QUARTIC GRAPHS WITH EVERY EDGE IN A TRIANGLE

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Abstract. We characterise the quartic (i.e. 4-regular) multigraphs with the property that every edge lies in a triangle. The main result is that such graphs are either squares of cycles, line multigraphs of cubic multigraphs, or are obtained from the line multigraphs of cubic multigraphs by a number of simple subgraph-replacement operations. A corollary of this is that a simple quartic graph with every edge in a triangle is either the square of a cycle or a graph obtained from the line graph of a cubic graph by replacing triangles with copies of $K_{1,1,3}$.

1. Introduction

One of the most fundamental properties of a graph is whether it contains triangles, and if so, whether it has many triangles or few triangles, and many authors have studied classes of graphs that are extremal in some sense with respect to their triangles. While studying an unrelated graphical property, the authors were led to consider the class of regular graphs with the extremal property that *every edge* lies in a triangle; a property that henceforth we denote *the triangle property.*

Although it seems impossible to characterise regular graphs of arbitrary degree with the triangle property, there are considerable structural restrictions on a graph with the triangle property when the degree is sufficiently low. In particular, when the degree is 4, these restrictions are so strong that we can obtain a precise structural description of the family of quartic graphs with the triangle property, indeed even the class of quartic *multigraphs* with the triangle property.

To state the result, we first need two basic families of quartic multigraphs with the triangle property. The *squared n-cycle* C_n^2 is usually defined to be the graph obtained from the cycle C_n by adding an edge between each pair of vertices at distance 2. However for our purposes, we want to be more precise about multiple edges, and so for $n \geq 3$, we define C_n^2 as the Cayley *multigraph*[1] $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm 2\})$. For $n = 3$ and $n = 4$, this creates graphs with multiple edges (see Figure 1) but for $n \geq 5$ the graph is simple and either definition suffices. Inspection of Figure 1 makes it clear that in all cases the graph is a quartic multigraph with the triangle property.

The second basic family is the family of *line multigraphs* of cubic multigraphs. For a multigraph G, we define the line multigraph $L(G)$ to have the edges of G as its vertices, and where two edges of G are connected by k edges in $L(G)$ if they are mutually incident to k vertices in G. In particular, if e and f are parallel edges in G, then there is a double edge in $L(G)$ between the vertices corresponding to e and f; an example is shown in Figure 2. The edge set of $L(G)$ can be partitioned into cliques, with each vertex of degree d in G corresponding to a clique of size d in $L(G)$. Therefore if G is a *cubic* multigraph,

[1]Thus the "connection set" of the Cayley graph is viewed as a multiset.

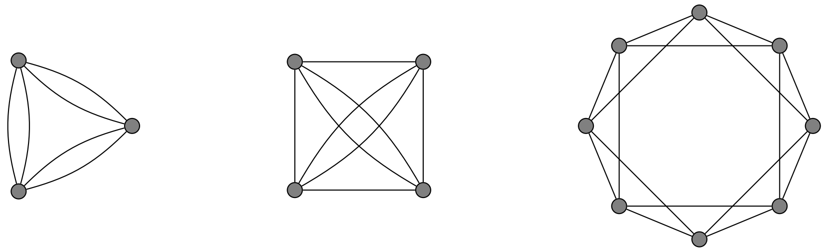

FIGURE 1. Squared cycles for $n = 3$, $n = 4$ and $n = 8$.

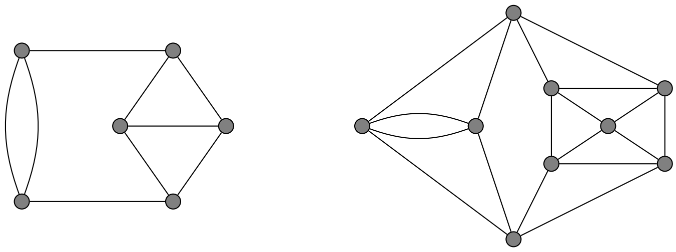

FIGURE 2. A cubic multigraph and its line graph

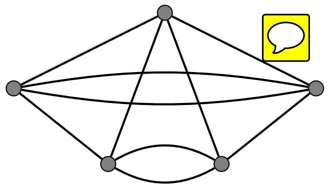

FIGURE 3. A 5-vertex quartic multigraph with the triangle property

the edge set of $L(G)$ can be partitioned into triangles, showing in a particularly strong way that $L(G)$ has the triangle property.

