An Application of a Lagrangian Penalty Function to Obtain Optimal Redundancy

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A system is defined whereby there are $N_s$ subsystems in passive redundancy, and each subsystem has $N_A$ components in active redundancy. The mean time to failure is derived under the assumption of independent failure time, and this is used as a measure of system effectiveness. A budget constraint limits the number of components and switches which may be used to construct a system. This paper finds the maximum mean time to failure for different values of the budget parameters.

Consider a system with $N$ components. The system is operative when any component is operating. Otherwise, the system is said to have failed to operate. We assume that each component has the same average failure rate, denoted by $\lambda$. Further, we assume that the time to failure for any component is an independent random variable with a negative exponential probability distribution.

There are two types of redundancy we wish to consider. First, active (i.e., parallel) redundancy is defined as components which operate simultaneously. A system with active redundancy fails when all components are inoperative. The mean time failure for an active system is given by:

$$MTFA = \frac{1}{\lambda} \sum_{i=1}^{N} \frac{1}{i}$$  \hspace{1cm} (1)

The second type of redundancy is passive or stand-by. In this system a component operates while the others do not. In the event of a failure, a switch, whose reliability is denoted $r$, activates the next component (if there are any remaining). The system fails if any switch fails or if all components fail. The mean time failure for a passive system with $N$ components (and $N - 1$ switches) is given by:

$$MTFP = \frac{1}{\lambda} \frac{(1 - r^N)}{1 - r}$$

We wish to investigate a general active-passive system. Consider a subsystem with $N_A$ components in active redundancy. The total system has $N_s$ sub-systems in passive redundancy. The active-passive system is uniquely defined by the configuration couple $(N_A, N_s)$. Figure 1 illustrates an active-passive system with a configuration couple (3, 2).

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From (1) the mean time to failure is given by:

\[ \text{MTF} (N_A, N_p) = \frac{1}{\lambda} \frac{1 - r^{N_p}}{1 - r} \sum_{i=1}^{N_A} \frac{1}{i} \]

We shall designate the harmonic sum as

\[ H(N_A) = \sum_{i=1}^{N_A} \frac{1}{i} \]

The mean time to failure as a multiple of component failure rate is thus given by:

\[ f(N_A, N_p) = \lambda \text{MTF} (N_A, N_p) = \frac{1 - r^{N_p}}{1 - r} H(N_A). \]

Various approaches have been taken to find optimal redundancy when \( N_A = 1 \) (e.g., Giesler and Karr [5]) or when \( N_p = 1 \) (e.g., Kettelle [6]). In this paper we seek to optimize redundancy for the general active-passive system. That is, we seek to maximize \( f(N_A, N_p) \). However, we must limit the number of components and switches due to a cost constraint. Let \( C_s \) be the cost of each switch and let \( C_c \) be the cost of each component. Further, let the total cost be given by

\[ C(N_A, N_p) = C_s N_A N_p + C_c (N_p - 1). \]

Hence, we have the non-linear integer program:

Maximize \( f(N_A, N_p) \) subject to

1. \( C(N_A, N_p) \leq b' \)
2. \( N_A, N_p \geq 1 \)
3. \( N_A, N_p \) integer-valued.
We wish to solve parametrically on $b'$. This will provide a cost-effectiveness curve. The inputs are $r$, $C_s$, and $C_*$. We shall choose our cost units so that $C_* = 1$ (i.e., divide $C(N_A, N_*)$ by $C_*$). We thus have the problem (for $r < 1$):

$P$: Maximize $H(N_A)(1 - r^{N_*)})$ subject to

1. $(N_A + C_*)N_* \leq b$

Note that constraint (1) has been modified somewhat so that

$$b = b' + C_*$$

The procedure employed to solve the non-linear integer program is the penalty function approach as described in M. Bellmore, H. J. Greenberg, and J. J. Jarvis [2]. For further reading we suggest H. Everett [3] and W. Nunn [8].

In an effort to solve our problem ($P$) we solve:

$P'(y)$: Maximize $\log [H(N_A)(1 - r^{N_*)}] - y \log [(N_A + C_*)N_*]$ subject to

1. $N_A, N_* \geq 1$
2. $N_A, N_*$ integer-valued

where $y$ is specified so that $y > 0$.

Everett [2] has shown that if $(N_A^*, N_*^*)$ solves $P'(y)$, then

1. $(N_A^*, N_*^*)$ solves $P$ for $b = (N_A^* + C_*)N_*^*$
2. For all $(N_A, N_*) \geq 1$ we have

$$H(N_A)(1 - r^{N_*}) \leq \left(\frac{b}{b(y)}\right)^y H(N_A^*)(1 - r^{N_*})$$

where $b = (N_A + C_*)N_*$.

