Computing Market Equilibria with Price Regulations Using Mathematical Programming

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One approach to modeling and solving for economic equilibria relies on mathematical programming. These models solve for competitive equilibria. However, policy analysis often requires measuring the impacts of government price regulations that differ from the competitive equilibrium. In this paper we provide a unified framework for computing market equilibrium in mathematical programming models in the presence of government price regulations. The iterative procedure that we use is essentially a Gauss-Seidel algorithmic strategy. The paper concludes by showing how to represent tax/rebate programs, average-cost pricing, and price ceilings.

ECONOMIC equilibrium models are being used to study the consequences of a wide range of government programs. The equivalence between mathematical programming and certain economic equilibrium problems has become a basic tool for both formulating and solving equilibrium models. (See, for example, Arrow, Hurwicz and Uzawa 1958; Koopmans 1951; Samuelson 1952; Gale 1960; Intriligator 1971; Manne and Markowitz 1963; Takayama and Judge 1971; and Shapiro 1978.) This paper provides a unified framework for computing equilibria in mathematical programming models involving government policies that affect market prices.

In a mathematical programming framework, the market prices in the model's solution are set by the marginal costs of production and distribution. This practice is consistent with an unregulated, competitive economy. Special techniques are necessary to adjust prices to reflect government interventions in the marketplace. In this paper, we synthesize in a unified framework several techniques used to capture the effects of government programs. The underlying concept in our development is a set of procedures for adjusting the unregulated model so that the price and quantity variables converge to a regulated market equilibrium.

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Since 1974 some energy models, such as the Project Independence Evaluation System (PIES) and Hogan (1975), have used ad hoc methods. These methods have worked well in practice, but they are not couched in any formal theory of convergence. The object of this paper is to develop such a theory.

Some previous literature has treated tax programs from a computational perspective. Shoven (1977) has adopted this approach; however, he treated a different issue—how to compute a general equilibrium, balancing tax revenues with expenditures. He developed an extension of Scarf's (1982) algorithm to solve the system of equations. Here, we are concerned with a different dimension of complexity—how to impose taxes when the supply curve is not necessarily a simple function and might be a subprogram in a mathematical program. Furthermore, our computational framework includes other forms of government price regulations.

The paper is organized as follows. First, we describe the unregulated model, using a principle of mathematical programming equivalence. Then, we present a unifying framework for representing government interventions and the main convergence theorem in an abstract form. Since the goal of this paper is algorithmic, we do not discuss the subject of incentives and potential biases induced by regulation.

Given a set of supply and demand curves, one can think of a family of the unregulated model solutions as being defined by an implicit price function of the transportation tariffs. That is, manipulating the tariffs modifies the model solution. The basic approach to representing price regulations involves manipulating the tariffs to reflect the effects of regulation. The convergence theorem is based on two results: (i) the implicit price function is (under mild assumptions) a contractor on the tariff space, and (ii) a Gauss-Seidel algorithmic strategy is convergent if the procedure for adjusting tariffs is Lipshitz continuous, and retains the contraction property. Mathematically, the convergence theorem follows from known results (see, for example, Altman 1977), but the employment of a tariff adjustment function provides a unifying theme to the ad hoc procedures. (See Ahn and Hogan 1982; Hogan, Sweeney and Wagner 1978; Greenberg and Murphy 1980; and Murphy et al. 1981.)

Within the context of the general framework, we use regulatory structures to model three regulatory programs: (1) a tax/rebate program, (2) average-cost pricing, and (3) producer price ceilings. In each case we define a tariff adjustment function and show that it satisfies the appropriate Lipschitz condition that ensures convergence.

1. UNREGULATED EQUILIBRIUM MODEL

In this section we present a central result, namely the mathematical programming equivalency theorem: for a model that expresses the equi-
librium problem in the form of equations and inequalities containing the quantity and price variables to be determined, a mathematical program is constructed that has the same solution.

We describe the equilibrium model here in terms of aggregate agents of the market: producers, consumers, and distributors. The supply or demand curves, which represents price responses, might be a simple function or a complex submodel. While the model presented here is not the general case, it satisfies the central goal of this analysis: to show how to represent government policies in a modeling discipline that is based intrinsically on perfect competition. We call the representation of a policy a "regulatory structure," and we refer to the "regulated model/solution."

