The Inversion of 2-Step Graphs

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The 2-step graph $S_2(G)$ of a given graph $G$ was introduced by Exoo and Harary. We now solve the inverse problem. That is, given a graph $H = (V, L_H)$ and the class of connected graphs $\mathcal{C}$ we get $C_H = \{G \in \mathcal{C} : S_2(G) = H\}$ and we determine for what $H$ is $C_H \neq \emptyset$. We show that if $H$ has 2 components and $G \in C_H$, then $G$ is bipartite, and for connected $H$ we find equivalent conditions for $C_H \neq \emptyset$. Our work is based upon ideas about relationships between graphs and rectangular matrices which we developed in previous papers.

1. Introduction

In [1] Exoo and Harary defined the 2-step graph of a given graph, showed that every graph $G$ has a limiting 2-step graph $L$, and determined the structure of $L$. The 2-step graph $H = S_2(G)$ of a given graph $G$ has the same points as $G$ with two points adjacent in $H$ whenever they are connected by a path of length 2 in $G$.

Here we are interested in inverting 2-step graphs. That is, given a graph $H = (V, L_H)$ and a class of graphs $\mathcal{G} = \{G = (V, L_G)\}$, find $G \in \mathcal{G}$ such that $S_2(G) = H$, or ascertain that none exists. In particular, we consider the case where $\mathcal{G} = \mathcal{C}$, the set of connected graphs. Let $C_H = \{G \in \mathcal{C} : S_2(G) = H\}$. We determine for what $H$ is $C_H \neq \emptyset$ and characterize $C_H$ for these cases.

First, we show that if $H$ has more than two components, then $C_H = \emptyset$. Next we show that when $H$ has exactly two components, if $G \in C_H$, then $G$ is bipartite. Furthermore, those $H$ satisfying $C_H = \emptyset$ are determined by using Theorem 2 of [3]. Finally, for connected $H$ we find equivalent conditions for $C_H \neq \emptyset$.

Most of our work in this paper uses ideas about relationships between graphs and rectangular matrices developed by the authors in previous papers [2, 3, 4]. Given an $m \times n$ matrix $A$, we define two sets of points $R = \{r_1, \ldots, r_m\}$ and $C = \{c_1, \ldots, c_n\}$ to represent the rows and columns of $A$, respectively. The three basic graphs are:
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FUNDAMENTAL BIGRAPH. \( BG \) is a bipartite graph (bigraph) on \( R \) and \( C \). The lines correspond to the nonzeros of \( A \), i.e., \([r_i, c_j]\) is a line in \( BG \) iff \( a_{ij} \neq 0 \).

ROW GRAPH: \( RG \) has point set \( R \). The line \([r_i, r_j]\) belongs to \( RG \) if there exists \( c_j \in C \) such that \([r_i, c_j]\) and \([r_k, c_j]\) are lines of \( BG \).

COLUMN GRAPH. \( CG \) has point set \( C \). The line \([c_i, c_j]\) belongs to \( CG \) if there exists \( r_i \in R \) such that \([r_i, c_j]\) and \([r_i, c_k]\) are lines of \( BG \).

2. PRELIMINARY RESULTS

First we establish some relationships between 2-step graphs and graphs associated with matrices.

**Theorem 2.1.** Let \( A \) be an \( m \times n \) matrix. Then

\[
S_2(BG(A)) = RG(A) \cup CG(A)
\]

**Proof.** Let \( R = \{r_1, \ldots, r_m\} \) be the row points and \( C = \{c_1, \ldots, c_n\} \) the column points. Since a path of length 2 in \( BG(A) \) must be of the form \([r_i, c_k, r_j]\) or \([c_i, r_k, c_j]\), row points are not adjacent in \( S_2(BG(A)) \) and column points are not adjacent in \( S_2(BG(A)) \). Furthermore, \([r_i, r_k]\) is a line in \( RG(A) \) iff there exists a path of the form \([r_i, c_k, r_j]\) in \( BG(A) \), and \([c_i, c_j]\) is a line in \( CG(A) \) iff there exists a path of the form \([c_i, r_k, c_j]\) in \( BG(A) \). Hence, \( S_2(BG(A)) = RG(A) \cup CG(A) \).

