Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs

Jennifer Diemunsch[*§] Michael Ferrara[*¶] Allan Lo[†]
Casey Moffatt[*] Florian Pfender[‡] and Paul S. Wenger[*]

Abstract

A *rainbow matching* in an edge-colored graph is a matching in which all the edges have distinct colors. Wang asked if there is a function $f(\delta)$ such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must have a rainbow matching of size δ. We answer this question in the affirmative; an extremal approach yields that $f(\delta) = 13\delta/3 - 2$ suffices. Furthermore, we give an $O(\delta(G)|V(G)|^2)$-time algorithm that generates such a matching in a properly edge-colored graph of order at least 6.5δ.

Keywords: Rainbow matching, properly edge-colored graphs

1 Introduction

All graphs under consideration in this paper are simple, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G, respectively. In this paper, we consider edge-colored graphs and let $c(uv)$ denote the color of the edge uv. An edge coloring of a graph is *proper* if the colors on edges sharing an endpoint are distinct. An edge-colored graph is *rainbow* if all edges have distinct colors. Rainbow matchings are of particular interest given their connection to transversals of Latin squares: each Latin square can be converted to a properly edge-colored complete bipartite graph, and a transversal of the Latin square is a rainbow perfect matching in the graph. Ryser's conjecture [7] that every Latin square of odd order has a transversal can be seen as the beginning of the study of rainbow

[*]Dept. of Mathematical and Statistical Sciences, Univ. of Colorado Denver, Denver, CO; email addresses `jennifer.diemunsch@ucdenver.edu`, `michael.ferrara@ucdenver.edu`, `casey.moffatt@ucdenver.edu`, `paul.wenger@ucdenver.edu`.

[†]School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK, `s.a.lo@bham.ac.uk`

[‡]Institut für Mathematik, Univ. Rostock, Rostock, Germany; `Florian.Pfender@uni-rostock.de`.

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matchings. Stein [8] later conjectured that every Latin square of order n has a transversal of size $n-1$; equivalently every proper edge-coloring $K_{n,n}$ has a rainbow matching of size $n-1$. The connection between Latin transversals and rainbow matchings in $K_{n,n}$ has inspired additional interest in the study of rainbow matchings in triangle-free graphs. A thorough survey of rainbow matching and other rainbow subgraphs in edge-colored graphs appears in [5].

Several results have been attained for rainbow matchings in arbitrarily edge-colored graphs. The *color degree* of a vertex v in an edge-colored graph G, written $\hat{d}(v)$, is the number of distinct colors on edges incident to v. We let $\hat{\delta}(G)$ denote the minimum color degree among the vertices in G. Wang and Li [10] proved that every edge-colored graph G contains a rainbow matching of size at least $\left\lceil \frac{5\hat{\delta}(G)-3}{12} \right\rceil$, and conjectured that a rainbow matching of size $\left\lceil \hat{\delta}(G)/2 \right\rceil$ exists if $\hat{\delta}(G) \geq 4$. LeSaulnier et al. [6] then proved that every edge-colored graph G contains a rainbow matching of size $\left\lfloor \hat{\delta}(G)/2 \right\rfloor$. Finally, Kostochka and Yancey [4] proved the conjecture of Wang and Li in full, and also that triangle-free graphs have rainbow matchings of size $\left\lceil 2\hat{\delta}(G)/3 \right\rceil$.

Since the edge-colored graphs generated by Latin squares are properly edge-colored, it is of interest to consider rainbow matchings in properly edge-colored graphs. In this direction, LeSaulnier et al. proved that a properly edge-colored graph G satisfying $|V(G)| \neq \delta(G) + 2$ that is not K_4 has a rainbow matching of size $\lceil \delta(G)/2 \rceil$. Wang then asked if there is a function f such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must contain a rainbow matching of size δ [9]. As a first step towards answering this question, Wang showed that a graph G with order at least $8\delta/5$ has a rainbow matching of size $\lfloor 3\delta(G)/5 \rfloor$.

Since there are $n \times n$ Latin squares with no transversals when n is even (see [1, 11]), for such a function f it is clear that $f(\delta) > 2\delta$ when δ is even. Furthermore, since maximum matchings in $K_{\delta,n-\delta}$ have only δ edges (provided $n \geq 2\delta$), there is no function for the order of G depending on $\delta(G)$ that can guarantee a rainbow matching of size greater than $\delta(G)$.

In this paper we answer Wang's question from [9] in the affirmative.

