Extremal Graphs Having No Stable Cutsets

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Abstract

A stable cutset in a graph is a stable set whose deletion disconnects the graph. It was conjectured by Caro and proved by Chen and Yu that any graph with n vertices and at most $2n-4$ edges contains a stable cutset. The bound is tight, as we will show that all graphs with n vertices and $2n-3$ edges without stable cutset arise recursively glueing together triangles and triangular prisms along an edge or triangle. As a by-product, an algorithmic implication of our result will be pointed out.

Keywords: stable cutset; independent cutset; fragile graph; extremal graph

1 Introduction

All graphs considered are finite and have no loops or multiple edges. For a graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$, write $|G| = |V(G)|$ and $\|G\| = |E(G)|$. A *stable set* (or an *independent set*) in G is a set of pairwise non-adjacent vertices. A *cutset* (or *separator*) of G is a set S of vertices such that $G - S$ is disconnected. A *stable cutset* in G is a cutset of G which is also a stable set. Throughout this paper, we consider the empty set as stable. So in particular, every disconnected graph contains a stable cut set. It is naturally expected that graphs with few edges would have stable cutsets. Indeed, the following theorem was conjectured by Caro and proved by Chen and Yu.

Theorem 1 ([4]). *Let G be a graph with $\|G\| \leqslant 2|G| - 4$. Then G contains a stable cutset.*

Small stable cutsets are discussed in [3], and algorithmic and complexity aspects of stable cutsets are discussed in [5, 1, 7, 8, 9]. The importance of stable cutsets in connection to perfect graphs are demonstrated in [6, 11]. In [2] it is noted that graphs containing stable cutsets play a role in some decomposition algorithms.

Actually, Chen and Yu proved the following stronger result.

Theorem 2 ([4]). *Let G be 2-connected a graph with $\|G\| \leqslant 2|G| - 4$. Then for every vertex $x \in V(G)$, there is a stable cutset not containing x.*

This implies immediately the following corollary; a vertex x is a cut vertex if $\{x\}$ is a cutset.

Corollary 3. *Let G be a graph with $\|G\| \leqslant 2|G| - 4$, and $x \in V(G)$. Unless x is the unique cut vertex in G, there is a stable cutset not containing x.*

The bound in Theorem 1 is tight. In the next section we describe all graphs with n vertices and $2n - 3$ edges that have no stable cutset (Theorem 5). In the last section we will point out an algorithmic implication of our result.

Notation and definitions. Let G be a graph. The complement of G is written $\overline{G}$. The neighborhood of a vertex v in G, denoted by $N_G(v)$, is the set of all vertices in G adjacent to v; if the context is clear, we simply write $N(v)$. Set $\deg(v) = |N(v)|$, the degree of the vertex v. For a subset $W \subseteq V(G)$, $N(W) = \bigcup_{w \in W} N(w) \setminus W$, and $G[W]$ is the subgraph of G induced by W; write $G - W = G[V(G) \setminus W]$ and $G - w = G - \{w\}$. Given another graph H, an H-cutset S of G is a cutset such that $G[S]$ is isomorphic to H, while a k-cutset is a k-element cutset. An edge cut of G is a set M of edges such that $G - M = (V(G), E(G) \setminus M)$ is disconnected. A *matching cut* in G is an edge cut of G that is also a matching.

P_k stands for the path with k vertices and $k - 1$ edges, C_k is the cycle with k vertices and k edges. A complete graph with k vertices is denoted K_k; K_k^- is K_k minus an edge. The K_3 is also called a triangle and the $\overline{C_6}$ is also called a triangular prism; see Figure 1.

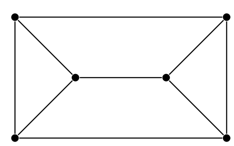

Figure 1: The triangular prism $\overline{C_6}$.

We will make use of the following well-known graph operation. A clique in a graph is a set of pairwise adjacent vertices. Let G_1, G_2 be disjoint graphs which each have nonempty cliques Q_1, respectively, Q_2 of the same size. A graph obtained from G_1 and G_2 by first choosing a bijection $f : Q_1 \to Q_2$ and then identifying each x in Q_1 with $f(x)$ in Q_2 is said to arise from G_1 and G_2 by clique identification. If the chosen cliques have two, respectively, three vertices, we also speak of *edge identifications*, respectively, *triangle identifications*. Finally, for convenience, we consider G_1 and G_2 as induced subgraphs of the graph arising from G_1 and G_2 by clique identification. Thus, a graph G arises from two graphs by clique identification if and only if there exist induced subgraphs G_1 and G_2 in G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique.