The model is a partial equilibrium model since it uses costs rather than incorporates all resources. In addition, so that we might reduce the amount of notation and eliminate side issues, demands are for a selected set of goods, considered as one product. The extension to multiproduct models is immediate using, for example, the PIES solution technique, where the policies and model solution techniques described here were employed.

Model Description

Consider a market composed of a single product and three agents:

\begin{align*}
  & m \quad \text{Producers} \\
  & n \quad \text{Consumers, and} \\
  & 1 \quad \text{Distributor.}
\end{align*}

The producers and consumers can each be distinguished by region, sector, or any other relevant classification.

Each aggregate producer \((i)\) has an associated supply function of price, \(S_i(C_i)\), and each aggregate consumer \((j)\) has a demand function of price, \(D_j(P_j)\). We assume that \(S_i(C_i)\) is strictly monotonically increasing and continuous, and that \(D_j(P_j)\) is strictly monotonically decreasing and continuous. These assumptions guarantee the invertibility of the supply and demand functions. We refer to \(S_i(C_i)\) and \(D_j(P_j)\) as quantity functions, and let \(C_i(S_i)\) and \(P_j(D_j)\) denote the associated (inverse) price functions. We also let \(C(i) = C_i(S_i)\) and \(P(j) = P_j(D_j)\) denote actual solution values and let \(C, P, S, D\) denote the vectors of solution values.

Let \(X(i, j)\) denote the flow from producer \(i\) to consumer \(j\), and let \(X\) be the matrix of flows. Let \(t(i, j)\) be constants representing the transportation costs, i.e., tariffs. Consequently, each consumer's price for a transaction (i.e., for \(X(i, j) > 0\)) equals the producer's price plus the transportation

\[ P(j) = C(i) + t(i, j). \]
The equilibrium problem is to determine

Producer prices and quantities = \( C, S(C) \)
Consumer prices and quantities = \( P, D(P) \)
Distribution = \( X \)

that satisfy the following conditions:
1. Supply balance: \( \sum_{i=1}^{m} X(i, j) = S(i) \), for \( i = 1, \ldots, m \).
2. Demand balance: \( \sum_{i=1}^{n} X(i, j) = D(j) \), for \( j = 1, \ldots, n \).
3. No backflow: \( X \geq 0 \).
4. Fixed distributor tariffs: \( \sum_{i,j} (P(j) - C(j) - t(i, j))X(i, j) = 0 \).
5. Consumers will not pay more than the delivered cost:
\[
P(j) \leq C(i) + t(i, j).
\]

An equilibrium solution must satisfy the complementary slackness condition:
\[
X(i, j) > 0 \text{ implies } P(j) = C(i) + t(i, j).
\]

This condition can be established by direct substitution in the other stated conditions. One can replace the fixed tariff condition (4) by the complementary slackness conditions.

Consider the following mathematical program.

MP: Minimize
\[
\sum_{i} t(i, j)X(i, j) - \sum_{i} \int_{Q(i)}^{D(j)} P_{i}(z) \, dz + \sum_{i} \int_{0}^{S(i)} C_{i}(z) \, dz
\]
subject to
\[
\sum_{j} X(i, j) - S(i) \leq 0 \quad \text{(multiplier = } v(i); \, i = 1, \ldots, m)\]
\[
\sum_{i} X(i, j) - D(j) \geq 0 \quad \text{(multiplier = } w(j); \, j = 1, \ldots, n)\]
\[
X, S \geq 0 \quad \text{and } \quad D > Q.
\]

The parameter \( Q(j) \) is a lower bound on the demand by consumer \( j \). Generally, \( Q(j) = 0 \) is possible, but \( Q(j) > 0 \) might be necessary for the integral to exist. Monotonicity of the supply and demand functions implies MP is convex, and a solution exists if at some prices supply exceeds demand. We assume that MP has a solution.