Observe that if \( G \) is any bipartite graph with sets \( R \) and \( C \) determining the bipartition, then we can construct a binary matrix \( A \) with rows corresponding to the points in \( R \) and columns to the points in \( C \) so that \( BG(A) = G \). Hence, the 2-step graph of any bipartite graph is determined by Theorem 2.1.

For a graph \( G \), let \( A(G) \) denote the adjacency matrix of \( G \). Then \( A(G) \) is symmetric, and if \( G \) is connected and nontrivial, then each row and column of \( A \) has a nonzero entry.

**Theorem 2.2.** Let \( G \) be a connected graph, \( A = A(G) \), and \( H = S_2(G) \). Then \( H \cong RG(A) \cong CG(A) \).

**Proof.** If \( G \) has only one point, the result is trivial, so we may assume \( G \) has at least 2 points. Since \( A \) is symmetric, \( RG(A) \cong CG(A) \). Let \( B(A) \) denote the binary matrix obtained from \( A \). Since each row and column of \( A \) has a nonzero, the adjacency matrix for \( RG(A) \) is \( B(AAT) - I \) by Theorem 1 of [4]. Hence \([i, j]\) is an edge in \( RG(A) \) iff the \( i, j \) entry in \( B(AAT) - I \) is 1 iff the \( i, j \) entry in \( AAT \) is \( \geq 1 \). But \( AAT = A \), and the \( i, j \) entry of \( A^2 \) is \( k \geq 1 \) iff there are \( k \) paths of length 2 between \( i \) and \( j \) in \( G \) iff \([i, j]\) is a line of \( H \).

In particular, we consider the binary matrix \( M \) and associate with it the adjacency matrix.
\[ A(BG(M)) = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} \]

as in [4]. Suppose \( M \) is regular, that is, there are no null rows or columns. Define \( M^TM \) and \( MM^T \) using boolean addition, then,

\[ A(S_2(BG(M))) = \begin{bmatrix} MM^T - I & 0 \\ 0 & MM^T - I \end{bmatrix} = A(RG(M)) \begin{bmatrix} 0 \\ 0 & A(CG(M)) \end{bmatrix} \]

Let \( v_i \leftrightarrow v_j \) mean that there exists a path connecting \( v_i \) to \( v_j \) and \( v_i \leftrightarrow v_j \) mean that there exists a simple path with an odd number of points connecting \( v_i \) to \( v_j \). The following lemma will be useful below.

**Lemma 2.1.** If \( v_i \) and \( v_j \) are points in a graph \( G \in \mathcal{C}_H \) then \( v_i \leftrightarrow v_j \) in \( G \) iff \( v_i \leftrightarrow v_j \) in \( H \).

**Proof.** Let \( G \in \mathcal{C}_H \). Suppose \( v_i \leftrightarrow v_j \) so that there exist points \( x_h, y_k \) such that \((v_i, x_1, y_1, x_2, y_2, \ldots, x_p, y_p, x_{p+1}, v_j)\) is a path in \( G \). Then, \((v_i, y_1, x_2, \ldots, y_p, v_j)\) is a path in \( H \).

Conversely, suppose \( v_i \leftrightarrow v_j \). Then we can find \( y_l \) such that \( v_i, y_1, y_2, \ldots, y_p, v_j \) is a simple path in \( H \). Since \( G \in \mathcal{C}_H \), there exist points \( x_t \) such that \((v_i, x_1, y_1, x_2, y_2, \ldots, y_p, x_{p+1}, v_j)\) is a path in \( G \) with an odd number of points. If \( x_m = x_n \) for \( m < n \), then we can remove all pairs \((x_m, y_m), (x_{m+1}, y_{m+1}), \ldots, (x_{n-1}, y_{n-1})\) and still have a path in \( G \) with an odd number of points. Hence, \( v_i \leftrightarrow v_j \) in \( G \).

Let us now determine \( \mathcal{C}_G \) when \( H \) is not connected. First we show that \( \mathcal{C}_H = \emptyset \) if \( H \) has more than 2 components.

