighted degree. At most one vertex v_ℓ with $j > \ell \notin X$ can have its weighted degree in that pair. Adjust the weights $\omega_2(v_k v_j)$ so that $d_\omega(v_j)$ is in that pair. If now $d_\omega(v_j) = d_\omega(v_\ell)$, we may either change some weight $\omega_2(v_k v_j)$ with $k \neq \ell$ to move $d_\omega(v_j)$ to the other value in that pair, or change both $\omega_2(v_k v_j)$ and $\omega_2(v_\ell v_j)$ to keep $d_\omega(v_j)$ and to move $d_\omega(v_\ell)$ to the other value in that pair (which may be necessary if $v_\ell v_j \in E$). This concludes the proof. □

References

1. M. Aigner and E. Triesch, Irregular assignments of trees and forests. SIAM J. Discrete Math. 3(1990), 439–449.
2. G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular Networks. Congressus Numerantium 64 (1988), 187 – 192.
3. B. Cuckler and F. Lazebnik, Irregularity Strength of Dense Graphs, J. Graph Theory 58 (2008), 299–313.
4. R. Diestel, "*Graph Theory*," Springer Verlag, Heidelberg, 2005.
5. R.J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs. Colloq. Math. Soc. János Bolyai, 52, Combinatorics, Eger. North Holland, Amsterdam, 1987, 247–256.
6. A. Frieze, R. Gould., M. Karoński, F. Pfender, On Graph Irregulaity Strength, J. Graph Theory 41 (2002), 120–137.
7. M. Kalkowski, M. Karoński, F. Pfender, Vertex-Coloring Edge Weightings: Towards the 1-2-3-Conjecture, J. Combin. Theory (B), to appear.
8. T. Nierhoff, A tight bound on the irregularity strength of graphs. SIAM J. Discrete Math. 13 (2000), 313–323.

9. Przybyło, J. Irregularity strength of regular graphs, Electronic J. Combinatorics 15 (2008) (1)#R82
10. Przybyło J., Linear bound for on the irregularity strength and the total vertex irregularity strength of graphs, SIAM J. Discrete Math., to appear.

Current address, M. Kalkowski: Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Poznań, Poland
E-mail address: `kalkos@atos.wmid.amu.edu.pl`

Current address, M. Karoński: Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Poznań, Poland, and Emory University, Department of Mathematics and Computer Science, Atlanta, GA, USA
E-mail address: `karonski@amu.edu.pl`

Current address, F. Pfender: Universität Rostock, Institut für Mathematik, Rostock, Germany
E-mail address: `Florian.Pfender@uni-rostock.de`