RECTANGULAR MATRICES AND SIGNED GRAPHS*

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Abstract. This paper extends the theory of graphs associated with real rectangular matrices to include information about the signs of the elements. We show when signed row and column graphs can be defined for the matrix $A$. We also deduce conditions under which these graphs are balanced. This leads to a definition of the class of quasi-Morishima rectangular matrices $A$. It is shown that the Perron–Frobenius theorem applies to the matrices $AA^T$ and $A^TA$ when $A$ is a quasi-Morishima matrix. Finally we examine the applications of our results to several classes of matrices occurring in energy economic models. All results in this paper are purely qualitative in character.

1. Introduction. This paper continues the development of the theory and use of graphs and digraphs associated with rectangular matrices which we initiated in [1] and [2]. Our aim here is to construct a theory which will adequately exploit the sign information contained in real matrices. We touched briefly upon this topic in the previous papers, but no systematic presentation was attempted. It turns out that, under certain conditions, a satisfactory theory of positivity can be devised for rectangular real matrices.

Our work in this paper has been motivated by our efforts to study the important properties of two special classes of matrices introduced by H. J. Greenberg in [3]. Greenberg has identified physical flows matrices (PFM) and physical flows with feedback matrices (PFFM) as important components of energy economic models.

Before introducing the PFM and PFFM we remind the reader how the basic graphs associated with the $m \times n$ matrix $A$ are defined.

Given the $m \times n$ matrix $A$ we define two sets of points, $R = \{r_1, \cdots, r_m\}$ and $C = \{c_1, \cdots, c_n\}$ to represent the rows and columns of $A$, respectively. We then have the following definitions.

- **Fundamental bipartite graph (bigraph):** $BG$ is a bigraph on the point sets $R$ and $C$. The line $[r_i, c_j]$ belongs to $BG$ if $a_{ij} \neq 0$.
- **Row graph.** $RG$ is defined on $R$. The line $[r_i, r_j]$ belongs to $RG$ if there exists $c_k \in C$ such that $[r_i, c_k]$ and $[r_j, c_k]$ are in $BG$. Thus two rows are adjacent if they have a common column intersection in $A$.
- **Column graph.** $CG$ is defined on $C$. The line $[c_i, c_k]$ belongs to $CG$ if there exists $r_k \in R$ such that $[c_i, r_k]$ and $[c_k, r_k]$ belong to $BG$. In other words, two columns are adjacent if they have a common row intersection in $A$.

A rectangular matrix $A$ will be called **regular** if each row and column of $A$ contains at least one nonzero element.

Now it is clear that the sign information in the real matrix $A$ can be immediately incorporated into the bigraph $BG$. In fact, we label the line $[r_i, c_j]$ positive if $a_{ij} > 0$ and negative if $a_{ij} < 0$. The resulting signed graph will be denoted by $BG^+$.

The usefulness of signed graphs and digraphs has been demonstrated by several authors (see Harary [4], [5], Maybee and Quirk [6], and Roberts [7], for example). Let us therefore show that for PFM and PFFM we can define the signed graphs $RG^+$ and $CG^+$.

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Physical flows matrix. The matrix $A$ is a PFM if the rows can be partitioned into disjoint sets $S$ and $M$ and the columns into disjoint sets $P$, $T$ and $K$ such that:

1. Every element $a_{ij}$ with $i \in S, j \in P$ is nonnegative.
2. Every element $a_{ij}$ with $i \in M, j \in K$ is nonpositive.
3. Every element $a_{ij}$ with $i \in S, j \in T$ is nonnegative.
4. Every element $a_{ij}$ with $i \in M, j \in T$ is nonnegative.
5. All other elements of $A$ are zero.

It is therefore true that $A$ is a PFM if and only if there exist permutation matrices $P$ and $Q$ such that

\[
PAQ = \begin{bmatrix}
    A_{11} & A_{12} & 0 \\
    0 & A_{22} & A_{23}
\end{bmatrix},
\]

where $A_{11} \geq 0$, $A_{12} \leq 0$, $A_{22} \geq 0$, $A_{23} \leq 0$.

The matrix $A$ in PFM will be called regular if each of the blocks $A_{11}$, $A_{12}$, $A_{22}$ and $A_{23}$ is nonempty and regular.

Physical flows with feedback matrix. The matrix $A$ is a PFFM if the rows can be partitioned into disjoint sets $I$, $S$, $M$ and the columns into disjoint sets $P$, $T$, $K$ such that (1) through (5) hold and:

6. Every $a_{ij}$ with $i \in I, j \in P$ is nonpositive.
7. Every $a_{ij}$ with $i \in I, j \in K$ is nonnegative.
8. Every $a_{ij}$ with $i \in I, j \in T$ is zero.