**Theorem 2.3.** If \( H \) has more than 2 components, then \( \mathcal{C}_H = \emptyset \).

**Proof.** Suppose \( \mathcal{C}_H \neq \emptyset \) and let \( G \in \mathcal{C}_H \) and \( v_i, v_j \), with \( v_i \) in different components of \( H \). We will reach a contradiction by showing that \( v_i \leftrightarrow v_j \) or \( v_i \leftrightarrow v_j \) in \( H \). Since \( G \) is connected, \( v_i \leftrightarrow v_j \) in \( G \). By Lemma 2.1 we can't have \( v_i \leftrightarrow v_j \) in \( G \), so we can assume the path is simple of the form \((v_i, x_1, y_1, x_2, y_2, \ldots, x_p, y_p, v_j)\).

Since \( G \) is connected, we can assume \( v_i \leftrightarrow v_j \) in \( G \). Note that if \((v_i, v_j)\) is a line in \( G \), then \( v_i \leftrightarrow v_j \) in \( G \), so by Lemma 2.1, \( v_i \leftrightarrow v_j \) in \( H \), a contradiction. Hence we can assume there is a simple path \((v_i, w_1, w_2, \ldots, w_q, v_j)\).

If the \( w_i \)s are distinct from the \( x_i \)s and \( y_i \)s, then if \( q \) is odd, \( v_i \leftrightarrow v_j \), and if \( q \) is even, \( v_i \leftrightarrow v_j \), so by Lemma 2.1, we have either \( v_i \leftrightarrow v_j \) or \( v_i \leftrightarrow v_j \) in \( G \).
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\( H \), a contradiction. If they are not distinct, choose the smallest \( k \) such that \( w_k \) is either an \( x_i \) or \( y_j \). Suppose \( w_k = x_i \) and consider the simple paths

\((v_1, w_1, \ldots, w_k = x_i, y_j, \ldots, x_p, y_p, v_j)\) and \((v_i, w_1, \ldots, w_k = x_i, y_{i-1}, x_{i-1}, \ldots, y_1, x_1, v_j)\). One of these paths must be odd, so by Lemma 2.1, either \( v_i \leftrightarrow v_j \) or \( v_i \leftrightarrow v_j \) in \( H \), a contradiction. A similar argument leads to a contradiction if \( w_k = y_j \).

In Theorem 2.1 we saw that if \( G \) is bipartite then \( S_2(G) \) has two components. Next we see that if \( H \) has two components and \( G \in C_H \), then \( G \) is bipartite.

**THEOREM 2.4.** If \( H \) has exactly two components, every solution in \( C_H \) is a bipartite graph with the two sets of points in the bipartition of \( G \) determined by the components of \( H \).

**Proof.** Suppose \( H = (V_1, L_1) \cup (V_2, L_2) \) and \( G \in C_H \). We will prove that \( L_2 \) cannot contain a line with both end points in \( V_1 \) (or with both end points in \( V_2 \)). Assume, to the contrary, that \((v_1, v_2) \in L_2 \) where \( v_1, v_2 \in V_1 \). Let \( (v_2, v_3, \ldots, v_k) \) be a shortest path from \( v_2 \) to any point \( v_k \) in \( V_2 \). Thus, \( v_1, v_2, \ldots, v_{k-1} \in V_1 \) and \( v_k \in V_2 \). This is possible since \( G \) is connected. This implies that \((v_{k-2}, v_k) \in L_H \), which is a contradiction.

Thus, if \( H \) has exactly two components, we can confine \( C_H \) to the bipartite graphs. However, if \( H \) has 2 components, it is not necessarily true that \( C_H \neq \emptyset \). Let \( H \) have two components \( H_1 \) and \( H_2 \) and suppose \( G \in C_H \). Then \( G \) is bipartite by Theorem 2.4 and from the remark following Theorem 2.1, we can construct a \( \{0, 1\} \) matrix \( A \) with rows corresponding to the points in \( H_1 \) and columns corresponding to the points in \( H_2 \). By Theorem 2.1 we have that \( H_1 \cong RG(A) \) and \( H_2 \cong CG(A) \). In particular, from Theorem 2.1 and 2.4 we have that \( C_H \neq \emptyset \) iff there exists a matrix \( A \) such that \( H_1 = RG(A) \) and \( H_2 = CG(A) \). In [3] we determined conditions on \( H_1 \) and \( H_2 \) for this to happen using clique cover graphs.