Now there are certain *subgraph-replacement* operations that can be performed on a graph while preserving the triangle property. A triangle T (viewed as a set of three edges) is called *eligible* if it can be removed without destroying the triangle property, or if one of the three edges belongs to a triple edge.The first two operations apply to graphs with eligible triangles; in each an eligible triangle T is removed, leaving the vertices and any other edges connecting them, and a new subgraph is attached in a specific way to the vertices of the removed triangle. Operation 1 subdivides the three edges, and joins the three new vertices with a new triangle, while Operation 2 replaces the triangle with a copy of $K_{1,1,3}$. Figure 1 depicts these operations.

The third and fourth operations replace specific subgraphs with larger subgraphs, and are best described by Figure 5 rather than in words. The named vertices are the points of attachment of the subgraph to the remainder of the graph and remain unchanged. In Operation 3, the left-hand subgraph is necessarily an induced subgraph, but in Operation 4 it is possible that x and y are connected by a double edge, in which case the left-hand

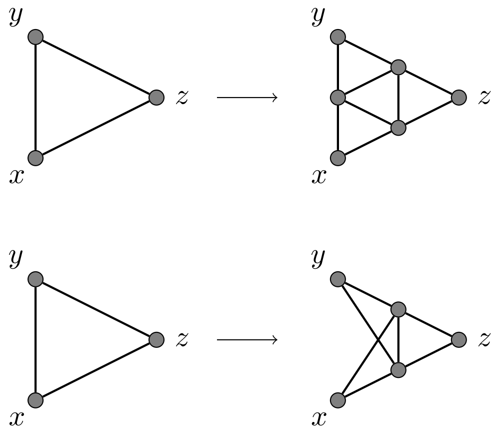

FIGURE 4. Operations 1 and 2 where $\{xy, yz, zx\}$ is an eligible triangle

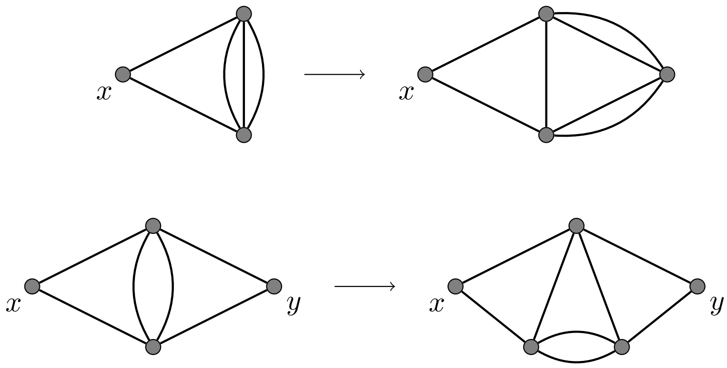

FIGURE 5. Operations 3 and 4

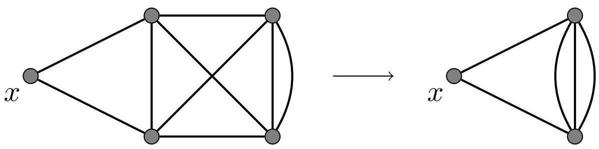

FIGURE 6. Operation 5, which creates a triple edge

subgraph is the entire squared 4-cycle and the right-hand graph one of the 5-vertex quartic graphs with the triangle property. It is not possible for x and y to be connected by a single edge (see Lemma 2 below).

The final operation *decreases* the number of vertices, and is used only to create triple edges. It is shown in Figure 6 and again the named vertex is the point of attachment of this subgraph to the graph. The left-hand subgraph is necessarily an induced block of the original graph.

Finally we can state the main theorem of the paper.

Theorem 1. *If G is a connected 4-regular multigraph with the triangle property, then either*

(1) *G is the square of a cycle of length at least 3, or*
(2) *G is the 5-vertex multigraph shown in Figure 3, which is obtained by applying Operation 4 once to the squared 4-cycle (see Figure 3), or*
(3) *G can be obtained from the line multigraph of a cubic multigraph by a sequence of applications of Operations 1–5.*

The remainder of the paper proves this theorem.

2. Proof of the main theorem

We start with some elementary observations that will be repeatedly used in what follows:

Lemma 1. *A graph G has the triangle property if and only if for every vertex $v \in V(G)$, the graph induced by the neighbourhood of v contains no isolated vertices.*

Proof. If w is an isolated vertex in $N(v)$, then the edge vw does not lie in a triangle and conversely. □

Lemma 2. *If G is a quartic graph with the triangle property, and H is a subgraph of G such that every vertex of H has degree 4 other than two non-adjacent vertices v and w of degree 3, then $G = H + vw$.*

Proof. Suppose that v and w are the two non-adjacent vertices of degree 3 in H. If the fourth edge from v leads to a vertex x *outside* H, then x is isolated in $N(v)$ (because all the other neighbours of v already have full degree). Thus the fourth edge from v must join v and w and then $H + vw$ is quartic and hence equal to G. □

Theorem 2. *The class of connected quartic multigraphs with the triangle property is closed under Operations 1–5 and their reversals.*

Proof. For each of the five operations, it is easy to check that every edge shown in either the left-hand or right-hand subgraph lies in a triangle completely contained within the subgraph, and so the replacement in either direction does not create any "bad" edges not in triangles.