Thus $A$ is a PFFM if and only if there exist permutation matrices $P$ and $Q$ such that

\[
PAQ = \begin{bmatrix}
    A_{11} & 0 & A_{13} \\
    A_{21} & A_{22} & 0 \\
    0 & A_{32} & A_{33}
\end{bmatrix},
\]

where $A_{11} \geq 0$, $A_{13} \leq 0$, $A_{21} \geq 0$, $A_{22} \leq 0$, $A_{32} \geq 0$, $A_{33} \leq 0$.

The matrix $A$ will be called a regular PFFM if each of the blocks $A_{11}$, $A_{13}$, $A_{21}$, $A_{22}$, $A_{32}$, $A_{33}$ is nonempty and regular.

Now let us observe that, when $A$ is a PFM or a PFFM the scalar product of any two columns is positive, negative or zero independently of the magnitudes of the elements because all terms in the scalar product are weakly of the same sign. The same is true for the scalar product of two row vectors. Consequently to such matrices we can associate the signed graphs $CG^+$ and $RG^+$ in which the line $[c_n, c_j]$ ($[r_n, r_j]$) is positive if the corresponding column (row) vectors have a positive scalar product and negative if the scalar product is negative. The smallest regular PFM is shown in Fig. 1 together with $CG^+$ and $RG^+$. The smallest regular PFFM is illustrated in Fig. 2. In drawing the graphs we have followed the convention of using dashed lines to represent negative lines as introduced in [8].

\[
A = \begin{bmatrix}
    1 & -1 & 0 \\
    0 & 1 & -1
\end{bmatrix}
\]

\[
\begin{array}{ccc}
P & T & K \\
CG^+: & \begin{array}{c}
P \\
Q
\end{array} & \begin{array}{c}
RG^+: \\
O-----O
\end{array}
\end{array}
\]

\[
A = \begin{bmatrix}
    0 & 1 & -1 \\
    0 & 1 & -1
\end{bmatrix}
\]

\[
\begin{array}{ccc}
P & T & K \\
CG^+: & \begin{array}{c}
P \\
Q
\end{array} & \begin{array}{c}
RG^+: \\
O-----O
\end{array}
\end{array}
\]

FIG. 1. Smallest PFM.

Since the graphs $CG^+$ and $RG^+$ can be defined for PFM and PFFM, it seems natural to seek to determine the class of real rectangular matrices for which these graphs, or at least one of them, can be defined. Section 2 is devoted to the determination of this class of matrices and to some of their properties.
Now it turns out that among the matrices for which the signed graphs \( CG^+ \) and \( RG^+ \) exist there is a subclass with the property that these graphs are balanced (all cycles positive). This subclass includes the PFM but not the PFFM. For such matrices we can develop a satisfactory theory of positivity and we can apply the Perron–Frobenius and its corollaries to the matrices \( AA^T \) and \( A^T A \). This is the subject matter of \( \S \) 3.

Finally \( \S \) 4 is devoted to applications of our results to the classes PFM and PFFM and to certain generalizations of these classes.

All of our results are purely qualitative in character, i.e., they hold regardless of the magnitudes of the matrix elements.

2. Signed matrices. We begin with an embellishment of some fundamental ideas introduced in the paper [6] (see also [9] where similar concepts are used). Let \( x = (x_1, \cdots, x_n) \) and \( y = (y_1, \cdots, y_n) \) be vectors in the euclidean space \( \mathbb{R}^N \). We shall call \( x \) and \( y \) conformal if \( x_i y_i \geq 0, \ 1 \leq i \leq N \), and anticonformal if \( x_i y_i \leq 0, \ 1 \leq i \leq N \). In the terminology of [6] \( x \) and \( y \) are conformal if they lie in the closure of the same qualitative cone \( Q \) in \( \mathbb{R}^N \) and anticonformal if one vector lies in a closed qualitative cone and the other in the corresponding negative closed cone.

Let \( A = [a_{ij}] \) be an \( m \times n \) real matrix. We will call \( A \) row signed if the row vectors of \( A \) regarded as elements of \( \mathbb{R}^n \) all lie in the same closed qualitative cone or in its negative, i.e., if they are pairwise either conformal or anticonformal. We define \( A \) to be column signed if the column vectors of \( A \) regarded as elements of \( \mathbb{R}^m \) are pairwise either conformal or anticonformal.

Lemma 1. A is column signed if and only if \( A \) is row signed.

Proof. Assume \( A \) is column signed. Suppose \( A \) is not row signed. Then there exist rows \( r_i \) and \( r_j \) such that for some \( i \) and \( j \), \( a_{ij}a_{ij} > 0 \) and \( a_{ij}a_{ij} < 0 \). Now consider the products \( a_{ij}a_{ij} \) and \( a_{ij}a_{ij} \). We have \( a_{ij}a_{ij}a_{ij}a_{ij} < 0 \) so these products have different signs. Therefore \( A \) is not column signed, a contradiction. It follows that \( A \) must be row signed. The proof is similar if we assume \( A \) is row signed.