Given a graph \( G \), a finite set \( S = \{s_1, \ldots, s_n\} \) of cliques of \( G \) is called a clique cover if every point and line of \( G \) belongs to at least one clique in \( S \). We associate with \( S \) a clique cover graph, \( Q(S) \), in the following way. \( Q(S) \) is a graph on the points \( 1, 2, \ldots, n \) and the line \([i, j]\) belongs to \( Q(S) \) iff \( S_i \) and \( S_j \) contain at least one common point.

The following theorem now follows immediately from the above discussion and Theorem 2 of [3].

**THEOREM 2.5.** Let \( H \) have two components \( H_1 \) and \( H_2 \). Then \( C_H \neq \emptyset \) iff \( H_1 \) is isomorphic to a clique cover graph of \( H_2 \).

It is important to note that we only need to show that one of the graphs is isomorphic to a clique cover graph of the other. Once one has the

\[ J r. \text{ Comb., Inf. \\& Syst. Sci.} \]

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isomorphism, the results of [3] provide a method for constructing \( G \in C_H \).
However, the isomorphism can be difficult to find.

3. The Case Where \( H \) is Connected

In this section we determine \( C_H \) when \( H \) is connected. Since if \( H \) is trivial (has only one point), then \( G = H \) is the unique solution, we may assume that \( H \) is not trivial (has at least two points). As in section 2, let \( A(G) \) denote the adjacency matrix of \( G \). We will need the following two lemmas.

**Lemma 3.1.** Suppose \( A \) is an \( m \times m \) symmetric \( (0, 1) \) matrix with \( a_{ii} = 0 \) for all \( i \). If \( BG(A) \) is connected, then the graph \( G \) such that \( A(G) = A \) is connected.

*Proof.* Since \( BG(A) \) is connected and all \( a_i = 0, m > 1 \). Let \( i, j \in G \). We will show that there exists a path from \( i \) to \( j \). Since \( BG(A) \) is connected, there exists a path connecting \( r_i \) to \( r_j \) in \( BG \). If we let \( i = i_k \) and \( j = j_k \), then the path in \( BG \) is of the form \((r_{i_1}, e_{j_1}, r_{i_2}, e_{j_2}, \ldots, r_{i_k}, e_{j_k})\). We get a path in \( G \) as follows:

\[
[r_{i_1}, e_{j_1}] \in BG \Rightarrow a_{i_1, j_1} = 1 \text{ and } i_1 \neq j_1 [i_1, j_1] \in G
\]
\[
[c_{j_1}, r_{i_2}] \in BG \Rightarrow a_{i_2, j_2} = 1 \text{ and } i_2 \neq j_1 [i_2, j_1] \in G
\]
\[
\vdots
\]
\[
[r_{i_k}, e_{j_k}] \in BG \Rightarrow a_{i_k, j_k} = 1 \text{ and } i_k \neq j_k [i_k, j_k] \in G
\]

Hence, \((i = i_1, j_1, i_2, j_2, \ldots, i_k, j_k = j)\) is a path in \( G \) connecting \( i \) to \( j \). □

**Lemma 3.2.** If a graph \( H \) is isomorphic to a clique covering graph of itself, then there exists a \((0, 1)\)-matrix \( A \) such that \( H \cong RG(A) \cong CG(A) \).

*Proof.* Suppose \( H \) is isomorphic to a clique covering graph of itself. Let \( G_1 = H = G_2 \), then by Theorem 2 of [3] there exists a \((0, 1)\)-matrix \( A \) such that \( RG(A) = G_1 \) and \( CG(A) = G_2 \). Hence \( H \cong RG(A) \cong CG(A) \). □

If \( S \) is a clique cover of \( H \) and \( H \) is isomorphic to the clique graph of \( S \), then the matrix \( A \) in the lemma can be constructed using \( S \). Denote this matrix by \( \mathcal{A}(S) \). If a matrix \( A \) is symmetric with zeros on the diagonal, we will call \( A \) a zero-symmetric matrix. Now we determine when \( \mathcal{C}_H \neq \emptyset \).