Theorem 1. *If G is a properly edge-colored graph satisfying $|V(G)| \geq 13\delta(G)/3 - 2$, then G contains a rainbow matching of size $\delta(G)$.*

The proof of Theorem 1 utilizes extremal techniques akin to those that appear in [4, 6, 9] and [10]. We also implement a greedy algorithmic approach to demonstrate that it is possible to efficiently construct a rainbow matching of size δ in a properly edge-colored graph with minimum degree δ having order at least 6.5δ. To our knowledge, an algorithmic approach of this type has not been previously employed in the study of rainbow matchings.

Theorem 2. *If G is a properly edge-colored graph with minimum degree δ satisfying $|V(G)| > \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$, then there is an $O(\delta(G)|V(G)|^2)$-time algorithm that produces a rainbow matching of size δ in G.*

2 Proof of Theorem 1

Let G be a properly edge-colored n-vertex graph with minimum degree δ and $n \geq 13\delta/3 - 2$. The theorem trivially holds if $\delta = 1$, so we may assume that $\delta \geq 2$. By way of contradiction, let G be a counterexample with δ minimized; thus G does not contain a rainbow matching of size δ. Further, we may assume that $|E(G)|$ is minimized, so in particular the vertices of degree greater than δ form an independent set, as otherwise we could delete an edge without lowering the minimum degree. We break the proof into a series of simple claims.

Let $\Delta(G) = d_1 \geq d_2 \geq \ldots \geq d_n = \delta$ with $d(v_i) = d_i$ be the degree sequence of G.

Lemma 3. *For $1 \leq k \leq 2\delta/3$, there exists an $i \leq k$ such that $d_i \leq 3\delta - k - 2i$.*

Proof. Suppose that for some $k \leq 2\delta/3$, $d_i \geq 3\delta + 1 - k - 2i$ for all $1 \leq i \leq k$. Delete the vertices $v_1, v_2, \ldots, v_k$ from G, and note that $\delta(G \setminus \{v_1, \ldots, v_k\}) \geq \delta - k$, so there exists a rainbow matching M_k on $\delta - k$ edges in $G \setminus \{v_1, \ldots, v_k\}$ by the minimality of G.

The vertex v_k has at most $2(\delta - k)$ neighbors in M_k, and is incident to at most $\delta - k$ edges colored with a color occurring in M_k. Thus, v_k has a neighbor $w_k \notin V(M_k)$ such that the color of $v_k w_k$ does not occur in M_k, so we can extend M_k to M_{k-1} by adding the edge $v_k w_k$. Note that $w_k \neq v_i$ for $i \leq k$ as $\{v_1, \ldots v_k\}$ contains no vertices of degree δ and is therefore an independent set.

Continuing, v_{k-1} has at most $2(\delta - k) + 1$ neighbors in M_{k-1}, and is incident to at most $\delta - k + 1$ edges colored with a color occurring in M_{k-1}. Thus, we can find $w_{k-1} \in N(v_{k-1})$ to extend M_{k-1} to M_{k-2}. Continuing backwards, we can extend the matching step by step to matchings M_i all the way to M_0, which is a rainbow matching of size δ, a contradiction finishing the proof. □

As a corollary of Lemma 3, we get the following lemma.

Lemma 4. *For $1 \leq k \leq 2\delta/3$, we have $\sum_{i=1}^{k} d_i \leq k(3\delta - 2 - k)$, with equality only if $d_1 = d_k = 3\delta - 2 - k$.*

Proof. We proceed by induction on k. For $k = 1$, the statement follows from Lemma 3. Now let $k > 1$ and let $i \le k$ such that $d_i \le 3\delta - k - 2i$. Then by induction,

$$\sum_{j=1}^{i-1} d_j \le (i-1)(3\delta - 1 - i), \text{ and}$$

$$\sum_{j=i}^{k} d_j \le (k-i+1)d_i \le (k-i+1)(3\delta - k - 2i).$$

Thus,

$$\sum_{j=1}^{k} d_j \le (i-1)(3\delta-1-i)+(k-i+1)(3\delta-k-2i) = 3k\delta-k^2-k+1-i(k+2-i) \le k(3\delta-2-k),$$

where equality can hold only for $i = 1$ and $d_k = d_1$. □

Claim 5. *The largest color class in G contains at least two edges.*

Proof. It suffices to prove that G has a matching of size δ, since two edges in such a matching must have the same color. Let G_1 be a component of G. Then either G_1 contains a path of length $2\delta - 1$, or a hamiltonian path. This is an easy consequence of the standard proof (see e.g. [2]) of Dirac's minimum degree condition for hamiltonian cycles [3].