2 The Result

Let $\mathcal{G}_{sc}$ be the the class of graphs one gets by recursively glueing together triangles and triangular prisms along an edge or triangle. More precisely,

1. $K_3 \in \mathcal{G}_{sc}$ and $\overline{C_6} \in \mathcal{G}_{sc}$.
2. If $G_1, G_2 \in \mathcal{G}_{sc}$ and G is obtained from G_1 and G_2 by edge identification, then $G \in \mathcal{G}_{sc}$.
3. If $G_1, G_2 \in \mathcal{G}_{sc}$ and G is obtained from G_1 and G_2 by triangle identification, then $G \in \mathcal{G}_{sc}$.

Notice that we may restrict to $G_2 \in \{K_3, \overline{C_6}\}$ in the above definition without changing the class $\mathcal{G}_{sc}$, which effects the complexity of the algorithm considered in the last section of the paper.

Proposition 4. *Any graph $G \in \mathcal{G}_{sc}$ has $\|G\| = 2|G| - 3$ edges and no stable cutset.*

Proof. The statement is obvious for triangles and triangular prisms. Let G arise from $G_1, G_2 \in \mathcal{G}_{sc}$ by edge or triangle identification, and write $G = G_1 \cup G_2$ with clique $Q = G_1 \cap G_2$ of size two or three. Then

$$|G| = |G_1| + |G_2| - |Q| \quad \text{and} \quad \|G\| = \|G_1\| + \|G_2\| - \|Q\|,$$

and hence, by induction,

$$\|G\| = (2|G_1| - 3) + (2|G_2| - 3) - \|Q\| = (2|G| - 3) + (2|Q| - 3 - \|Q\|) = 2|G| - 3.$$

Note that, as Q is a clique, any stable cutset in G is also a stable cutset in G_1 or G_2. Hence, by induction again, G has no stable cutset. □

Theorem 5. *Let G be a graph with $\|G\| \leqslant 2|G| - 3$. Then G contains a stable cutset or $G \in \mathcal{G}_{sc}$.*

Proof. Our proof starts with a number of claims. For the sake of contradiction, we assume that G is a minimal counterexample to Theorem 5. Then, by Theorem 1,

Claim 6. $\|G\| = 2|G| - 3$.

Claim 7. *Every vertex v lies in a triangle.*

Otherwise, $N(v)$ would be a stable cutset in G.

Claim 8. *G contains no K_2-cutset and no K_3-cutset.*

Otherwise, let G contain a cutset Q isomorphic to K_2 or K_3. Write $G = G_1 \cup G_2$ with $G_1 \cap G_2 = Q$. Since G has no stable cutset and Q is a clique, G_1 and G_2 have no stable cutset. By Theorem 1, $\|G_i\| \geqslant 2|G_i| - 3$, $i = 1, 2$, hence, by Claim 6, $\|G_i\| = 2|G_i| - 3$. Therefore, by the minimality of G, $G_i \in \mathcal{G}_{sc}$, and thus $G \in \mathcal{G}_{sc}$, a contradiction.

Claim 9. *G is 3-connected.*

Otherwise, by Claim 8, G would contain a stable cutset.

Claim 10. *G contains no 3-edge matching cut.*

Otherwise, let $M = \{x_1y_1, x_2y_2, x_3y_3\}$ be a matching cut of G. Since G is 2-connected, $G - M$ has exactly two components, say G_1 and G_2. Then the set of all edges between G_1 and G_2 is exactly M, and we may assume that $x_1, x_2, x_3 \in V(G_1)$, $y_1, y_2, y_3 \in V(G_2)$. Now, if $\{x_1, x_2, x_3\}$ is not a clique, say $x_1x_2 \notin E(G)$, then $\{x_1, x_2, y_3\}$ is a stable cutset of G, a contradiction. So, $\{x_1, x_2, x_3\}$ and, by symmetry, $\{y_1, y_2, y_3\}$ are cliques. Since $G \neq \overline{C_6}$, at least one of these cliques must be a cutset of G, contradicting Claim 8.

Claim 11. *G contains no K_4^-.*

Otherwise, contract the edge between the two vertices of degree 3 in this (not necessarily induced) subgraph, resulting in a graph G'. By Claim 9, G' is 2-connected. By Claim 6, $\|G'\| \leqslant \|G\| - 3 = 2|G| - 6 = 2|G'| - 4$. By Theorem 2, G' contains a stable cutset not containing the new vertex, which is also a stable cutset in G.

Claim 12. *For any two non-adjacent vertices x, y we have $|N(x) \cap N(y)| \leqslant 2$.*

Otherwise, contract the two vertices, and get a (2-connected) graph G' with $\|G'\| \leqslant 2|G'| - 4$. Then, G' has a stable cutset by Theorem 1 which yields a stable cutset in G.