**Theorem 3.1.** Let \( H \) be a connected graph with more than one point. Then \( \mathcal{C}_H \neq \emptyset \) iff \( H \) is isomorphic to a clique covering graph of itself and there exist permutation matrices \( P \) and \( Q \) such that \( PA(S)Q \) is zero-symmetric.

*Proof.* Suppose \( \mathcal{C}_H \neq \emptyset \) and let \( G \in \mathcal{C}_H \). Then by Theorem 2.2 we have \( H \cong RG(A) \cong CG(A) \) for \( A = \mathcal{A}(G) \). Furthermore, \( A \) has a nonzero in each row and column since \( G \) is connected with more than one point. But then if \( G_1 = H \) and \( G_2 = H \), then \( RG(A) = G_1 \) and \( CG(A) = G_2 \). Hence, by Theorem 2 of [3], \( G_1 \) is isomorphic to a clique covering graph of \( G_2 \), so

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that $H$ is isomorphic to a clique covering graph of itself. Furthermore, $A = \mathcal{A}(S)$ is zero-symmetric since it is the adjacency matrix for $G$.

Suppose $H$ is isomorphic to a clique covering graph of itself and there exist permutation matrices $P$ and $Q$ such that $A = P \mathcal{A}(S) Q$ is zero-symmetric. By construction (see Theorem 2 of [3]), each row and column of $\mathcal{A}(S)$ has a nonzero. Let $G$ be the graph satisfying $A(G) = A$. Since $RG(A) = H$ is connected, $BG(A)$ is connected by Theorem 2.2 of [2], so $G$ is connected by Lemma 3.1. By Theorem 2.2, $S_G(G) \cong RG(A)$, but $RG(A) \cong H$ by Lemma 3.3. Hence, $G \in \mathcal{C}_H$ so that $\mathcal{C}_H \neq \emptyset$.

**Remark.** Note that if $S = \{S_i\}$ is the clique cover of the theorem and if $|S_i|$ is the number of points in $S_i$, then $\sum |S_i|$ must be even since $P \mathcal{A}(S) Q$ is zero-symmetric.

To illustrate the conditions in Theorem 3.1 we consider the simple graph

$$
\begin{array}{c}
1 & 2 & 3 \\
\hline
H: & & \\
\end{array}
$$

If one attempts to construct a 2-step graph inverse for $H$, a logical inconsistency is reached:

1. line $(1, 2)$ in $H$ implies $G$ contains lines $(1, 3)$ and $(2, 3)$;
2. line $(2, 3)$ in $H$ implies $G$ contains lines $(1, 2)$ and $(1, 3)$; but
3. line $(1, 3)$ not in $H$ implies $G$ cannot contain both lines $(1, 2)$ and $(2, 3)$.

What condition in Theorem 3.1 was violated? The clique covers for $H$ contain the line set $\{(1, 2), (2, 3)\}$ which is a minimal clique cover [3]. But consider the matrix $(S)$ extended to have 3 cliques:

$$
\mathcal{A}(S) = \begin{bmatrix}
1 & 0 & x \\
1 & 1 & y \\
0 & 1 & z
\end{bmatrix}
$$

The clique corresponding to the last column of $\mathcal{A}(S)$ must satisfy $xz = 0$ since, if $x = z = 1$, line $(1, 3)$ is created. But it is not hard to see that $\mathcal{A}(Q)$ cannot be rearranged to a 3 zero-symmetric form.

This example can be easily extended to the form of path $H = P_n$,

$$
\begin{array}{c}
1 & 2 & \cdots & n \\
\hline
H: & & \\
\end{array}
$$

for which there is also no 2-step inverse. In the next section we present larger classes of pathologies.
By contrast suppose $H$ is an odd cycle $C_n$ with $2K + 1$.