Result (1) gives us a solution for some $b$, and result (2) gives us an upper bound for all other $b$.

Everett has demonstrated that not necessarily every value of $b$ can be obtained by some choice of $y$. However, Tables I through IV give the values obtained, and we shall discuss interpolation in the next section.

Another result is useful when determining how to search $y$ in order to obtain certain values $b = b(y)$. Everett has shown that if $y_1 > y_2$, then $b(y_1) \leq b(y_2)$. This result was employed in the algorithm designed to solve this problem.

The algorithm is coded in FORTRAN IV for the IBM 360. Tables I through IV give solutions for some values of $b$. Before presenting some examples, note that $P'(y)$ is separable. That is, we solve 2 problems as

$P_1(y)$: Maximize $(1 - r^{N_*)}/(N_*^*)$ subject to $N_* \geq 1$ and $N_*$ integer.

$P_2(y)$: Maximize $[H(N_A)]/[(C_* + N_A)^*]$ subject to $N_A \geq 1$ and $N_A$ integer.
### Table I

$r = .5$

<table>
<thead>
<tr>
<th>CS</th>
<th>$y$</th>
<th>$N_A^*$</th>
<th>$N_P^*$</th>
<th>$b-N_P^* C_S$</th>
<th>$H(N_A^*)/(1-r^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 to</td>
<td>.50</td>
<td>2</td>
<td>2</td>
<td>4.0</td>
<td>1.125</td>
</tr>
<tr>
<td>1.5</td>
<td>.45</td>
<td>3</td>
<td>2</td>
<td>6.0</td>
<td>1.375</td>
</tr>
<tr>
<td></td>
<td>.40</td>
<td>5</td>
<td>2</td>
<td>10.0</td>
<td>1.712</td>
</tr>
<tr>
<td></td>
<td>.35</td>
<td>8</td>
<td>3</td>
<td>24.0</td>
<td>2.378</td>
</tr>
<tr>
<td></td>
<td>.30</td>
<td>13</td>
<td>.3</td>
<td>39.0</td>
<td>2.783</td>
</tr>
<tr>
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<td>.25</td>
<td>28</td>
<td>3</td>
<td>84.0</td>
<td>3.436</td>
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<tr>
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<td>1.0</td>
<td>0.500</td>
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<tr>
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<td>2.0</td>
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### Table II

$r = .75$

<table>
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<tr>
<th>CS</th>
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<th>$N_A^*$</th>
<th>$N_P^*$</th>
<th>$b-N_P^* C_S$</th>
<th>$H(N_A^*)/(1-r^2)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4</td>
<td>8.0</td>
<td>1.025</td>
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<td>.45</td>
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<td>6</td>
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<tr>
<td></td>
<td>.35</td>
<td>8</td>
<td>6</td>
<td>48.0</td>
<td>2.234</td>
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<td>.30</td>
<td>13</td>
<td>7</td>
<td>91.0</td>
<td>2.756</td>
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<td>.25</td>
<td>28</td>
<td>8</td>
<td>224.0</td>
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<tr>
<td>0.</td>
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<td>0.250</td>
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<tr>
<td></td>
<td>2.0</td>
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<td>0.5</td>
<td>1.6 to</td>
<td>1</td>
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<td>1.0</td>
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<td>2.0</td>
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<tr>
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<td>1.0 to 1.6</td>
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<td>2.0</td>
<td>0.375</td>
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<tr>
<td>1.0 to</td>
<td>1.0 to</td>
<td>2</td>
<td>1</td>
<td>2.0</td>
<td>0.375</td>
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<td>3.0</td>
<td>2.0</td>
<td></td>
<td></td>
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</table>
One can use methods of difference calculus [7] as a search scheme analogous to a gradient search. That is, we can search $N_A$ and $N_P$ as

$$N^*_s = N_o + sD$$

where $s$ is a direction (1, 0, or -1) and $D$ is a step size.

The present program uses such a direct method\(^1\). We begin with $N_A = N_P = 1$. Using the forward difference operator, $N_A(N_A)$ is increased one unit at a time until the objective of $P1$ ($P2$) gets worse. Since we have unimodality, the maximum is achieved (in a finite number of steps). If $N^*_A$ (or $N^*_P$) exceeds 30, we stop the incrementation (see for example Table IV). When a parameter is changed (viz., $\lambda$, $r$, or $C_s$), the starting value is the previous solution. In that case backward differences may be used to investigate decreasing $N_A$ or $N_P$.

We use no particular search scheme for $\lambda$. The choices given in the tables were selected to give a range for $b$. More specific search schemes to solve for a given $b$ appear in Bellmore, Greenberg, and Jarvis [2].

\(^1\) There is no implication that our scheme is most efficient. Tables I-IV were generated in approximately 5 minutes total time on the IBM 360.
Now consider the following example:
For \( r = 0.5 \), what is the optimal system for \( C_s = 1.0 \) and \( b = 12 \)?