In view of Lemma 1 we shall say henceforth that \( A \) is signed without using the adjectives column or row.

The following lemma is a complement to Lemma 1.

Lemma 2. Let \( G^+ \) be a signed graph with \( n \) points. Then there exists a matrix \( A \) such that \( CG^+(A) = G^+ \).

Proof. Let \( e_1, \cdots, e_m \) be the lines of \( G^+ \) and \( p_1, \cdots, p_m \) the points. Construct \( A \) as follows:

Column \( j \) of \( A \) corresponds to point \( p_j \) of \( G^+ \).
If \( e_i = [c_{ij}, c_{ij}] \), then \( a_{ij} = 1 \) and

\[
\begin{align*}
a_{ij} &= 1 \quad \text{if } e_i \text{ is positive}, \\
a_{ij} &= -1 \quad \text{if } e_i \text{ is negative}.
\end{align*}
\]

Observe that each row of \( A \) has only two nonzero elements and each row corresponds to a unique line of \( CG^+(A) \). Thus \( CG^+(A) = G^+ \).
The next result relates the property that $A$ is signed to properties of $AA^T$ and $A^TA$. We require first some preliminary ideas. Recall from [6] that, if $x \in \mathbb{R}^n$, \( \text{sgn} \, x = (\text{sgn} \, x_1, \cdots, \text{sgn} \, x_N) \) where \( \text{sgn} \) is the usual signum function. In the same way we can associate with any real $m \times n$ matrix $A$ the matrix $\text{sgn} \, A = [\text{sgn} \, a_{ij}]$ (see [6] for more detail). Let us introduce the addition table

<table>
<thead>
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in which we use an $x$ to denote an indeterminate (as to sign) entry. Define $A^T \circ A$ to be the matrix product formed using this addition table.

**Theorem 1.** Let $A$ be an $m \times n$ matrix with at least one nonzero element in each row and column. The following are equivalent:

1. $A$ is signed.
2. $\text{sgn} \, (AA^T) = \text{sgn} \, A \circ (\text{sgn} \, A)^T$.
3. $\text{sgn} \, (A^TA) = (\text{sgn} \, A)^T \circ \text{sgn} \, A$.

**Proof.** The proof is left to the reader.

The interpretation of (2) and (3) is that the left-hand side is defined if and only if the right-hand side is. Since $\text{sgn} \, A \circ (\text{sgn} \, A)^T$ is symmetric and each row has a nonzero element, the diagonal elements are all positive and we need only calculate the elements above the principal diagonal. If any of these elements equals $x$; then $A$ is not signed. Otherwise $A$ is signed.

For very large matrices Theorem 1 may not provide a useful test of whether or not $A$ is signed, especially if $AA^T$ or $A^TA$ is not sparse. The next result provides another criterion.

**Theorem 2.** $A$ is signed if and only if every 4-cycle of $BG^+$ is positive.

3. **Positivity.** For the deeper study of signed matrices we will require some background. First we recall that a signed graph is called balanced if every cycle is positive. Secondly the signed graph $G^+$ is balanced if and only if the points of $G^+$ can be partitioned into disjoint subsets $S_1$ and $S_2$ (one of which may be empty) such that every line joining two points in the same set is positive and every line joining points in different sets is negative (see Harary [4] for further details).

Let $A = [a_{ij}]^T$ be a square sign-symmetric matrix. Then we associate with $A$ a graph $G(A)$ as follows: $G$ has $n$ points labelled $1, 2, \cdots, n$ and a line joining points $i$ and $j$ ($i \neq j$) if $a_{ij}$ (and $a_{ji}$) is nonzero. The line $[ij]$ will be given a positive label if $a_{ij} > 0$ and a negative label if $a_{ij} < 0$. In this way we arrive at the signed graph $G^+(A)$. The matrix $A$ is called a Morishima matrix if $G^+(A)$ is balanced. Moreover, the square matrix $A$ is a Morishima matrix if and only if there exists a permutation matrix $P$ such that

$$P^T A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \geq 0, A_{22} \geq 0, A_{12} \leq 0, A_{21} \leq 0$ and where each of the blocks $A_{11}$ and $A_{22}$ is square. In this representation the block $A_{22}$ may be $0 \times 0$ in which case $A$ itself is nonnegative which is a special case of the Morishima class.