The clique cover is $S = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1)\}$ with $H \cong Q(S)$ and the associated matrix can be rearranged to be zero-symmetric:

\[
\begin{bmatrix}
1 & 1 & & & \\
& 1 & 1 & & \\
& & 1 & \ddots & \\
& & & \ddots & 1 \\
& & & & 1
\end{bmatrix}
\]

(Note that $(n, 1)$ corresponds to column $p + 1$.)

4. **Classes of Invertible/Noninvertible Graphs**

We now make use of our previous results to determine if $C_H \neq \emptyset$ for various types of connected graphs.

**Theorem 4.1.** If $H$ is connected with more than one point and $C_H = \emptyset$, then $H$ must contain an odd cycle.

**Proof.** Suppose $G \subseteq C_H$. If $G$ contains no odd cycles, it is bipartite, in which case $H$ has 2 components, a contradiction. If $G$ contains the odd cycle $(x_0, y_1, x_1, y_2, x_2, \ldots, y_p, x_p, x_0)$, then $H$ contains the odd cycle $(x_0, x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p, x_0)$.

**Theorem 4.2.** $C_H \neq \emptyset$ for the following connected graphs $H$:
1. $H$ is complete with more than 2 points.
2. $H$ is an odd cycle.

**Proof.** (1) is obvious since $H \subseteq C_H$. For (2), let $H$ be the odd cycle $(x_0, x_1, x_2, \ldots, x_p, y_1, \ldots, y_p, x_0)$. Then $G \subseteq C_H$ is the odd cycle $(x_0, y_1, x_1, y_2, x_2, \ldots, y_p, x_p, x_0)$.

It is also easy to construct $G \subseteq C_H$ for either (1) or (2) using Theorem 3.1.

**Theorem 4.3.** $C_H = \emptyset$ for the following connected graphs $H$: 

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(1) $H$ is an even cycle.

(2) $H$ is bipartite.

(3) $|L_H| = p > 2k + 1$, where $H = (p, L_H)$ contains exactly one odd cycle with $2p + 1$ points.

Proof. (1) and (2) are immediate consequences of Theorem 4.1 since $H$ does not contain an odd cycle. The proof of (3) is more involved and uses Theorem 3.1.

First suppose $2k + 1 > 3$ and let $S = S_i$ be a clique cover. Then each $S_i$ must be a line. Furthermore, if $(x_{i_{1}}, \ldots, x_{i_{k}}, x_{i_{k+1}})$ is the cycle in $H$, then the cliques $S_i = \{x_{i_{1}}, x_{i_{j}}, x_{i_{k+1}}\}$, $S_j = \{x_{i_{j}}, x_{i_{k}}, x_{i_{k+1}}\}$ determine a cycle of length $k = 2k + 1$ in the clique covering graph. Since $2k + 1 < p$ and $H$ is connected, there is a point $x$ adjacent to some $x_{i_{j}}$. But then $x_{i_{j}}$ belongs to the cliques $(x_{i_{j-1}}, x_{i_{j}}), (x_{i_{j}}, x_{i_{j+1}}), (x_{i_{j}}, x)$ and so the clique covering graph has a 3 cycle, a contradiction. Now, if $2k + 1 = 3$ and all the cliques are lines, the above argument produces two 3-cycles, a contradiction. Otherwise, one of the cliques is the 3-clique, each line is a clique, and since we need $p$ cliques, there are two cliques which are equal to cliques already used. Now if the 3-clique is in the cover once or three times, then $\sum |S|$ is odd which contradicts the remark following Theorem 3.1. Hence, the 3-clique must occur twice in the clique cover, say $S_i$ and $S_j$, and some line must occur twice, say $S_m$ and $S_n$. As above, there is a point $x$ adjacent to a point $y$ in the 3-clique, and the cliques $S_i, S_j$, and $(x, y)$ determine a 3-cycle in the clique covering graph. If $S_m = S_n = \{x, z, y\}$, one of these points must be in another clique, say $\{z, y\}$ and then these cliques determine another 3-cycle in the clique covering graph, a contradiction.

If $H$ is an even cycle it is easy to see that $H$ is isomorphic to a clique covering graph of itself. Hence, from (1) in Theorem 4.3 we see the necessity of the zero-symmetric condition of Theorem 3.1. From (2) we see that if $H$ is a tree with more than one point, $C_H = \emptyset$.

