TECHNICAL NOTE

A Lagrangian Property for Homogeneous Programs

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Abstract. Euler's equation is applied to the response function of a mathematical program whose maximand and constraint functions are positively homogeneous over the policy cone. An algorithmic advantage (using GLM) is cited, and certain posynomial geometric programs are distinguished.

1. Introduction

In this short note, we point out a property of mathematical programs whose maximand and constraint functions are homogeneous. If, in addition, the program is convex, then this property can be used to find a solution with the generalized Lagrange multiplier method (GLM) requiring at most two Lagrangian maximizations. Some of this work was influenced by the earlier report of Nunn (Ref. 1).

Define $f^*(b) = \sup\{f(x) \mid g(x) \leq b \text{ and } x \in C\}$, where we assume the following: (i) $C$ is a nonempty, closed cone; (ii) $f$ is positively homogeneous of degree $p$, that is, $f(tx) = t^p f(x)$ for all $x \in C$ and $t \geq 0$; (iii) $g$ is positive homogeneous of degree $q \neq 0$; and (iv) $f^* < +\infty$ for all $b \in \mathbb{R}^m$. Then, the set of feasible right-handsides (i.e., $B \equiv$ range $g + \mathbb{R}_+^m$) is a cone and $f^*$ is positively homogeneous of degree $p/q$.

2. Single Constraint

Consider the case of a single constraint, so $m = 1$. Note that $B = (-\infty, \infty)$ or $B = [0, \infty)$. Further, we have for $b \neq 0$

$$f^*(b + \epsilon) - f^*(b) = f^*(b)[(1 + \epsilon/b)^{p/q} - 1],$$

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so $f^*$ is differentiable at each $b \neq 0$, and
\[ b[\frac{df^*(b)}{db}] = (p/q)f(b) \quad \text{for each} \quad b \neq 0. \]

This is known as Euler's equation.

By our assumption that $f^*(0) < +\infty$, we see that $f^*(0) = 0$, so $f^*$ is continuous and has the closed form given by
\[ f^*(b) = \begin{cases} C_1b^{p/q} & \text{if } b > 0, \\ 0 & \text{if } b = 0, \\ C_2b^{p/q} & \text{if } b < 0, \end{cases} \]

where the case $b < 0$ applies only if $B = (-\infty, \infty)$. Further, $C_1 = C_2 = 0$ if $p = 0$.

If $f^*$ is concave and has a concave conjugate (Ref. 2), then we can choose a Lagrange multiplier $\lambda$, such that the supremum of the Lagrangian on $C$ is finite. If this is achieved at $x^*$, where $g(x^*) \neq 0$, then $C_1$ or $C_2$ is determined. This means that, with at most two Lagrangian maximizations, the entire response curve is obtained. A policy for some $b$ can be obtained by then maximizing the Lagrangian for the correct multiplier given by the derivative of $f^*$ at the desired resource level.

For example, suppose that
\[ C = R_+^n, \quad f(x) = \sum_{j=1}^n c_jx_j^p, \quad g(x) = \sum_{j=1}^n a_jx_j^q, \]

where
\[ c_0 \geq 0, \quad a_0 > 0, \quad 0 < p \leq 1, \quad q \geq 1. \]

Then, the Lagrangian is given by
\[ L(x, \lambda) = \sum_{j=1}^n (c_jx_j^p - \lambda a_jx_j^q). \]

For $q > p$, the maximum occurs at
\[ x^*_j(\lambda) = \frac{p c_j}{q \lambda a_j} (q - 1)^{1/(q - p)} \]
for $\lambda > 0$. Our set of feasible right-handsides is only the positive halfline, so $C_1$ is determined, whence all of $f^*$ is obtained. To obtain an optimal policy for a particular $b > 0$, we merely set
\[ \lambda = C_1(p/q)b^{p/q - 1} \]
and compute $x^*(\lambda)$. For $p = q = 1$, we have a simple linear program, and $x^*(\lambda)$ will generally not be unique.

3. Multiple Cones

For more than two cones, still provides the best result at $b$.