With this material as background we can proceed. First note that, if $BG^+$ is balanced, it has nonnegative 4-cycles and so $RO^+$ and $CG^+$ are defined.
Theorem 3. The following are equivalent:
(a) \( BG^+ \) is balanced.
(b) \( CG^+ \) is balanced.
(c) \( RG^+ \) is balanced.

Proof. We will only establish the equivalence of (a) and (b), the proof that (a) is equivalent to (c) being similar. Suppose \( BG^+ \) is balanced. Then the points of \( BG^+ \) can be partitioned into two disjoint subsets \( B_1 \) and \( B_2 \) such that positive lines join points of the same subset and negative lines join points of different subsets. Define \( C_1 = B_1 \cap C \) and \( C_2 = B_2 \cap C \).

Suppose first that \( c_i \) and \( c_j \) in \( C_1 \) are joined in \( CG^+ \). Then there exists a row point \( r_i \) such that \( [c_i, r_i] \) and \( [c_j, r_i] \) are lines in \( BG^+ \). If \( r_i \in B_1 \) both lines are positive and if \( r_i \in B_2 \) both are negative. In the latter case \( a_{ii} < 0 \) and \( a_{ij} < 0 \) so \( [c_i, c_j] \) is positive. A similar argument works if both points are in \( C_2 \).

Now suppose \( c_i \in C_1 \), \( c_j \in C_2 \) and \( [c_i, r_i] \) and \( [c_j, r_i] \) in \( BG^+ \). If \( r_i \in B_1 \), then \( [c_i, r_i] \) is positive, \( [c_j, r_i] \) is negative and it follows that \( [c_i, c_j] \) is negative. A similar result holds if \( r_i \in B_2 \).

Thus we have shown that \( CG^+ \) is balanced and so (a) implies (b).

Assume next that \( CG^+ \) is balanced so that the points in \( CG^+ \) can be partitioned into disjoint subsets \( C_1 \) and \( C_2 \) such that positive lines join points of the same subset and negative lines join points of different subsets. We construct disjoint subsets \( B_1 \) and \( B_2 \) of \( BG^+ \) as follows:

\[
\tilde{R} = \{ r_i \in R : [r_i, c] \text{ is positive for some } c \in C_1 \},
\]=
\[
\tilde{\tilde{R}} = \{ r_i \in R : [r_i, c] \text{ is negative for some } c \in C_2 \text{ and } r_i \text{ is not adjacent to any } c \in C_1 \},
\]

\[
R_1 = \tilde{R} \cup \tilde{\tilde{R}}, \quad R_2 = R - R_1,
\]

\[
B_1 = R_1 \cup C_1, \quad B_2 = R_2 \cup C_2.
\]

It is clear that \( B_1 \cap B_2 = \emptyset \) and \( B_1 \cup B_2 = R_1 \cup C_1 \cup R_2 \cup C_2 = R \cup C \).

First we show that all lines joining points in \( B_1 \) are positive. Suppose \( [r_i, c_j] \) is negative for some \( c_i, c_j \in B_1 \). Then \( r_i \notin \tilde{R} \), so \( r_i \notin \tilde{\tilde{R}} \) and there is a point \( c_i \in C_1 \) such that \( [c_i, r_i] \) is positive. But then \( a_{ii} < 0 \) and \( a_{ij} > 0 \) so that \( [c_i, c_j] \) is negative in \( CG^+ \), a contradiction. Next we show that all lines between points in \( B_2 \) are positive. Suppose \( [r_i, c_j] \) is negative for some \( r_i, c_j \in B_2 \). Since \( r_i \notin \tilde{R} \), then there is a point \( c_i \in C_1 \) adjacent to \( r_i \) and \( [r_i, c_i] \) is negative. But then \( a_{ii} < 0 \) and \( a_{ij} < 0 \), so that \( [c_i, c_j] \) is a positive line in \( CG^+ \), a contradiction.

Finally we show that all lines between points in \( B_1 \) and \( B_2 \) are negative. Suppose there is a line from a point in \( R_2 \) to a point in \( C_1 \). Then this line must be negative by definition. Suppose there is a line from a point \( r_i \in R_1 \) to a point \( c_j \in C_2 \). If \( r_i \in \tilde{R} \), then \( [r_i, c_j] \) is a line for some \( c_i \in C_1 \). Then \( [r_i, c_i] \) is positive and \( [c_i, c_j] \) is a line in \( CG^+ \) so it must be negative. Thus, since \( a_{ii} > 0 \), we have \( a_{ij} < 0 \) so that \( [r_i, c_i] \) is negative. If \( r_i \in \tilde{\tilde{R}} \), then \( [r_i, c_j] \) is negative for some \( c_i \in C_2 \). But this means \( c_i \) and \( c_j \) are adjacent in \( CG^+ \), so \( [c_i, c_j] \) must be positive and hence \( [r_i, c_i] \) must be negative.