A path of length $2\delta - 1$ contains a matching of size δ, so we may assume that every component of G has hamiltonian path of length at most $2\delta - 2$. As $|G| > 2\delta - 1$, G must contain at least two components. Since every component contains at least $\delta + 1$ vertices, the hamiltonian paths in each component contain matchings of size at least $\delta/2$, combining to a matching of size δ. □

Let C be a maximum color class in G and let $|C| = a$. By the minimality of G, there exists a rainbow matching $M = \{x_i y_i : 1 \le i \le \delta - 1\}$ of size $\delta - 1$ in $G - C$. Without loss of generality, we may assume that $c(x_i y_i) = i$ for $1 \le i \le \delta - 1$ and the edges in C have color δ. Let $W = V(G) \backslash V(M)$; observe that $|W| = n - 2(\delta - 1)$. If there is an edge c in $G[W]$ with $c(e) \notin \{1, \ldots, \delta - 1\}$ then we can add e to M to obtain a rainbow matching of size δ. Thus we may assume that every edge whose color is not in $\{1, \ldots, \delta - 1\}$ has an endpoint in $V(M)$. We say that an edge uv is *good* if its color is not in $\{1, \ldots, \delta - 1\}$ and one of its endpoints is in W. A vertex $v \in V(M)$ is *good* if v is incident with at least seven good edges.

Claim 6. *For $i \in \{1, \ldots, \delta - 1\}$, if x_i is incident with at least three good edges, then no good edge is incident with y_i, and vice versa.*

Proof. Suppose that $y_i u$ is a good edge. If x_i is incident with at least three good edges, then x has a neighbor w such that vw is a good edge, $w \neq u$, and $c(x_i w) \neq c(y_i u)$. Thus $(M \cup \{x_i w, y_i u\}) \backslash \{x_i y_i\}$ is a rainbow matching of size δ, a contradiction. □

By Claim 6, we may assume without loss of generality that $\{x_1, \ldots, x_r\}$ is the set of good vertices for some $r \geq 0$. Let $W' = W \cup \{y_1, \ldots, y_r\}$.

Claim 7. *No edge uv in $G[W']$ has color in $\{1, \ldots, r\}$.*

Proof. By way of contradiction, assume that there is an edge uv in $G[W']$ such that $c(uv) \in \{1, \ldots, r\}$. Let M' be the subset of M consisting of the edge with color $c(uv)$ and any edges with an endpoint in $\{u, v\}$. There are at most three such edges (the edge with color $c(uv)$ and possibly one for each endpoint); without loss of generality, let $M' = \{x_1 y_1, \ldots, x_t y_t\}$ (here $1 \leq t \leq 3$). Note that x_j is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that $x_j w_j$ is a good edge for $1 \leq j \leq t$ and the colors on the edges $uv, x_1 w_1, \ldots, x_t w_t$ are distinct. Thus $(M \cup \{uv, x_1 w_1, \ldots, x_t w_t\}) \backslash \{x_1 y_1, \ldots, x_t y_t\}$ is a rainbow matching of size δ, a contradiction. □

We say that the edge uv is *nice* if its color is not in $\{r+1, \ldots, \delta - 1\}$ and one of its endpoints is in W'. Note that every good edge is nice. Recall that every good edge has an endpoint in $V(M)$. By Claim 6 and Claim 7, no nice edge lies in $G[W']$. Hence, every nice edge joins vertices in W' and $V(G) \backslash W'$. A vertex $v \in V(M) \backslash \{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ is *nice* if v is incident with at least seven nice edges. Note that if there is no good vertex (i.e. $r = 0$), then the definitions of good and nice vertices are the same and so there is also no nice vertex. Next, we show the analogue of Claim 6 and Claim 7 for nice vertices and edges.