Consider the Lagrangian conditions:

(i) a) \( v(i) - C_{i}(S(i)) \geq 0 \)
    b) \( v(i) - C_{i}(S(i)) > 0 \Rightarrow S(i) = 0 \)
(ii) \( P_{i}(D(j)) - w(j) = 0 \)
(iii) a) \( t(i, j) + v(i) - w(j) \geq 0 \)
     b) \( t(i, j) + v(i) - w(j) > 0 \Rightarrow X(i, j) = 0 \).
From (i)a) and (ii), \( v = C \) and \( w = P \), i.e., producer and consumer prices equal the corresponding Lagrange multipliers of the supply and demand constraints. The equivalence between the mathematical programming solution and the economic equilibrium can be stated as follows:

**Theorem 1 (MP Equivalency).** Suppose the supply and demand curves are representable by integrable price functions, \( C_i(S_i) \) and \( P_j(D_j) \), respectively. Then every equilibrium solution comprises a solution to MP with prices \( (C, P) \) as Lagrange multipliers for the associated supply and demand constraints.

By the assumption of strict monotonicity, the associated equilibrium price (i.e., Lagrange multiplier) is unique.

### 2. Algorithmic Framework for Regulatory Structures

The price regulations treated in this paper are of three types: taxes and rebates that are functions of prices, average-cost pricing, and producer price ceilings. Examples of proposed or actual policies in these categories that we modeled include National Energy Plan taxes on natural gas consumption, various oil taxes, electricity pricing and the oil entitlements program, and natural gas wellhead price regulations.

The forms of price controls analyzed in the next section can be stated with our notation as follows.

1. **Taxes and rebates:**

   \[
   C_i = C_i^p - T_i(C_i^p) \quad \text{or,} \quad \text{Net price received (producer = from price distributor)}
   \]

   \[
   P_j = P_j^d - R_j(P_j^d) \quad \text{or,} \quad \text{Net price (consumer = paid to price distributor)}
   \]

2. **Average-cost pricing:**

   \[
   X(i, j) > 0 \Rightarrow P_j = A_i(S_i(C_i)) + t(i, j),
   \]

   where the average cost \( A_i(Q) \) is calculated as follows:

   \[
   A_i(Q) = \left( \int_0^Q C_i(q) \, dq \right) / Q.
   \]

3. **Producer price ceilings:**

   \[
   C \leq C^*,
   \]

   that is, regulations prohibit producers from charging a price above \( C^* \).
We show that all three of these price controls can be represented within the following framework.

Suppose there exists a tariff adjustment function,

$$T: \mathbb{R}^{m+n}_+ \rightarrow \mathbb{R}^{m+n},$$

so that finding the regulated equilibrium corresponds to solving the unregulated model where the tariffs $t$ used in the unregulated model are consistent with $T(C, P)$, i.e.,

$$t = T(C, P).$$

That is, suppose a regulated equilibrium can be stated as the following problem.

Find market variables $(C, S, P, D, X)$ and tariffs $t$ such that

1. $(C, S, P, D, X)$ solve MP;
2. $t = T(C, P)$.

Consider the following algorithmic framework:

(I) Given the tariff matrix $t$, solve MP and obtain

$$(C, S, P, D, X).$$

(II) Given prices $(C, P)$, reset the tariff matrix:

$$t = T(C, P)$$

and return to (I).

This procedure is a Gauss–Seidel strategy which uses the latest trial values from the previous step in the next step.

In two dimensions, the Gauss–Seidel strategy becomes the cobweb algorithm (see Leontief 1934 or Day 1978) for finding an equilibrium. Say we have a supply curve $S(C)$, a demand curve $D(P)$, and transportation tariffs of zero, so $C = P$ in an equilibrium solution. Figure 1 illustrates how the cobweb algorithm works.

Starting at a trial price $P_0$, evaluate demand $Q_1 = D(P_0)$, find the supply price $P_1$ where $Q_1 = S(P_1)$, evaluate demand at $P_1$, and so on. It is well-known that when $S'(P) > |D'(P)|$ for all $P$ over a compact set, the cobweb algorithm converges, i.e., a contraction is guaranteed. The results presented here are of a similar nature.

For example, let $T$ be a tax function on supply, which depends only on supply prices $C$. For brevity, let $T(C)$ denote the taxes, dropping $P$ from the domain. The supply function becomes $S(C - T(C))$. The algorithm becomes one of iterating as follows:

1. Set $k = T(C)$
2. Solve $S(C - k) = D(C)$ for $C$ and return to 1.

Graphically the algorithm is a sequence of supply curve shifts (Figure 2). The only difference between Figure 2 and the usual cobweb is that
Figure 1. The cobweb algorithm.