We have thus shown that \( B_1 \) and \( B_2 \) satisfy the conditions for \( BG^+ \) to be balanced, so (b) implies (a).

Theorem 3 adds to the list of properties shared by the various graphs associated with a regular rectangular matrix. We now connect the form of the matrix \( A \) to balance.
Theorem 4. \(BG^+\) is balanced if and only if there exist permutation matrices \(P\) and \(Q\) such that

\[
PAQ = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where \(A_{11} \geq 0, A_{22} \geq 0, A_{12} \leq 0, A_{21} \leq 0\).

Proof. Suppose \(BG^+\) is balanced. Then the points of \(B\) can be partitioned into disjoint subsets \(B_1\) and \(B_2\) (one of which may be empty) such that positive lines join points of the same subset and negative lines join points of different subsets. Furthermore, \(B_1 = R_1 \cup C_1\) and \(B_2 = R_2 \cup C_2\). If one of the sets, say \(B_2 = \emptyset\), then \(A \equiv 0\) and the result is trivial. In the remaining cases there exist permutation matrices \(P\) and \(Q\) such that

\[
PAQ = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}.
\]

From the restrictions on the lines in \(BG^+\) we clearly have that \(A_{11} > 0, A_{12} < 0, A_{21} < 0, A_{22} > 0\). Since any of the sets \(R_i\) or \(C_i\), \(1 < i < 2\) may be empty, \(PAQ\) may contain one block, two blocks or all four blocks. For the converse suppose there exist permutation matrices \(P\) and \(Q\) such that \(PAQ\) has the above form. Then choosing \(R_i\) and \(C_i\) as before and setting \(B_1 = R_1 \cup C_1, B_2 = R_2 \cup C_2\) we see that \(BG^+\) is balanced.

The following simple lemma shows that whenever \(CG^+\) and \(RG^+\) are defined, certain cliques in these graphs must be balanced.

Lemma 3. If a \(k\)-clique in \(CG^+\) or \(RG^+\) arises from the \(k\) nonzeros in a row (column) of \(A\), then the \(k\)-clique is balanced.

Proof. Suppose a \(k\)-clique in \(CG^+\) is determined by \(k\) nonzeros in a row. Let \(C_1\) be the set of columns having a positive entry in the row and \(C_2\) the set of columns having a negative entry. Clearly every line joining points in the same set is positive and every line joining points in different sets is negative. Therefore the \(k\)-clique is balanced.

We caution the reader that the lemma does not imply that every clique of \(CG^+\) or \(RG^+\) is balanced; it only identifies the existence of a spanning set of balanced cliques. To show this consider the PFFM \(A\) with

\[
\text{sgn } A = \begin{bmatrix}
- & - & 0 & 0 & 0 & + & + \\
0 & - & - & 0 & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 \\
+ & + & + & 0 & - & 0 \\
0 & 0 & 0 & + & + & - \\
0 & 0 & 0 & + & 0 & -
\end{bmatrix}.
\]

The graph \(RG^+\) is shown in Fig. 3. The cliques \(\langle 1, 4, 5 \rangle, \langle 1, 2, 4, 5 \rangle\) and \(\langle 2, 4, 5 \rangle\), for example, are not balanced. Yet the spanning set \(S = \{\langle 1, 3, 4 \rangle, \langle 1, 2, 4 \rangle, \langle 2, 3, 4 \rangle, \langle 2, 4, 5 \rangle\}\),
(3, 5, 6), (3, 4, 5, 6), (1, 2, 5), (1, 6)\) consists of balanced cliques. (Here we are using the notation of Harary [11] for the subgraph generated by the set \(X\), namely \(\langle X \rangle\).

In view of our results in Theorems 3 and 4 we are tempted to call the \(m \times n\) matrix \(A\) a Morishima matrix if \(CG^+\) is balanced. This will, however, introduce an inconsistency when \(m = n\). As an example consider the \(4 \times 4\) matrix \(A\) with sign patterns

\[
\text{sgn } A = \begin{bmatrix}
- & 0 & - & 0 \\
+ & + & + & 0 \\
0 & + & 0 & + \\
+ & 0 & + & +
\end{bmatrix}.
\]

The graphs \(CG^+\), \(RG^+\), and \(\bar{D}^+(A)\), the signed directed graph of \(A\) with loops omitted, are illustrated in Fig. 4. Since the graph \(\bar{D}^+(A)\) has negative cycles, this matrix is not a Morishima matrix, yet both \(CG^+\) and \(RG^+\) are balanced. Therefore we introduce the following basic concept.

**Definition 1.** The \(m \times n\) matrix \(A\) will be called a quasi-Morishima matrix if \(BG^+\) is balanced.

The remainder of our results will be valid for all quasi-Morishima matrices including the case \(m = n\).