Claim 8. *For $i \in \{r+1, \ldots, \delta - 1\}$, if x_i is incident with at least three nice edges, then no nice edge is incident with y_i, and vice versa.*

Proof. Suppose $y_i u$ is a nice edge for some $i \in \{r+1, \ldots, \delta - 1\}$. If x_i is incident to at least three nice edges, then x_i has a neighbor v such that $x_i v$ is a nice edge, $v \neq u$, and $c(x_i v) \neq c(y_i u)$. Let M' be the subset of M consisting of edges with an endpoint in $\{u, v\}$ or a color in $\{c(x_i v), c(y_i u)\}$. There are at most four such edges (possibly one with each endpoint and one with each color); without loss of generality, let $M' = \{x_1 y_1, \ldots, x_t y_t\}$ (here $0 \leq t \leq 4$). Note that x_j is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that $x_j w_j$ is a good edge for $1 \leq j \leq t$ and the colors on the edges $x_i v, y_i u, x_1 w_1, \ldots, x_t w_t$ are distinct. Thus $(M \cup \{x_i v, y_i u, x_1 w_1, \ldots, x_t w_t\}) \backslash \{x_i y_i, x_1 y_1, \ldots, x_t y_t\}$ is a rainbow matching of size δ, a contradiction. □

By Claim 8, we may assume that $\{x_{r+1}, x_{r+2}, \ldots, x_{r+s}\}$ is the set of nice vertices for some $s \geq 0$.

Claim 9. *No edge uv in $G[W']$ has color in $\{1, \ldots, r+s\}$.*

Proof. By Claim 7, the claim holds if $s = 0$. Assume that $s \geq 1$, and consequently $r \geq 1$. Without loss of generality, suppose that there is an edge uv in $G[W']$ with $c(uw) = r+1$. Because x_{r+1} is nice, it has a neighbor v' in W' such that $x_{r+1}v'$ is a nice edge and $v' \neq u, v$. The rest of the proof is essentially the same as the proofs of Claims 7 and 8. Let M' be the subset of M consisting of those edges an endpoint in $\{u, v, v'\}$ or color $c(x_{r+1}v')$. Again there are at most four edges in M' and we let $M' = \{x_1y_1, \ldots, x_ty_t\}$. Defining $w_1, \ldots, w_t$ as before, $(M \cup \{uv, x_{r+1}v', x_1w_1, \ldots, x_tw_t\}) \setminus \{x_{r+1}y_{r+1}, x_1y_1, \ldots, x_ty_t\}$ is a rainbow matching of size δ, a contradiction. □

Next, we count the number of nice edges in G.

Claim 10. *There are at most* $\max\{(3\delta - 8 - r + s)r + 6(\delta - 1), (7\delta/3 - 7 + s)\,r + 6(\delta - 1)\}$ *nice edges in G.*

Proof. Recall that $V(G) \setminus W' = \{x_1, \ldots, x_{\delta-1}, y_{r+1}, \ldots, y_{\delta-1}\}$ and every nice edge joins vertices from W' and $V(G) \setminus W'$. For $r \leq 2\delta/3$, the set of good vertices is incident to at most $r(3\delta - 2 - r)$ nice edges by Lemma 4. Similarly, for $r > 2\delta/3$, the set of good vertices is incident to at most $r(3\delta - 2 - \lfloor 2\delta/3 \rfloor) \leq r(7\delta/3 - 1)$ nice edges. For $i \in \{r+1, \ldots, r+s\}$, since x_i is nice, by Claim 8 x_i is incident to at most $r+6$ nice edges and y_i is incident to none. For $i \in \{r+s+1, \ldots, \delta-1\}$, by Claim 8 there are at most six nice edges with an endpoint in $\{x_i, y_i\}$. Therefore, the number of nice edges is at most

$$(3\delta - 2 - r)r + (r+6)s + 6(\delta - 1 - r - s) = (3\delta - 8 - r + s)r + 6(\delta - 1) \text{ for } r \leq 2\delta/3,$$

and

$$(7\delta/3 - 1)\,r + (r+6)s + 6(\delta - 1 - r - s) = (7\delta/3 - 7 + s)\,r + 6(\delta - 1) \text{ for } r > 2\delta/3. \quad \square$$

Recall that C is the color class with color δ, $|C| = a$, and C is a maximum size color class. Therefore there are at least $2(a - \delta + 1)$ vertices in W incident to an edge in C. Since every edge in C has an endpoint in $V(M)$ it follows that there are at least $2(a - \delta + 1)$ vertices in $V(M)$ joined to W by edges in C. Without loss of generality, let $\{r+s+1, \ldots, r+s+t\}$ be the set of indices $i \in \{r+s+1, \ldots, \delta-1\}$ such that x_i or y_i is incident to an edge with color δ. By Claim 6 and Claim 8, we have

$$t \geq a - \delta + 1 - \frac{r+s}{2} \text{ and } r + s + t \leq \delta - 1. \tag{1}$$

Claim 11. *For $i \in \{r+s+1, \ldots, r+s+t\}$, there is at most one edge of color i in $G[W]$.*