Figure 2. The algorithm framework with supply taxes.

the supply curve $S(P)$ replaces the horizontal lines in the usual cobweb algorithm. Using the inverse supply and demand functions, we can construct an equivalent convergence result.

**Theorem 2.** In the one-dimensional case, let $T(C(Q))$ be the tariff adjustment function in the tax algorithm. If

$$(T'(C(Q)) - 1)C'(Q) < -P'(Q),$$

for all $Q$ over some compact set $K$ containing the equilibrium, then the algorithmic framework converges on $K$.

Although Theorem 2 follows from the main result to be presented, it is instructive to prove this special case. First, subtract $C(Q)$ from the
supply and demand price functions in Figure 2 (treating quantity as the independent variable). We then have the cobweb algorithm. Using the convergence conditions,

\[ [(T(C(Q)) + C(Q)) - C(Q)]' < | P''(Q) - C'(Q) |. \]

That is,

\[ T'(C(Q))C'(Q) < -P'(Q) + C'(Q) \]

or

\[ (T'(C(Q)) - 1)C'(Q) < -P'(Q). \]

Note that if \( T' < 0 \), which could correspond to a producer subsidy that decreases as the price increases, then the algorithm is guaranteed to converge. Equivalent results can be constructed for rebates to consumers.

We now investigate the general case. This is done in two steps. First, we establish the sensitivity of prices to tariffs in MP. Then with a Lipschitz condition assumed for the tariff adjustment function \( T \), we establish the main convergence theorem.

**Implicit Price Function**

For a given solution \( (C, S, P, D, X) \) to MP, the flows comprise an optimal distribution for the embedded transportation problem. Let these flows correspond to a basic solution represented by a flow network, \( N = (V_1, V_2, A) \), as follows. The points in \( V_1 \) correspond to the \( m \) producers, and the points in \( V_2 \) correspond to the \( n \) consumers. The arcs in \( A \) correspond to the \( m + n - K \) basic activities that account for all flow, where \( K = \) number of components (i.e., trees) of \( N \). Note:

\[ X(i, j) > 0 \] implies \( (i, j) \in A \)

and

\[ (i, j) \in A \] implies \( P(j) = C(i) + t(i, j). \)

In general, these implications are not reversible, owing to degeneracy and alternative solutions.

Define

\[ F(X) = \{ (i, j) : X(i, j) > 0 \} \]

and

\[ G(C, P) = \{ (i, j) : P(j) = C(i) + t(i, j) \}. \]

In general,

\[ F(X) \subset A \subset G(C, P), \]
but the following condition implies equality throughout (and hence unique prices and flows).

**Nondegeneracy Condition.** $F(X) = G(C, P)$.

The main result of this section is a theorem that establishes Lipschitz continuity of the prices as functions of the tariff matrix. The proof also establishes that under a mild additional assumption, the *implicit price function* is a contractor.

The key to this result is a simple property of dual variables in transportation models that may be illustrated by an example. Let the arcs in Figure 3 represent the positive flows in the optimal solution. The dual solution satisfies four price equations:

- $P_1 = C_1 + t_{11}$ from arc(S1, D1)
- $P_1 = C_2 + t_{21}$ from arc(S2, D1)
- $P_2 = C_2 + t_{22}$ from arc(S2, D2)
- $P_2 = C_3 + t_{32}$ from arc(S3, D2).

**Figure 3.** An optimal solution to a transportation problem.
We can reformulate these four equations (with five prices) as functions of one price, say \( C_1 \):

\[(1') \quad P_1 = C_1 + t_{11} = C_1 + k_1 \]
\[(2') \quad C_2 = C_1 + t_{11} - t_{21} = C_1 + k_2 \]
\[(3') \quad P_2 = C_1 + t_{11} - t_{21} + t_{22} = C_1 + k_3 \]
\[(4') \quad C_3 = C_1 + t_{11} - t_{21} + t_{22} - t_{32} = C_1 + k_4. \]

In this example, increasing \( t_{22} \) increases \( k_3 \) and \( k_4 \), while \( k_1 \) and \( k_2 \) remain the same.