**Theorem 5.** Let \(A\) be a quasi-Morishima matrix. Then \(A^TA\) and \(AA^T\) are (symmetric) Morishima matrices. If \(A\) has a nonzero element in each row and column then \(A^TA\) and \(AA^T\) have positive diagonal elements.

**Proof.** The last statement of the theorem is trivial so we prove only the first statement. Since \(A\) is quasi-Morishima there exist permutation matrices \(P\) and \(Q\) such that

\[
B = PAQ = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix},
\]

where \(B_{ij} \geq 0\), \(B_{22} \geq 0\), \(B_{12} \leq 0\), \(B_{21} \leq 0\). Then

\[
B^T = (PAQ)^TPAQ = Q^TA^TP^TPAQ; \quad \text{hence} \quad B^T = Q^TA^TAQ.
\]

Calculating \(B^T\), we set

\[
B^T = \begin{bmatrix}
B_{11}^T & B_{12}^T & B_{13}^T & B_{14}^T \\
B_{21}^T & B_{22}^T & B_{23}^T & B_{24}^T \\
B_{31}^T & B_{32}^T & B_{33}^T & B_{34}^T \\
B_{41}^T & B_{42}^T & B_{43}^T & B_{44}^T
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}.
\]

From the signs in the block form of \(A\) we get that \(C_{11} \geq 0\), \(C_{22} \geq 0\), \(C_{12} \leq 0\) and \(C_{21} \leq 0\). Furthermore, if \(B_{11}\) is \(r \times s\), it is easy to show that \(C_{11}\) is \(s \times s\) and \(C_{22}\) is \((n-s) \times (n-s)\). It follows that \(A^TA\) is a Morishima matrix. A similar argument shows that \(AA^T\) is a Morishima matrix.

This theorem shows that the class of quasi-Morishima matrices is closely connected to the class of Morishima matrices. We will show how the Perron–Frobenius theorem applies to these matrices, but we require some preliminary results first.
THEOREM 6. Let $A$ be a regular quasi-Morishima matrix. Then there exist diagonal matrices $D_1$ and $D_2$ with diagonal elements $\pm 1$ such that $D_2AD_1 \equiv 0$.

Proof. Without loss of generality we can assume that $A$ itself is nonnegative; otherwise we can choose $D_1$ to be the $n \times n$ identity matrix and $D_2$ the $m \times m$ identity matrix. By Theorem 4 we can find $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix},$$

where all of the blocks are nonnegative. Suppose $A_{11}$ is $r \times s$. Let $D_{10} = \text{diag}(1, \cdots , 1, -1, \cdots , -1)$ where there are $s$ 1's and $n - s$ -1's, and let $D_{20} = \text{diag}(1, \cdots , 1, -1, \cdots , -1)$ where there are $r$ 1's and $m - r$ -1's. Then $D_{20}PAQD_{10}$ is nonnegative. Define $D_2 = P^{-1}D_{20}P$ and $D_1 = QD_{10}Q^{-1}$, then $D_2AD_1 = P^{-1}D_{20}PAQD_{10}Q^{-1} \equiv 0$, proving the theorem.

Note that the matrices $D_1$ and $D_2$ are in general not unique if $CG^+$ is not connected. We will discuss this and related matters in a subsequent paper.

Before applying this theorem let us consider an example. Let $A$ be the quasi-Morishima matrix with sign pattern

$$\text{sgn } A = \begin{bmatrix} + & 0 & - & - & + & 0 \\ 0 & + & - & 0 & 0 & - \\ - & 0 & 0 & + & - & + \\ 0 & 0 & + & + & 0 & + \end{bmatrix}.$$ 

We can compute $D_1$ directly from the balanced graph $CG^+$ by partitioning its points into subsets $C_1$ and $C_2$ so that the lines have the usual properties. We obtain $C_1 = \{1, 2, 5\}$, $C_2 = \{3, 4, 6\}$. If we choose signs so that the scalar product of $D_1$ with row 1 is positive, we obtain $D_1 = \text{diag}(1, 1, -1, -1, 1, -1)$ and

$$\text{sgn } AD = \begin{bmatrix} + & 0 & + & + & + & 0 \\ 0 & + & + & 0 & 0 & + \\ - & 0 & 0 & - & - & - \\ 0 & 0 & - & - & 0 & - \end{bmatrix},$$

so that the sign pattern of the columns is $[+ + - -]$. Thus we choose $D_2 = \text{diag}(1, 1', -1, -1)$ and $D_2AD_1 \equiv 0$.