Proof of claim. Suppose uv and $u'v'$ are edges of color i in $G[W]$ for some $i \in \{r+s+1, \ldots, r+s+t\}$. Without loss of generality, we may assume that there exists $w \in W$ such that $c(x_i w) = \delta$ and $w \neq u, v$. Hence, $(M \cup \{uv, x_i w\}) \setminus \{x_i y_i\}$ is a rainbow matching of size δ, a contradiction. □

Now, we count the number of nice edges from W' to $V(G) \setminus W'$. Recall that each color class in G contains at most a edges. By Claim 9, there is no edge in $G[W']$ of color $i \in \{r+1, \ldots, r+s\}$. Thus, for $i \in \{r+1, \ldots, r+s\}$ there are at most $a-1$ vertices in W' that are incident with an edge of color i. By Claim 11, for $i \in \{r+s+1, \ldots, r+s+t\}$, there are at most a vertices in W that are incident with an edge of color i. Recall that $W' \setminus W = \{y_1, \ldots, y_r\}$. Hence, for $i \in \{r+s+1, \ldots, r+s+t\}$, there are at most $a+r$ vertices in W' that are incident with an edge of color i. Since every color class has size at most a, for $i \in \{r+s+t+1, \ldots, \delta-1\}$, there are at most $2(a-1)$ vertices in W' that are incident with an edge of color i. It then follows, using the fact that $|W'| = |W| + r = n - 2(\delta-1) + r$ and (1), that the number of nice edges from W' to $V(G) \setminus W'$ is at least

$$\begin{aligned} & \delta|W'| - (a-1)s - (a+r)t - 2(a-1)(\delta-1-r-s-t) \\ = \; & \delta n - 2\delta(\delta-1) - (a-1)(2\delta-2-2r-s) + (a-2)t + (\delta-t)r \\ \geq \; & \delta n - 2\delta(\delta-1) - (a-1)(2\delta-2-2r-s) + (a-2)t + (r+s+1)r. \end{aligned}$$

Now assume that $r \leq 2\delta/3$. Since there are at most $(3\delta-8-r+s)r + 6(\delta-1)$ nice edges in G by Claim 10,

$$\delta n \leq (3\delta-9-2r)r - (a-2)t + 2(\delta+3)(\delta-1) + (a-1)(2\delta-2-2r-s). \tag{2}$$

To finish the proof we bound the right hand side of (2) to obtain a contradiction. Note that $-(a-2)$, the coefficient of t, is nonpositive by Claim 5. Thus the right hand side of (2) is maximized when t is minimized. By (1), $t \geq \max\{a-\delta+1-(r+s)/2, 0\}$.

If $a \leq \delta - 1 + (r+s)/2$, then we let $t = 0$. The coefficient of a becomes $2\delta-2-2r-s \geq 2(\delta-1-r-s) \geq 0$. Thus (2) is maximized when a is maximized, and evaluating at $a = \delta-1+(r+s)/2$ yields

$$\delta n \leq 2(2\delta+1)(\delta-1) + (2\delta-6-3r)r - (3r+s-2)s/2.$$

Recall that if $s \geq 1$, then $r \geq 1$. Hence, $(3r+s-2)s \geq 0$ and so

$$\delta n \leq 2(2\delta+1)(\delta-1) + (2\delta-6-3r)r.$$

This is maximized when $r = \delta/3 - 1$, yielding $n \leq 13\delta/3 - 4 + 1/\delta$. Since $\delta \geq 2$ this is a contradiction.

If $a \geq \delta - 1 + (r+s)/2$, we let $t = a - \delta + 1 - (r+s)/2$. Then, (2) becomes

$$\delta n \leq (3\delta - 1 - (3r+s)/2 - a)a + (3\delta - 8 - 2r)r + 2(\delta+1)(\delta-1).$$

If $(3\delta - 1)/2 - (3r+s)/4 \leq \delta - 1 + (r+s)/2$, then the right hand side is maximized when $a = \delta - 1 + (r+s)/2$, which corresponds to the case when $a \leq \delta - 1 + (r+s)/2$ and so we are done. Letting $a = (3\delta-1)/2 - (3r+s)/4$ yields

$$\begin{aligned}\delta n \leq& \frac{1}{4}\left(3\delta - 1 - \frac{3r+s}{2}\right)^2 + (3\delta - 8 - 2r)r + 2(\delta+1)(\delta-1)\\ =& \left(-\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4}\right)r + \left(\frac{3r}{8} + \frac{s}{16} - \frac{3\delta}{4} + \frac{1}{4}\right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4}\\ =& \left(-\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4}\right)r + \left(\frac{6r + s + 4 - 12\delta}{16}\right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4}\end{aligned}$$

Since $\delta \geq r + s + 1$, this is maximized when $s = 0$, yielding

$$\delta n \leq \left(-\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4}\right)r + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4}. \tag{3}$$

The right hand side of (3) is maximized at $r = 2(29 - 3\delta)/23 \leq 2$, yielding

$$n \leq 17\delta/4 - \min\{2, 22/\delta\} < 13\delta/3 - 2,$$

a contradiction.