The result of the next theorem shows what happens to the price relationships in an equilibrium when the \( t_{ij} \) are varied. Assume there is neither a supply nor a demand response and our example is the standard transportation problem except for a fixed supply price of \( C_1 \). Then, with a change in \( t_{22} \) of \( A \), the new equilibrium prices are \( C_1, P_1, C_2, P_2 + \Delta, C_3 + \Delta \). Here, the derivatives of the prices are \( \dot{C}_1 = \dot{P}_1 = \dot{C}_2 = 0 \) and \( \dot{P}_2 = \dot{C}_3 = 1 \). Suppose \( \Delta > 0 \) and there is some supply and demand response. For supply to equal demand, then the prices \( C_1, P_1, \) and \( C_2 \) must fall while \( P_2 \) and \( C_3 \) must increase. This result is valid because merely increasing some prices increases supply and lowers demand, throwing the market out of equilibrium. The following theorem is the general statement of this result.

**Theorem 3 (Implicit Price Function Theorem).** Suppose the price-quantity curves are represented by differentiable, strictly monotonic functions \((S, D)\), and assume the Nondegeneracy Condition holds for all \( t \in \eta(t^*) \), where \( \epsilon > 0 \). Then the prices are differentiable functions on \( \eta(t^*) \) and the derivatives \( \dot{C}_i = \partial C_i / \partial t_{pq} \) and \( \partial P_j / \partial t_{pq} = \dot{P}_j \) satisfy

\[ \| (\dot{C}, \dot{P}) \| < \lambda, \]

where \( \lambda \leq 1 \), and where the norm is the maximum absolute value of the derivatives.

**Proof.** The Nondegeneracy Condition ensures that the network \( N \) remains invariant over the neighborhood \( \eta(t^*) \). Thus, if \( (p, q) \notin A \), \( (\dot{C}, \dot{P}) = 0 \). So consider \( (p, q) \in A \). The MP conditions, which are necessary and sufficient over \( \eta(t^*) \), are

1. \( m + n - K \) price relations:
   \[ P_j(t) = C_j(t) + t(i, j) \quad \text{for all} \quad (i, j) \in A; \]
2. \( K \) supply/demand balance equations:
   \[ \sum_{i \in V_i}^* S_i(C_i(t)) = \sum_{j \in V_j^*} D_j(P_j(t)) \quad \text{for} \quad k = 1, \ldots, K \]

(where \( V_i^k, V_j^k, A^k \) is the \( k \)-th tree of \( N \)). Since these relations decouple
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into $K$ independent systems, we have

$$C_i = P_j = 0 \quad \text{for} \quad i \notin V^h_1 \quad \text{and} \quad j \notin V^h_2.$$  

Let us thus drop the superscript used to denote a component and simply examine $m + n$ equations for a single component:

$$P_j(t) = C_i(t) + t(i, j) \quad \text{for all} \quad (i, j) \in A;$$

$$\sum_{i=1}^{m} S_i(C_i(t)) = \sum_{j=1}^{n} D_j(P_j(t)).$$

The tree structure enables us explicitly to eliminate $m + n - 1$ of the price functions according to the following recursion. Define the root as the point (in $V_1$) corresponding to the $p$th producer, and let $C_p$ be the derivative of the implicit price function. We shall demonstrate that $C_i$ ($i \neq p$) and $P_j$ are simple linear functions of $C_p$.

We partition the supply and demand nodes into two groups, those nodes that are connected to $p$ through arc $pq$ and those that are not. Formally, let $I_0 = \{p\}$, $J_0 = \{q\}$ and $l = 0$.

Given $I_l$, define

$$J_l = \{j: (i, j) \in A, j \notin J_{l-1}, \text{for some } i \text{ in } I_l\}$$

$$I_{l+1} = \{i: (i, j) \in A \text{ for some } j \text{ in } J_l\}.$$  

Given $J_l$, define

$$\tilde{I}_l = \{i: (i, j) \in A, i \notin I_{l-1}, \text{for some } j \in J_l\}$$

$$\tilde{J}_{l+1} = \{j: (i, j) \in A \text{ for some } i \in \tilde{I}_l\}.$$  

Since $(V_1, V_2, A)$ is a tree, this recursion is well defined, and it induces partitions

$$V_1 = I \cup \tilde{I} \quad \text{and} \quad V_2 = J \cup \tilde{J},$$  

where

$$I = I_0 \cup I_1 \cup \ldots \quad \tilde{I} = \tilde{I}_0 \cup \tilde{I}_1 \cup \ldots$$

$$J = J_0 \cup J_1 \cup \ldots \quad \tilde{J} = \tilde{J}_0 \cup \tilde{J}_1 \cup \ldots.$$  

(Note: $p \in I$ and $q \in J$.)