Claim 13. *G contains no P_3-cutset.*

Otherwise, let $\{x, y, z\}$ be a cutset of G such that $xy, yz \in E(G)$, and let G_1 and G_2 be induced subgraphs of G with $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{x, y, z\}$. Then

$$\|G_1\| + \|G_2\| = \|G\| + 2 = 2|G| - 1 = 2|G_1| + 2|G_2| - 7.$$

Thus, by symmetry $\|G_1\| \leqslant 2|G_1| - 4$, and note that, by Claim 9, y is not a cut vertex of G_1. Therefore, by Corollary 3, G_1 contains a stable cutset not containing y. But this is then also a stable cutset in G.

Claim 14. *In every triangle, at least two vertices belong to other triangles as well.*

Proof of Claim 14: Assume that xyz is a triangle and y and z are in no other triangles. Then there is an edge $y'z'$ with $y' \in N(y)$ and $z' \in N(z)$ as otherwise, by Claim 11, $(N(y) \cup N(z)) \setminus \{y, z\}$ is a stable cutset. Contracting $\{y, z'\}$ to a new vertex v yields a graph G' with $\|G'\| \leqslant 2|G'| - 3$. Since every stable cutset in G' yields a stable cutset in G, G' has no stable cutset. So $G' \in \mathcal{G}_{sc}$ by the minimality of G.

Now assume that G' contains a 3-edge matching cut M. Then by Claim 10, v is in one of the edges in M, say $M = \{au, bv, cw\}$, and further, we have $by', bz \in E(G)$. Let the two triangles in G' enclosing M be abc and uvw, where $b \in \{y', z\}$ by Claim 12.

Let G_1 be the component of $G - \{au, by', bz, cw\}$ containing abc. By Claim 8, $G_1 = abc$. Let $G_2 = G - G_1$, and note that $\|G_2\| = 2|G_2| - 4$. Further, G_2 is 2-connected as any 1-cutset in G_2 would yield a 2-cutset in G if we add b to it. We may assume by symmetry that $z'w \in E(G)$ (we will not use the fact that y lies in only one triangle, so y and z' are symmetric in the following argument). Further assume that $yu \in E(G)$. By Theorem 2, G_2 contains a stable cutset X not containing w. As X is not a stable cutset of G, y is in a different component of $G_2 - X$ than w. Thus, $u \in X$ and $X \cup \{b\}$ is a cutset of G, and as this is not a stable cutset, $z' \in X$. But then $X \cup \{c\}$ is a stable cutset in G, a contradiction. So $yu \notin E(G)$ and therefore $z'u \in E(G)$. By Theorem 2, G_2 contains a stable cutset X not containing z', with y and z' in different components of $G_2 - X$. But then $X \cup \{b\}$ is a stable cutset in G, a contradiction. Therefore, G' contains no 3-edge matching cut.

As a result, G' can be built by starting with a triangle and recursively glueing on triangles along an edge (G' is a so-called 2-tree). As G is 3-connected, every such 2-cutset in G' has the form uv, and u is connected to exactly one of y and z' by Claim 13. Further, every vertex of degree at least 3 in G' lies in such a cutset.

On the other hand, there are at least two vertices of degree 2 in G'. As G is 3-connected, such vertices must lie in $N_G(y) \cap N_G(z')$, but by Claim 12, $N_G(y) \cap N_G(z') = \{y', z\}$. Thus, exactly the two vertices y' and z have degree 2 in G', and $N_G(z) = \{x, y, z'\}$. By a symmetric argument using a contraction of $\{y', z\}$ instead of $\{y, z'\}$ in the beginning, $N_G(y) = \{x, y', z\}$. But this implies that $N_G(z') = V(G) \setminus \{y, z'\}$, as every vertex in $V(G) \setminus \{y, z'\}$ is in $N_{G'}(v) = N_G(y) \cup N_G(z')$. In particular, $xz' \in E(G)$. This contradicts Claim 11, as $G[\{x, y, z, z'\}]$ is then a K_4^-, hence Claim 14 follows.

Consider the vertex-triangle incidence graph H of G, i.e., the bipartite graph with partite sets $V(G)$ and the set of all triangles $T(G)$ in G, with an edge between a vertex $v \in V(G)$ and a triangle $T \in T(G)$ if $v \in V(T)$. By Claim 14, H is not a tree.