To complete the proof of the theorem, we are left with the case $r > 2\delta/3$. Similarly to (2), since we have at most $(7\delta/3 - 7 + s)\,r + 6(\delta - 1)$ nice edges in G by Claim 10, we have

$$\delta n \leq \;(7\delta/3 - 8 - r)\,r - (a-2)t + 2(\delta+3)(\delta-1) + (a-1)(2\delta - 2 - 2r - s). \tag{4}$$

Again the right hand side of (4) is maximized when t is minimized, and $t \geq \max\{a - \delta + 1 - (r+s)/2, 0\}$.

If $a \leq \delta - 1 + (r+s)/2$, (4) is maximized when $t = 0$ and $a = \delta - 1 + (r+s)/2$, which yields with $(3r + s - 2)s \geq 0$

$$\delta n \leq 2(2\delta+1)(\delta-1) + (4\delta/3 - 5 - 2r)r.$$

This is maximized when $r = 2\delta/3$, yielding $n < 4\delta$, a contradiction.

If $a \geq \delta - 1 + (r+s)/2$, we let $t = a - \delta + 1 - (r+s)/2$. Then, (4) becomes

$$\delta n \leq (3\delta - 1 - (3r+s)/2 - a)a + (7\delta/3 - 7 - r)r + 2(\delta+1)(\delta-1).$$

If $(3\delta - 1)/2 - (3r+s)/4 \leq \delta - 1 + (r+s)/2$, then the right hand side is maximized when $a = \delta - 1 + (r+s)/2$, which corresponds to the case when $a \leq \delta - 1 + (r+s)/2$ and so we are done. On the other hand, if $(3\delta - 1)/2 - (3r+s)/4 > \delta - 1 + (r+s)/2$, then

$$10\delta/3 \leq 5r \leq 5r + 3s < 2(\delta + 1),$$

a contradiction finishing the proof of the theorem. □

3 Proof of Theorem 2

We proceed by induction on $\delta(G)$. The result is trivial if $\delta(G) = 1$. We assume that G is a graph with minimum degree $\delta > 1$ and order greater than $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$.

Lemma 12. *If G has a color class containing at least $2\delta - 1$ edges, then G has a rainbow matching of size δ.*

Proof. Let C be a color class with at least $2\delta - 1$ edges. By induction, there is a rainbow matching M of size $\delta - 1$ in $G - C$. There are $2\delta - 2$ vertices covered by the edges in M, thus one of the edges in C has no endpoint covered by M, and the matching can be extended. □

It is also useful to note that we also have the following, which is identical to Lemma 3 when $k = 1$.

Lemma 13. *If G satisfies $\Delta(G) > 3\delta - 3$, then G has a rainbow matching of size δ.*

We begin by preprocessing the graph so that each edge is incident to at least one vertex with degree δ. To achieve this, arbitrarily order the edges in G and process them in order. If both endpoints of an edge have degree greater than δ when it is processed, delete that edge. In the resulting graph, every edge is incident to a vertex with degree δ. Furthermore, by Lemma 13 we may assume that $\Delta(G) \leq 3\delta - 3$; thus the degree sum of the endpoints of any edge is bounded above by $4\delta - 3$. After preprocessing, we begin the greedy algorithm.

In the ith step of the algorithm, a smallest color class is chosen (without loss of generality, color i), and then an edge e_i of color i is chosen such that the degree sum of the endpoints is minimized. All the remaining edges of color i and all edges incident with the endpoints of e_i are deleted. The algorithm terminates when there are no edges in the graph.

We assume that the algorithm fails to produce a matching of size δ in G; suppose that the rainbow matching M generated by the algorithm has size k. We let R denote the set of vertices that are not covered by M.