However, it remains to show that none of the edges that are *removed* in the operations or their reversals are essential for creating triangles involving edges that are not shown, either "optional edges" with both end vertices inside the subgraph or edges connecting the subgraph to the rest of the graph. For all five operations, any optional edges must connect pairs of named vertices, and it is clear that every pair of named vertices is at distance 2 in the subgraphs on both sides of each operation (thus forming the necessary triangle if the edge joining them was actually present).

Now consider edges connecting the subgraphs to the rest of the graph. For Operations 1 and 2 (forwards), the requirement that the triangle be eligible ensures that these edges lie in triangles using only edges that will not be removed, while for Operations 1 and 2 (backwards) and the remaining operations in either direction, the named vertices are adjacent only to vertices of degree four, and so there can be no triangles using an edge of the subgraph except those completely contained in the subgraph.

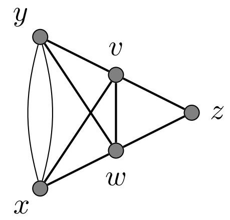

FIGURE 7. The right-hand graph of Operation 2 appears in Claim 2

□

The following proposition is the heart of the proof, as it characterises those quartic graphs with the triangle property that have *not* arisen as a consequence of applying Operations 1–4 to a smaller graph.

PROPOSITION 1. *Let G be a quartic graph with the triangle property on at least* 5 *vertices that contains none of the subgraphs on the right-hand sides of Operations* 1–4. *Then either*

(1) *G is the square of a cycle of length at least* 7, *or*
(2) *G is obtained from the line multigraph of a cubic multigraph by applications of Operation* 5.

Proof. The proof proceeds via a series of claims progressively restricting the structure of G.

Claim 1: The double edges of G form a matching.

Suppose for a contradiction that uv and vw are both double-edges. By Lemma 1, u is adjacent to w and because G has more than three vertices u is adjacent to a fourth vertex x. By Lemma 1, the vertex x is forced to be adjacent to w, thereby creating the subgraph on the right-hand side of Operation 3.

Claim 2: Every double edge is the diagonal of an induced K_4^-.

Let xy be a double edge, and let v be a common neighbor of x and y (which must exist by Lemma 1). Let x_1 and y_1 be the remaining neighbours of x and y, respectively, and note that by Claim 1, $x_1 \neq v \neq y_1$. If $x_1 \neq y_1$, then $x_1v, y_1v \in E(G)$ to create the required triangles, which creates the right-hand graph of Operation 4 which is a contradiction. Therefore we conclude that $x_1 = y_1$ and denote this vertex by w. If $vw \in E(G)$, then v and w must have an additional common neighbour, z. Therefore we have deduced the existence of the subgraph shown in Figure 7 which clearly contains $K_{1,1,3}$ (the right-hand graph of Operation 2) which is again a contradiction. Therefore $vw \notin E(G)$ and so $\{x, y, v, w\}$ form an induced K_4^-.

Claim 3: If G contains an induced K_4^- with no multiple edges, then G is a squared cycle of length at least 7.

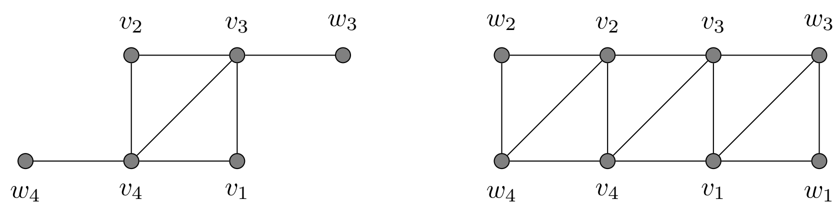

Figure 8. Two stages in the construction of Claim 3

Suppose that G contains an induced K_4^- on the vertices $\{v_1, v_2, v_3, v_4\}$ with all edges $v_iv_j \in E(G)$ except v_1v_2. Let w_3, w_4 be the remaining neighbors of v_3 and v_4, respectively. If $w_3 = w_4$, then $G[\{v_1, v_2, v_3, v_4, w_3\}]$ contains a $K_{1,1,3}$, which is the graph on the right side of Operation 2, a contradiction. Thus we can assume that $w_3 \neq w_4$ obtaining the first graph of Figure 8.