Next observe that $(D_2AD_1)^T = D_2^T A^T D_1^T$ hence

$$(D_2AD_1)^T D_2AD_1 = D_2^T A^T D_1^T D_2AD_1.$$ 

But $D_2^T = D_2$ so $D_2^T D_2$ is the identity matrix and

$$(D_2AD_1)^T D_2AD_1 = D_1 A^T AD_1 \equiv 0.$$ 

It follows that the matrix $A^T A$ is similar to a nonnegative matrix, the similarity being effected by the diagonal matrix $D_1$. Similarly we have

$$D_2AD_1(D_2AD_1)^T = D_2AA^T D_2 \equiv 0.$$ 

Now it is well known that the matrices $AA^T$ and $A^T A$ are symmetric and nonnegative definite (see [12], for example). Therefore the same will be true of the
matrices $D_1^T A^T A D_1$ and $D_2 A A^T D_2^T$. We are now ready to apply the Perron–Frobenius theory (see [12] for an elementary, but thorough, discussion).

**Theorem 7.** Let the $m \times n$ matrix $A$ be a quasi-Morishima matrix with at least one nonzero element in each row and column and suppose that $CG$ is connected. Then the matrix $A^T A$ has a simple eigenvalue $r_m$ equal to its spectral radius and, if $\lambda$ is any other eigenvalue of $A^T A$, we have $0 \leq \lambda < r_m$. Moreover, the eigenvector $y$ belonging to $r_m$ has all its components different from zero and its sign pattern is such that $D_1 y$ is a strictly positive vector $D_1 y > 0$, i.e., $y$ has the same sign pattern as $D_1$.

**Proof.** We apply the Perron–Frobenius theorem to the nonnegative matrix $A = D_1^T A^T A D_1$. Since $CG$ is connected, $A$ is irreducible. Also $a_{ii} > 0$, $1 \leq i \leq n$, because each column of $A$ has a nonzero element. Thus $A$ is primitive. The theorem follows.

For completeness we state the corresponding theorem for $AA^T$.

**Theorem 7’.** Let the hypotheses of Theorem 6 be satisfied. Then the matrix $AA^T$ has a simple eigenvalue $r_m$ equal to its spectral radius and, if $\lambda$ is any other eigenvalue of $AA^T$, we have $0 \leq \lambda < r_m$. Moreover, the eigenvector $y$ belonging to $r_m$ has all its components different from zero and the same sign pattern as $D_2$.

We note that it is known that $r_m = r_n$ and, more generally, that the spectrum of $AA^T$ is the same as that of $A^T A$ except that one of these matrices may have more zero eigenvalues than the other depending upon which of $m$ and $n$ is larger.

4. Applications. Let us begin with an examination of elements in the class PFM. Consider first the graph $RG^+(A)$ for $A \in\text{PFM}$. We can partition the points of $RG^+$ into two disjoint sets $S$ and $M$ with the property that lines joining points of $S$ or points of $M$ are positive and lines joining a point of $S$ and a point of $M$ are negative. It follows that $RG^+$ is balanced. We thus have as a consequence of Theorem 3 the following result.

**Theorem 8.** If $A \in\text{PFM}$, then $A$ is a quasi-Morishima matrix.

We now know that for $A \in\text{PFM}$ the graph $CG^+$ is balanced, but we are interested in the structure of this graph in any case. In fact the points of $CG^+$ can be partitioned into three disjoint sets $P$, $T$ and $K$ such that each line joining points in the same set is positive, each line joining points in different sets is negative, and no line joins a point in $P$ with a point in $K$.

If a matrix in the class PFM is regular the graphs $RG^+$ and $CG^+$ satisfy the following conditions:

(\(\alpha\)) Each $S$ point in $RG^+$ is adjacent to at least one $M$ point and each $M$ point to at least one $S$ point.

(\(\beta\)) Each $P$ point in $CG^+$ is adjacent to at least one $T$ point and each $K$ point to at least one $T$ point.

(\(\gamma\)) Each $T$ point in $CG^+$ is adjacent to at least one $P$ point and at least one $K$ point.

The effect of imposing regularity on a PFM is to insure that such matrices do describe a sort of flow. In fact the conditions guarantee that, when $A$ is permuted into the form (1), to each nonzero element in a row of $A_{11}$ there corresponds a nonzero element in the same row in $A_{12}$, to each nonzero element in a column of $A_{12}$ there corresponds a nonzero element in the same column of $A_{22}$, and to each nonzero element in a row of $A_{22}$ there corresponds a nonzero element in the same row of $A_{23}$. Thus there exists a connection (or flow) between elements of $A_{11}$ and elements of $A_{23}$.