Solution: \((N_A^*, N_P^*) = (5, 2)\) (Table 1).

Now let us consider interpolation when \( b \) is not one of the values listed. Let \( b_1 = \max \{ b_i : b_i \leq b \} \) and \( b_2 = \min \{ b_i : b_i \geq b \} \) where \( b_i \) are the listed values.

Hence, we have

\[ b_1 < b < b_2 \]

(since \( b \) is not listed). For example, suppose \( r = 0.5 \), \( C_s = 0.5 \), and \( b = 20 \). Then, \( b_1 = 11 \) and \( b_2 = 26 \).

Let \( f(b) = \max H(N_A) (1 - r^{N_P}) \) subject to

1. \( (C_s + N_A) N_P \leq b \)
2. \( N_A, N_P \geq 0 \)
3. \( N_A, N_P \) integer-valued.

By result (2) we have

\[ f(b) \leq \min \left[ f(b_1) \left( \frac{b}{b_1} \right)^{b_1}, f(b_2) \left( \frac{b}{b_2} \right)^{b_2} \right] \]

Hence,

\[ f(20) \leq \min \left[ 1.7125 \left( \frac{20}{11} \right)^4, 2.378 \left( \frac{20}{26} \right)^{35} \right] \]

or

\[ f(20) \leq 2.31. \]
We thus have immediate bounds as

\[ 1.7125 \leq f(20) \leq 2.31. \]

We shall start with \( N_A \) and \( N_\star \) given by the following convex combinations:

\[
N_A = \left[ N_A(b_1)/(b - b_1) + N_A(b_2)/(b_2 - b) \right]/[1/(b - b_1) + 1/(b_2 - b)]
\]

and

\[
N_\star = \left[ N_\star(b_1)/(b - b_1) + N_\star(b_2)/(b_2 - b) \right]/[1/(b - b_1) + 1/(b_2 - b)].
\]

For our example we have

\[
N_A = \frac{\frac{5}{9} + \frac{8}{6}}{\frac{1}{9} + \frac{1}{6}} = \frac{10 + 24}{5} = \frac{34}{5}
\]

and

\[
N_\star = \frac{\frac{2}{9} + \frac{3}{6}}{\frac{1}{9} + \frac{1}{6}} = \frac{4 + 9}{5} = \frac{13}{5}
\]

The solution may be among the neighboring integers as

1. (6, 2)
2. (6, 3)
3. (7, 2)
4. (7, 3)

Candidates number (2), (3) and (4) each has a cost greater than 20 and are therefore infeasible. Candidate (1) is feasible and

\[ H(6)(1 - r^2) = 2.45(.75) = 1.83. \]

Thus, we have improved the lower bound so that

\[ 1.83 \leq f(20) \leq 2.21. \]

We can stop here with the "good solution" \( (N_A, N_\star) = (6, 2) \), or we can proceed to try an improvement. If there are many missing values, then we can take a finer set of \( y \) values and therefore obtain more \( b \)-values. Otherwise, interpolation serves as an approximation.

**Examples**

1. Suppose the cost of each component is \$2, and the switch cost is also \$2. Further, let \( r = .75 \), and suppose we have a total budget of \$38. Find the optimum configuration.

**Solution:** We have that \( C(N_A, N_\star) \leq 38 \) so

\[
b = \frac{38 + C_\star}{C_c} = 20
\]
Table II for $C_s = C_c$ (i.e., $C_s = 1$ and our cost unit is $2$) gives for $y = .45$, $b = 15 + 5(1) = 20$. Hence, $(3, 5)$ is the solution.

(2) What is the percentage increase in MTF when the budget is increased by $p$ percent? Assume $C_s = 0$, $C_c = 1$, $r = .90$ and initial budget = 24.

Solution: From Table III we have

$$\text{MTF} = 1.076 \quad \text{for} \quad b = 24.$$ 

Let $p = 100$. Then the new maximum system effectiveness is 1.414. Hence, if $X$ is the percentage of increase in system effectiveness, we have

$$1.414 = 1.076(1 + X)$$

or $X = .312$. Thus, if the budget is doubled, the system effectiveness will increase by 30%.

For $b = 75$ we have that the system effectiveness is 1.813. Hence,

$$1.813 = (1 + X)1.414$$

or $X = .280$.

Thus, if the budget is increased by more than 75%, the system effectiveness will increase by nearly 30%.

Conclusions

The penalty function approach has provided the desired cost-effectiveness relations and the corresponding optimal configuration. It is noted that the assumption of independence is crucial to the analysis, and the optimal configuration couple observed here is an over-estimate of the true optimum if catastrophic modes of failure are present.

References