Let c_i denote the size of the smallest color class at step i. Since at most two edges of color $i+1$ are deleted in step i (one at each endpoint of e_i), we observe that $c_{i+1}+2 \geq c_i$. Otherwise, at step i color class $i+1$ has fewer edges. Let step h be the last step in the algorithm in which a color class that does not appear in M is completely removed from G. It then follows that $c_h \leq 2$, and in general $c_i \leq 2(h-i+1)$ for $i \in [h]$. Let f_i denote the number of edges of color i deleted in step i with both endpoints in R. Since $f_i < c_i$, we have $f_i \leq 2(h-i)+1$ for $i \in [h]$. Note that after step h, there are exactly $k-h$ colors remaining in G. By Lemma 12, color classes contain at most $2\delta - 2$ edges, and therefore the last $k-h$ steps remove at most $(k-h)(2\delta-2)$ edges. Furthermore, for $i > h$, the degree sum of the endpoints of e_i is at most $2(\delta-1)$.

For $i \in [h]$, let x_i and y_i be the endpoints of e_i, and let $d_i(v)$ denote the degree of a vertex v at the beginning of step i. Let $\mu_i = \max\{0, d_i(x_i)+d_i(y_i)-2\delta\}$; note that $2\delta \leq 2\delta + \mu_i \leq 4\delta - 3$. Thus, at step i, at most $2\delta + \mu_i + f_i - 1$ edges are removed from the graph. Since the algorithm removes every edge from the graph, we conclude that

$$|E(G)| \leq (k-h)(2\delta-2) + \sum_{i=1}^{h}(2\delta + \mu_i + f_i - 1). \tag{5}$$

We now compute a lower bound for the number of edges in G. Since the degree sum of the endpoints of e_i in G is at least $2\delta + \mu_i$, we immediately obtain the following inequality:

$$\frac{n\delta + \sum_{i\in[h]} \mu_i}{2} \leq |E(G)|.$$

If $f_i > 0$ and $\mu_i > 0$, then there is an edge with color i having both endpoints in R. Since this edge was not chosen in step i by the algorithm, the degree sum of its endpoints is at least $2\delta+\mu_i$, and one of its endpoints has degree at least $\delta+\mu_i$. For each value of i satisfying $f_i > 0$, we wish to choose a representative vertex in R with degree at least $\delta+\mu_i$. Since there are f_i edges with color i having both endpoints in R, there are f_i possible representatives for color i. Since a vertex in R with high degree may be the representative for multiple colors, we wish to select the largest system of distinct representatives.

Suppose that the largest system of distinct representatives has size t, and let T be the set of indices of the colors that have representatives. For each color $i \in T$ there is a distinct vertex in R with degree at least $\delta+\mu_i$. Thus we may increase the edge count of G as follows:

$$\frac{n\delta + \sum_{i\in[h]} \mu_i + \sum_{i\in T} \mu_i}{2} \leq |E(G)|. \tag{6}$$

We let $\{f_i^{\downarrow}\}$ denote the sequence $\{f_i\}_{i\in[h]}$ sorted in nonincreasing order. Since $f_i \leq 2(h-i)+1$, we conclude that $f_i^{\downarrow} \leq 2(h-i)+1$. Because there is no system of distinct representatives of size $t+1$, the sequence $\{f_i^{\downarrow}\}$ cannot majorize the sequence $\{t+1, t, t-1, \ldots, 1\}$. Hence there is a smallest value $p \in [t+1]$ such that $f_p^{\downarrow} \leq t+1-p$. Therefore, the maximum value of $\sum_{i=1}^{h} f_i^{\downarrow}$ is bounded by the sum of the sequence $\{2h-1, 2h-3, \ldots, 2(h-p)+3, t+1-p, \ldots, t+1-p\}$. Summing we attain

$$\sum_{i\in[h]} f_i \leq (p-1)(2h-p+1) + (h-p+1)(t+1-p).$$

Over p, this value is maximized when $p = t+1$, yielding $\sum_{i\in[h]} f_i \leq t(2h-t)$. Since $h \leq \delta - 1$, we then have $\sum_{i\in[h]} f_i \leq 2(\delta-1)t - t^2$.