Now, to avoid w_3 being isolated in the neighbourhood of v_3, it must be adjacent to either v_1 or v_2, and similarly for w_4. However if either v_1 or v_2 is adjacent to *both of* w_3 and w_4, then this creates the graph on the right-hand side of Operation 1. Therefore, by symmetry we can assume that $v_1w_3, v_2w_4 \in E(G)$ and $v_1w_4, v_2w_3 \notin E(G)$. Continuing, we see that the vertices v_1 and v_2 must each have a fourth neighbour w_1 and w_2, respectively. To avoid w_1 being isolated in $N(v_1)$ we must have $w_1w_3 \in E(G)$ and to avoid w_2 being isolated in $N(v_2)$ we must have $w_2w_4 \in E(G)$. If $w_1 = w_2$ then we have the situation of Lemma 2 and so G is the square of the 7-cycle.

If $w_1 \neq w_2$, then we have arrived at the second graph in Figure 8 and we can continue this process. Note that $w_3w_4 \notin E(G)$ as w_3 and w_4 have no common neighbor amongst the previously named vertices. So consider neighbours x_3 and x_4 of w_3 and w_4, respectively. If $x_3 = w_2$, then the final neighbour of w_2 is a neighbor of w_3 (to create a triangle for w_2w_3) and a neighbor of w_4 (to create a triangle for w_4x_4), and thus $x_4 = w_1$, and $w_1w_2 \in E(G)$. In this case, G is the square of an 8-cycle.

So assume that $x_3 \neq w_2$ and $x_4 \neq w_1$. If $x_3 = x_4$, this forces $w_1w_2 \in E(G)$, and G is the square of a 9-cycle. If $x_3 \neq x_4$, we continue with vertices x_1 and x_2, and so on. At each stage of this process, there is a chain of K_4^-s and we consider the two missing neighbours of the two vertices of degree three. Either the two neighbours are both in the chain already, in which case G is an even squared cycle, or the two neighbours coincide in a single new vertex, in which case G is an odd squared cycle, or they are two new vertices, in which case the chain is extended, and the argument repeated. Eventually this process must stop, producing a squared cycle.

Claim 4: If G does not contain an induced K_4^- with no multiple edges, then G can be obtained from the line multigraph of a cubic multigraph by applications of Operation 6.

Create a simple graph S from G as follows. For every double edge, delete one of the two end vertices (as these vertices are twins, it does not matter which one). For every triple edge, delete two of the three edges. This graph does not contain $K_{1,3}$ and K_4^- as induced subgraphs, and so by Harary & Holzmann [1] it is the linegraph of a unique triangle free graph $L^{-1}(F)$. Now G can be reconstructed from $L^{-1}(F)$ as follows: double every edge in $L^{-1}(F)$ that corresponds to a vertex in G whose twin was deleted. Add a double edge between the vertices of degree 1 on the edges corresponding to the end vertices of each triple edge in G. This forms a cubic multigraph, and we can construct G by taking the line multigraph of this graph and performing Operation 5 to recover the triple edges. □

With these results, we are now in a position to prove the main theorem.

Proof. (of Theorem 1) Suppose that G is a quartic graph with the triangle property, and repeatedly perform the reverse of Operations 1–4 until the resulting graph G' has no subgraphs isomorphic to any of the graphs on the right-hand side of Operations 1–4. If G' has at least 5 vertices, then by Proposition 1, it is either a squared n-cycle for $n \geq 7$ or has been obtained from the line multigraph of a cubic multigraph by applications of Operation 5. In the former case, G itself is equal to the squared n-cycle, because for $n \geq 7$, the squared n-cycles contain no eligible triangles. In the latter case, combining the applications of Operation 5 that transform the line multigraph of a cubic multigraph into G' with the applications of Operations 1–4 that transform G' into G shows that G has the required structure. If G' has fewer than 5 vertices then it is either the squared 3-cycle or the squared 4-cycle. The squared 3-cycle is the linegraph of a triple edge, while the only operation that can be applied to the squared 4-cycle is Operation 4 which creates the graph of Figure 3 (to which no further operations can be applied.) □

Remarks: Some graphs appear in more than one of the classes of Theorem 1. In particular, the squared 3-cycle is also the line graph of a triple edge, the squared 5-cycle (that is, K_5) is obtained by applying Operation 2 to the squared 3-cycle, while the squared 6-cycle is obtained by applying Operation 1 to the squared 3-cycle.

References

[1] F. Harary and C. Holzmann, Line graphs of bipartite graphs, *R*ev. Soc. Mat. Chile, **1**, 19–22, (1974)

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