Figure 4 illustrates this partitioning, where

$$C_i = \begin{cases} C_p & \text{if } i \in I \\ C_p + 1 & \text{if } i \in \tilde{I} \end{cases} \quad P_j = \begin{cases} C_p & \text{if } j \in J \\ C_p + 1 & \text{if } j \in \tilde{J} \end{cases}. \quad (1)$$

Finally, the supply/demand balance equation yields

$$C_p = (-\sum_{i \in \tilde{I}} S'_i + \sum_{j \in \tilde{J}} D'_j)/(\sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j)$$

where $S'_i = \partial S_i/\partial C_i$, $D'_j = \partial D_j/\partial P_j$.

Since $S' > 0$ and $D' < 0$, we have $-1 \leq C_p(t) < 0$ for all $t \in \eta_*(t^*)$.  

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Define $\delta(t) = -\text{Max}[|\dot{C}_p(t)|, \dot{C}_p(t) + 1]$, so $0 < \delta(t) \leq 1$. Then, define $\lambda = \text{Sup}\{\delta(t): t \in \eta_i(t^*)\}$, so $0 < \lambda \leq 1$, and

$$\| (\dot{C}, \dot{P}) \| \leq \lambda \text{ for all } t \in \eta_i(t^*).$$

From (1) for $\lambda = 1$, either $\dot{C}_i = 0$ for $i \in I$ and $\dot{P}_j = 0$ for $j \in J$ or $\dot{C}_i = 0$ for $i \in \tilde{I}$ and $\dot{P}_j = 0$ for $j \in \tilde{J}$. Letting $\dot{C}_i = 0$ for $i \in I$ and $\dot{P}_j = 0$ for $j \in J$ implies $\dot{C}_i = \dot{P}_j = 1$ for $i \in \tilde{I}$ and $j \in \tilde{J}$.

Given the strict monotonicity of the supply and demand curves, we have

$$S'_i > 0 \quad i \in \tilde{I}$$
$$S'_i = 0 \quad i \in I$$
$$D'_j < 0 \quad j \in \tilde{J}$$
$$D'_j = 0 \quad j \in J.$$

Consequently,

$$\sum_i S'_i - \sum_j D'_j > 0.$$
However, for the solution to remain in equilibrium, we must have

$$\sum_i S_i' - \sum_j D_j' = 0.$$ 

Therefore, $C_i < 0$ for $i \in I$ and $P_j < 0$ for $j \in J$. Similarly, $\bar{C}_i > 0$ for $i \in \bar{I}$ and $\bar{P}_j > 0$ for $j \in \bar{J}$, and $\lambda < 1$.

## 3. CONVERGENCE

Having established Lipschitz continuity of the implicit price function, it remains to establish (local) convergence of our Gauss-Seidel algorithm by a related assumption.

**Theorem 4 (Convergence Theorem).** Assume the conditions of the Implicit Price Function Theorem, and further assume the tariff adjustment function, $T$, is Lipschitz continuous, i.e.,

$$\| \nabla T \| \leq \mu$$

for all $(C, P) \in \mathbb{R}^{n+n}$. Then, if $\lambda \mu < 1$, the prices and tariffs converge to a regulated equilibrium.

**Proof.** Consider the composite function, where

$$t^{k+1} = T(C(t^k), P(t^k)).$$

Then

$$\| t^{k+1} - t^* \| \leq \| \nabla T \| (C, \bar{P}) \| t^k - t^* \| \leq (\lambda \mu) \| t^k - t^* \|.$$ 

Thus, we have the contraction property, and $\{t^k\} \to t^*$ linearly. Further, $\{(C^k, P^k)\} \to (C^*, P^*)$ linearly since

$$\| (C^k, P^k) - (C^*, P^*) \| \leq \lambda \| t^k - t^* \|.$$ 

The Convergence Theorem follows from standard results (Altman; Kirk 1981), but we included a sketch of a proof for completeness.