Suppose that $\lambda$ is a Lagrange multiplier.

By Euler's equation, we have
\[ (\lambda, g(x^*)) = (\lambda, \min_{x^*}) \]

Thus, even if $f^*$ is nonconcave, we can minimize $\lambda(b)$ over the entire response curve.

4. Posynomial (Ref. 5)

Duffin (Ref. 2) programs. First, arrive at a homoge

Here, we have

for each $i = 1, \ldots$
3. Multiple Constraints

For more than one constraint, Euler’s equation (see, Ref. 3, p. 109) still provides the relation \( b, \nabla f^*(b) = (p/q)f^*(b) \), provided that \( f^* \) is differentiable at \( b \). However, we cannot ensure that this will be the case.

Suppose that \( C = R_x^m \) and that \( x^* \) is a solution for a particular \( b \). If \( \lambda \) is a Lagrange multiplier, then we have

\[
(x^*, \nabla f(x^*) - \nabla g(x^*)\lambda) = 0.
\]

By Euler’s equation, this implies that

\[
pf(x^*) - q(g(x^*), \lambda) = 0.
\]

But \( (\lambda, g(x^*)) = (\lambda, b) \) and \( f(x^*) = f^*(b) \), so

\[
(p/q)f^*(b) = (\lambda, b).
\]

Thus, even if \( f^* \) is not differentiable at \( b \), the Lagrange multiplier function of \( b \) must satisfy the above equation. Note that, for \( p = q = 1 \), we can economically interpret the homogeneity, whereby the dual price \( \lambda(b) \) times the total resource \( b \) equals our maximum return. In the special case of a linear program, \( \lambda(b) \) is constant.

4. Posynomial Geometric Programs

Duffin (Ref. 4) described a linearization of posynomial geometric programs. First, he uses the fundamental mean-value inequality to arrive at a homogeneous form as follows:

\[
\begin{align*}
\text{minimize} & \quad \prod_{j=1}^{n} x_j^{a_{ij}}, \\
\text{subject to} & \quad \prod_{j=1}^{n} x_j^{a_{ij}} \leq 1, \quad i = 1, ..., m, \\
& \quad x_1, ..., x_n > 0.
\end{align*}
\]

Here, we have

\[
p = \sum_{j=1}^{n} a_{ij} \quad \text{and} \quad q = \sum_{j=1}^{n} a_{ij}
\]

for each \( i = 1, ..., m \). This restricts the class of homogeneous posynomial geometric programs that fit our assumptions.
In the case of a single constraint with feasible dual, we have \( B = (0, \infty) \) and
\[
f^*(b) = \left( \frac{\sum_{j=1}^{n} a_{ij}}{\sum_{j=1}^{n} a_{ij}} \right) (\lambda(b), b).
\]

A Lagrangian minimization will determine \( f^* \) on \( B \) and, with at most one more Lagrangian minimization, we can solve the original problem (with \( b = 1 \)). It should be noted that the lack of convexity structure in the geometric program does not affect the Lagrangian saddle point equivalence (at the heart of this study); this depends only upon convexity (or concavity) of \( f^* \) on \( B \). The usual transformation of variables shows posynomial geometric programs have this property.

Of course, the single constraint case with only one product is equivalent to a linear program by replacing \( x_j \) with its logarithm. However, we can depart from Duffin’s linearization and consider the following geometric program:

\[
\text{minimize} \quad \sum_{i=1}^{m} c_i \prod_{j=1}^{n} x_j^{a_{ij}},
\]

subject to \( \sum_{i=1}^{k} d_i \prod_{j=1}^{n} x_j^{b_{ij}} \leq 1 \),

\( x_1, \ldots, x_n > 0 \).

The homogeneity assumptions are met if
\[
\sum_{j=1}^{n} a_{ij} = p \quad \text{for all} \quad i = 1, \ldots, m \quad \text{and} \quad \sum_{j=1}^{n} b_{ij} = q \quad \text{for all} \quad i = 1, \ldots, k.
\]

References