The structure of a PFM as displayed in (1) suggests a generalization of this class which still retains the properties of balance in $CG^+$ and $RG^+$. A matrix $A$ will be
called a generalized physical flows matrix (GPFM) if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix}
A_{11} & A_{12} & 0 & \cdots & 0 & 0 \\
0 & A_{22} & A_{23} & & 0 & 0 \\
0 & 0 & 0 & & A_{pp} & A_{p,p+1}
\end{bmatrix},$$

where $A_{ii} \geq 0$, $A_{i,i+1} \leq 0$, $1 \leq i \leq p$. It is clear that the graphs $RG^+$ and $CG^+$ exist for $A \in$ GPFM. Let us show that $RG^+$ is balanced for $p \geq 3$.

Note that $RG^+$ can be partitioned into disjoint sets $S_i$, $1 \leq i \leq p$, such that lines joining two points of $S_i$ are positive for $1 \leq i \leq p$, lines joining a pair of points one in $S_i$ and one in $S_{i+1}$ are negative, $1 \leq i \leq p - 1$, and no line joins a point in $S_i$ to a point in $S_j$ if $|i - j| > 1$. Therefore we may partition the points of $RG^+$ into the disjoint sets $U$ (which is the union of the sets $S_i$ for $i$ odd) and $V$ (which is the union of the sets $S_i$ for $i$ even), and the sets $U$ and $V$ satisfy the conditions for balance. Thus $RG^+$, and hence $CG^+$ is balanced, and every element of GPFM is a quasi-Morishima matrix.

The form of $RG^+$ and $CG^+$ for $A \in$ GPFM suggests the following ideas which lead to another generalization of the class PFM still within the class of quasi-Morishima matrices. Define the signed graph $G^+$ to be a signed ladder graph of order $p \geq 2$ if the points of $G^+$ can be partitioned in $p$ disjoint sets $L_i$, $1 \leq i \leq p$, such that lines joining two points in $L_i$ are positive, lines joining two points one in $L_i$ and one in $L_{i+1}$ are negative, $1 \leq i \leq p - 1$, and no line joins a point in $L_i$ to a point in $L_j$ if $|i - j| > 1$. Then $G^+$ is balanced. Define the matrix $A$ to be a quasi physical flows matrix (QPFM) if $RG^+(A)$ is a signed ladder graph of order $p$ and if $CG^+$ is a signed ladder graph of order $p + 1$. It is clear that GPFM $\subseteq$ QPFM, but the converse is not true. It is also easy to see how to impose regularity conditions on the classes GPFM and QPFM. But even with regularity conditions imposed it is still not true that GPFM $=$ QPFM. Here is a simple example. Let

$$A = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 1 & -1
\end{bmatrix}.$$

Figure 5 illustrates $RG^+(A)$ and $CG^+(A)$. It is clear that $RG^+$ is a signed ladder graph of order $p = 2$ and $CG^+$ a signed ladder graph of order 3 so that $A \in$ QPFM. It is also clear that $A \notin$ GPFM, and that regularity conditions hold for $A$.

![Diagram](image)

**FIG. 5**

We do not have a characterization of matrices in the class QPFM in block form. This is an interesting open question and we feel that its solution would provide some useful insights for energy modelers and others concerned with large scale systems.
We will conclude with a brief discussion of the class PFFM. The example illustrated in Fig. 2 of the smallest regular PFFM shows that not every element in this class is a quasi-Morishima matrix. However the PFFM
\[
A = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]
satisfies the conditions for a QPFM so it is a quasi-Morishima matrix.

It turns out that even if \( A \) is a regular PFFM it can be a quasi-Morishima matrix. We do not have a simple criterion that will enable us to characterize which elements of this class are quasi-Morishima and which are not. This must be regarded as an open problem.

As a final application let us turn to an entirely different area. In the theory of Lanchester models of military combat between heterogeneous forces (see [10] for some interesting examples) systems of differential equations having the form
\[
\dot{x} = -Ay, \quad \dot{y} = -Bx
\]
must be solved. In the simplest cases \( A \) is an \( m \times n \) nonnegative matrix and \( B \) an \( n \times m \) nonnegative matrix. The matrix of the system is therefore the quasi-Morishima matrix
\[
a = \begin{bmatrix}
0 & -A \\
-B & 0
\end{bmatrix}
\]
The spectral properties of the system are most conveniently obtained by making use of Theorem 7.

A problem more general than the above is obtained when logistic considerations are incorporated into the equations of combat and, in particular, when one or both armies is being reinforced. In this case (see [13]) we obtain systems having the form
\[
\dot{x} = A_1 x + A_2 y, \quad \dot{y} = B_1 x + B_2 y,
\]
where \( A_1, A_2, B_1 \) and \( B_2 \) are rectangular matrices in general. When this happens one can identify certain key submatrices which are quasi-Morishima matrices. Here again the Perron–Frobenius theorem can be applied to obtain useful results.

REFERENCES