The convergence theorem may be weakened slightly to allow the composite map to be nonexpansive—that is, $\lambda \mu = 1$. Current fixed point theorems (see Altman; Kirk; Scarf 1973, 1982) require the composite function to be convex. The following theorem shows that the implicit price function is a convex function of tariffs, so one may obtain convergence results by assuming $T(\cdot)$ is convex and increasing.

**Theorem 5 (Convexity Theorem).** Assume the conditions of the Implicit Price Function Theorem. If $S$ is concave (and increasing) and $D$ is convex (and decreasing), then the implicit price functions ($P, C$) are convex on $\eta_c(t^*)$. 
Proof. The price relations imply we can reduce the dimensions to one price function, say $C_r(t)$, where

$$C_r(t) = C_r(t) + A_i t \quad \text{for} \quad i = 1, \ldots, m$$

$$P_j(t) = C_r(t) + B_j t \quad \text{for} \quad j = 1, \ldots, n.$$ 

Thus, it suffices to prove $C_r(t)$ is convex. The supply-demand balance equation gives

$$s(t) = \sum_i S_i(C_r(t) + A_i t) = \sum_j D_j(C_r(t) + B_j t) = d(t).$$

Suppose, to the contrary, there exist $t^1, t^2 \in \eta_*(t^*)$ and $\alpha \in (0, 1)$ satisfying

$$C_r(t) > \alpha C_r(t^1) + (1 - \alpha) C_r(t^2),$$

where $t = \alpha t^1 + (1 - \alpha) t^2$. Then, monotonicity and concavity of $S$ implies

$$s(t) > \alpha s(t^1) + (1 - \alpha) s(t^2).$$

Since $s(t^1) = d(t^1)$ and $s(t^2) = d(t^2)$,

$$s(t) > \alpha d(t^1) + (1 - \alpha) d(t^2).$$

Monotonicity and convexity of $D$ implies

$$d(t) < \alpha d(t^1) + (1 - \alpha) d(t^2),$$

so $s(t) > d(t)$, which contradicts the balance equation for $t$.

The properties just established provide a basis for analyzing the sensitivity of the equilibrium to parametric changes in the given functions ($S$, $D$ and $T$).

4. APPLICATIONS

In this section, we apply the general tariff adjustment function to three different types of regulatory programs. Specific programs of these types have been implemented, in some form, within PIES.

Tax/Rebate Programs

The regulatory structure for tax/rebate programs immediately fits into the algorithmic framework. The taxes $T_i(C_i)$ and rebates $R_j(P_j)$ are simply added to the transportation cost $t(i, j)$. That is, the tariff adjustment function is

$$T_{ij}(C, P) = t(i, j) + T_i(C_i) - R_j(P_j)$$

which becomes

$$T_{ij}(C, P) = t(i, j) + T_i(C_i) - R_j(C_i + t(i, j))$$

when $X(i, j) > 0$. The tariff adjustment function is a contractor when
\[| T_i'(C_i) - R_j'(P_j) | \leq \mu < 1. \]

For example, consider a proportionate tax/rebate rule:

\[ T_i(\pi) = a_i \pi \quad \text{and} \quad R_j(\pi) = b_j \pi. \]

Then, \( \mu = \max |a_i - b_j| < 1 \) and the contraction condition is satisfied if \( 0 \leq a, b < 1 \)—that is, producers pay only a portion \( a \) of their revenue to the government, and consumers receive a portion \( b \) of their costs from the government.

**Average-Cost Pricing**

This regulatory structure fits less obviously in the tariff adjustment framework. Average-cost pricing occurs with calculating electricity prices in energy models. The entitlements program for crude oil acquisition and "rolled-in" pricing for natural gas are two more examples (see Murphy 1983; Murphy et al.; and Murphy and Shaw 1983).