Consider the following illustration of (3) of Theorem 4.3:

![Diagram](https://via.placeholder.com/150)

$p = 2$. Suppose one attempts to construct a 2-step inverse of $H$ using theorem 3.1. The only clique cover is $S = S_i = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 6), (6, 7)\}$ and we have
However, $\mathcal{A}(Q(S))$ cannot be rearranged to be zero-symmetric. This follows immediately from the fact that row 1 has three nonzeros, but every column has exactly two nonzeros.

We can generalize from this example to state the following result.

**Theorem 4.3.** Suppose every clique cover of $H$ contains the line set. Then $\mathcal{C}_H = \emptyset$ if a point of $H$ has degree $\geq 3$.

*Proof.* By hypothesis, $\mathcal{A}(S)$ has a row with at least 3 nonzeros, whereas every column has 2 nonzeros. Thus symmetry is impossible.

We will conclude by showing an example of how Theorem 3.1 can be used to obtain a large class of nontrivial 2-step graphs. Observe that if $A$ is a zero symmetric matrix whose elements are zeros and one and $H$ is its row graph, then $H$ is isomorphic to a clique covering graph of itself and so, by Theorem 3.1, $\mathcal{C}_H \neq \emptyset$ if $H$ is connected. Thus by investigating classes of such matrices we can obtain classes of 2-step graphs. Here is one such class of matrices.

Recall that a partition of an integer is a division of the integer into positive integral parts. For $n \geq 3$ we consider partitions of $n$ such that at least one of the integral parts is greater than or equal to 3. Thus the possibilities for 6 are $6, 5 + 1, 4 + 1 + 1, 4 + 2, 3 + 2 + 1, 3 + 1 + 1 + 1$, $3 + 3$. We will construct a zero symmetric matrix of order $n$ for each such partition on $n$.

Observe first that the $k \times k$ matrix $B_k$ with zeros on the principal diagonal and ones everywhere else has a row graph which is a $k$-clique; suppose $P_n = k_1 + k_2 + \ldots + k_q$ is a partition of $n$ with $k_j \geq 3$ for at least one $j$. We first construct the matrix $A_0(P_n)$ which is the direct sum of the blocks $B_{k_1}, B_{k_2}, \ldots, B_{k_q}$. The row graph of $A_0(P_n)$ consists of $q$ disjoint cliques of sizes $k_1, \ldots, k_q$, respectively. In displayed form we have

$$A_0(P_n) = \begin{bmatrix} B_{k_1} & 0 & \ldots & 0 \\ 0 & B_{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_{k_q} \end{bmatrix}$$
Now we define \( A(P_n) \) as follows. Replace the \( k_i \times k_{i+1} \) blocks of zeros of \( A_0(P_n) \) on the first super-diagonal by blocks with all zeros except for a 1 in the lower left-hand corner, \( 1 \leq i \leq q - 1 \). Similarly replace the \( k_{i-1} \times k_i \) blocks of zeros on the first sub-diagonal by blocks with all zeros except for a 1 in the upper right-hand corner, \( 1 \leq i \leq q - 1 \). The resulting matrix will be zero symmetric and has a connected row graph by virtue of the fact that \( k_i \geq 3 \) for all \( i \). Hence we have a 2-step graph corresponding to a 3-part partition \( \pi \) satisfying this condition.

Now, in forming the partitions of the integer \( n \), the order of the integral parts is not important. Let us show, however, that different orderings of the integral parts of a partition can lead to non-isomorphic 2-step graphs. For \( n = 6 \) consider the partition \( 3 + 1 + 1 + 1 \) and its reordering into \( 1 + 1 + 3 + 1 \). Our construction gives

\[
A_{01} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
A_{02} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Filling in the 1's in the appropriate positions then yields

\[
A_1 = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

The corresponding row graphs are

\[
\text{RG}(A_1): \quad 1 \quad 4 \quad 6 \\
\text{RG}(A_2): \quad 3 \quad 6 \quad 4
\]

which are clearly not isomorphic. Corresponding inverses are, respectively,

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