We now combine bounds (5) and (6):

$$\frac{n\delta + \sum_{i\in[h]} \mu_i + \sum_{i\in T} \mu_i}{2} \leq (k-h)(2\delta-2) + \sum_{i=1}^{h}(2\delta + \mu_i + f_i - 1).$$

Hence, since $k \leq \delta - 1$,

$$\begin{aligned}\frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2}\sum_{[h]\setminus T} \mu_i + \sum_{i\in[h]} f_i \\ &\leq (2\delta-1)(\delta-1) + (\delta-1-t)(\delta-3/2) + 2(\delta-1)t - t^2 \\ &\leq 3\delta^2 - \frac{11}{2}\delta + \frac{5}{2} + \left(\delta - \frac{1}{2}\right)t - t^2.\end{aligned}$$

This bound is maximized when $t = (\delta - \frac{1}{2})/2$. Thus

$$n \leq \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

contradicting our choice for the order of G.

It remains to show that the proof given above provides the framework of a $O(\delta(G)|V(G)|^2)$-time algorithm that generates a rainbow matching of size $\delta(G)$ in a properly edge-colored graph G of order at least $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$. Given such a G, we create a sequence of graphs $\{G_i\}$ as follows, letting $G = G_0$, $\delta = \delta(G)$, and $n = |V(G)|$. First, determine $\delta(G_i)$, $\Delta(G_i)$, and the maximum size of a color class in G_i; this process takes $O(n^2)$-time. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ and the maximum color class has at most $2\delta(G_i) - 2$ edges, then terminate the sequence and set $G_i = G'$. If $\Delta(G_i) > 3\delta(G_i) - 3$, then delete a vertex v of maximum degree and then process the edges of $G_i - v$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ but a maximum color class C has at least $2\delta(G_i) - 1$ edges, then delete C and then process the edges of $G_i - C$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. Note that $\delta(G_{i+1}) = \delta(G_i) - 1$. If this process generates G_δ, we set $G' = G_\delta$ and terminate. Generating the sequence $\{G_i\}$ consists of at most δ steps, each taking $O(n^2)$-time.

Given that $G' = G_i$, the algorithm from the proof of Theorem 2 takes $O(\delta n^2)$-time to generate a matching of size $\delta - i$ in G'. The preprocessing step and the process of determining a smallest color class and choosing an edge in that class whose endpoints have minimum degree sum both take $O(n^2)$-time. This process is repeated at most δ times.

A matching of size $\delta - (i + 1)$ in G_{i+1} is easily extended in G_i to a matching of size $\delta - i$ using the vertex of maximum degree or maximum color class. The process of extending the matching takes $O(\delta)$-time. Thus the total run-time of the algorithm generating the rainbow matching of size δ in G is $O(\delta n^2)$. □

It is worth noting that the analysis of the greedy algorithm used in the proof of Theorem 2 could be improved. In particular, the bound $c_{i+1} \geq c_i - 2$ is sharp only if at step i there are an equal number of edges of color i and $i + 1$ and both endpoints of e_i are incident to edges with color $i + 1$. However, since one of the endpoints of e_i has degree at most δ, at most $\delta - 1$ color classes can lose two edges in step i. Since the maximum size of a color class in G is at most $2\delta - 2$, if G has order at least 6δ, then there are at least $3\delta/2$ color classes. Thus, for small values of i, the bound $c_i \leq 2(k - i + 1)$ can likely be improved. However, we doubt that such analysis of this algorithm can be improved to yield a bound on $|V(G)|$ better than 6δ.

References

[1] R. A. Brualdi and H. J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, Cambridge, UK, 1991.

[2] R. Diestel, Graph Theory, Springer-Verlag New York (1997).

[3] G. A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc.* (3) 2, (1952), 6981.

[4] A. Kostochka and M. Yancey, Large Rainbow Matchings in Edge-Colored Graphs. In Preparation.

[5] M. Kano and X. Li, Monochromatic and Heterochromatic subgraphs in edge-colored graphs – a survey, *Graphs Comb.* **24** (2008), 237-263.

[6] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, Rainbow matching in edge-colored graphs. *Electron. J. Combin.* 17 (2010), Note #N26.

[7] H. J. Ryser, Neuere Probleme der Kombinatorik, in "Vorträge über Kombinatorik Oberwolfach". Mathematisches Forschungsinstitut Oberwolfach, July 1967, 24-29.

[8] S. K. Stein, Transversals of Latin squares and their generalizations. *Pacific J. Math.* 59 (1975), no. 2, 567-575.

[9] G. Wang, Rainbow matchings in properly edge colored graphs. *Electron. J. Combin.* 18 (2011), Paper #P162.

[10] G. Wang and H. Li, Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.* 15 (2008), Paper #R138.

[11] I.M. Wanless, Transversals in Latin Squares: a survey, *Surveys in Combinatorics 2011*, London Math. Soc. (2011).