Let $x_1 T_1 x_2 T_2 \ldots x_k T_k x_1$ be a shortest cycle in H. By Claim 11, H has no cycles of length less or equal to 6, so $k \geqslant 4$. Then $C = x_1 x_2 \ldots x_k x_1$ is a cycle in G, and $V(T_i) \setminus V(C)$ consists of a distinct vertex for every $1 \leqslant i \leqslant k$.

If we contract $P = x_1 x_2 \ldots x_{k-1}$ to a new vertex v, we get a graph G' with $\|G'\| \leqslant 2|G'| - 4$. If v is not the unique cut vertex of G', then we can use Corollary 3 to find a stable cutset of G' not containing v, which is then also a stable cutset of G, a contradiction. Thus, v is the unique cut vertex of G'. Let Y be a component of $G \setminus V(P)$ and $1 \leqslant r \leqslant s \leqslant k-1$, such that

$$\{x_r, x_s\} \subseteq N(Y) \cap V(P) \subseteq \{x_r, \ldots, x_s\},$$

and

$$N(Z) \cap V(P) \setminus \{x_{r+1}, \ldots, x_{s-1}\} \neq \emptyset$$

for all components Z of $G \setminus V(P)$. Note that $s \geqslant r + 2$ as G is 3-connected. Now contract $x_{r+1} \ldots x_{s-1}$ to a new vertex x and call the resulting graph G''. Then $\|G''\| \leqslant 2|G''| - 3$, and let $G_1 := G''[Y \cup \{x_r, x, x_s\}]$ and $G_2 := G'' \setminus Y$. As in Claim 13, we have

$$\|G_1\| + \|G_2\| = \|G''\| + 2 \leqslant 2|G''| - 1 = 2|G_1| + 2|G_2| - 7.$$

Further, x is neither a cut vertex of G_1 nor of G_2. Thus, by Corollary 3, either G_1 or G_2 has a stable cutset not containing x. But this is also a stable cutset of G, a contradiction. $\square$

3 Complexity Issues

With STABLE CUTSET we mean the following decision problem: 'Does a given graph admit a stable cutset?' The computational complexity of STABLE CUTSET has been addressed in a number research papers, e.g., [5, 1, 7, 8, 9]. To sum up, STABLE CUTSET is NP-complete for graphs of maximum degree five (even for K_4-free planar graphs with maximum degree five [8] and for 5-regular line graphs of bipartite graphs [9]), and is trivial for graphs of maximum degree three (by Theorem 1, such graphs with more than seven vertices always have a stable cutset). The complexity status of STABLE CUTSET is still open for graphs with maximum degree four.

By Theorem 1, STABLE CUTSET for graphs with maximum degree four remains open only in four cases, namely for graphs with n vertices and m edges where $2n-3 \leqslant m \leqslant 2n$.

Thus, the following problem is of interest and has been addressed in [9, 10]:

STABLE CUTSET(n, m). *Given a graph G with n vertices and m edges. Does G have a stable cutset?*

It was shown in [9] that, for any given $\epsilon > 0$, STABLE CUTSET(n, m) is NP-complete for $m \geqslant (2 + \epsilon)n$. By Theorem 1, STABLE CUTSET(n, m) is trivial for $m \leqslant 2n - 4$. By Theorem 5, we obtain the following:

Corollary 15. STABLE CUTSET$(n, 2n - 3)$ *is solvable in polynomial time.*

Proof. Let G be a graph with n vertices and $m = 2n - 3$ edges. Then, by Theorem 5, G has a stable cutset, or else G must belong to $\mathcal{G}_{sc}$. Since the members of $\mathcal{G}_{sc}$ can be recognized in time $O(n^4)$ in an obvious way, Corollary 15 follows.

In fact, the recognition of $G \in \mathcal{G}_{sc}$ can be performed in quadratic time, based on the following observations. For every edge xy, we can in linear time test if $\{x, y\}$ is a cutset and determine the components of $G - \{x, y\}$. If $G \in \mathcal{G}_{sc}$, performing this for all $2n - 3$ edges, this process yields in quadratic time a set of at most $n-2$ components with at most a total of $3n - 6$ vertices, where each component is obtained from copies of K_3 and $\overline{C_6}$ via triangle identification. In particular, every vertex is in exactly one K_3. Further, every vertex of degree 2 is in a K_3-component, and the non-separating K_3 in other components consist exactly of the vertices of degree 3. This way, we can easily recover the building blocks, the $\overline{C_6}$, used to build up the components in quadratic time, by cutting off one $\overline{C_6}$ which includes a non-separating K_3 at a time (linear time for each step, linear number of steps).

With a bit more effort, one can show that one can decide if $G \in \mathcal{G}_{sc}$ in time $O(n \log n)$, but for the sake of exposition we do not present the argument here. □

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