With the average cost determined as follows:

\[ A_i(Q) = \int_0^Q C_i(q) \, dq, \]

the tariff adjustment function becomes

\[ T_{ij}(C, P) = t(i, j) - C(i) + A_i(S_i(C(i))). \]

Convergence is achieved when

\[ -1 + \frac{\partial A_i}{\partial S_i} \frac{\partial S_i}{\partial C(i)} \leq 1, \]

that is, when

\[ 0 \leq \frac{\partial A_i}{\partial S_i} \leq 2 \frac{\partial C_i}{\partial S_i}. \]

Convexity is a sufficient, but not necessary, condition for convergence, as shown next.

**THEOREM 7.** If \( C_i(Q) \) is convex and monotonic, then

\[ 0 \leq \frac{\partial A_i}{\partial Q} \leq \frac{\partial C_i}{\partial Q}. \]

**Proof.** By the convexity of \( C_i(Q) \),

\[ \frac{\partial C_i(Q)}{\partial Q} \frac{Q}{Q} + C_i(0) \geq C_i(Q). \]

Next, \( A_i(Q) \geq C_i(0) \) by the monotonicity of \( C_i(0) \). Also,

\[ \frac{\partial A_i(Q)}{\partial Q} = \frac{C_i(Q)}{Q} - \frac{A_i(Q)}{Q}. \]
Therefore,
\[
Q\left(\frac{\partial C_i(Q)}{\partial Q} - \frac{\partial A_i(Q)}{\partial Q}\right) = Q\left(\frac{\partial C_i(Q)}{\partial Q} - \frac{C_i(Q)}{Q} + \frac{C_i(Q)}{Q} - \frac{\partial A_i(Q)}{\partial Q}\right)
\]
\[
= Q \frac{\partial C_i(0)}{\partial Q} - C_i(Q) + A(q)
\]
\[
\geq Q \frac{\partial C_i(Q)}{\partial Q} - C_i(Q) + C_i(0)
\]
\[
\geq 0.
\]

Our approach for solving the regulated equilibrium is to use the general algorithm, converging upon a regulated equilibrium solution by a sequence of unregulated equilibria (i.e., solution of MP), distinguished by tariffs. The iterative adjustments to the tariffs are linear approximations that account for the differences between the average costs $A$ and the marginal costs $C$. That is, we subtract $B_t(C(i)) = C(i) - A_i(S_i(C(i)))$ from the original tariffs $t(i,j)$.

This Average-Cost Pricing Algorithm has cycled in PIES because the conditions of Theorem 7 are not always met. How this happened is illustrated in Figure 5. In the Pacific Northwest, there is a large amount of hydropower, the first step of the supply curve in Figure 5(A). Then, there is a supply step for nuclear power at a much greater cost. Initially, this marginal cost curve, without a tariff adjustment, is a supply curve in the linear program (LP). The unregulated solution uses all of the hydro but no nuclear, $Q_H$ in Figure 5(A). The dual variable, or marginal cost, is the price $P_1$ at which the demand curve cuts between the hydro and nuclear steps. $P_1 - C_H$ in Figure 5(A) is the difference between

![Figure 5](image_url)
marginal and average costs used to translate the supply curve downward (the dashed lines) by means of changing the transportation tariffs. The new equilibrium is at $Q_T$. Constructing the new $C'$ in Figure 5(B), and using $C_N - P_3$ to translate the supply curve downward, gets us back to $Q_H$ as the trial equilibrium and into a nonconvergent cycle. In practice, the difficulty was resolved by taking weighted averages of the old and new tariff adjustments, a common variant of successive approximation in fixed point computation (Krasnoselskii 1955; Mann 1953; Kirk).

**Producer Price Ceilings**

The regulatory structure for producer price ceilings is the last application we examined. (We present a very simple version. For two approaches to modeling natural gas markets see Murphy et al. and Murphy and Shaw.) We invoke two simplifying assumptions: (1) there is only one aggregate source of supply, and (2) all consumers share equally in the low cost sources of supply.

Define a new supply price function

$$C^*(S) = \min[C(S), C^*]$$

determined by a price ceiling $C^*$. Let $A^*(S)$ be the average cost curve derived from $C^*(S)$. Then the tariff adjustment function is

$$T(1, j) = t(1, j) + A^*(S) - C^*(S).$$

The convergence properties are the same as in the average-cost pricing regulatory structure.

Let $Q$ be the quantity satisfying $C(Q) = C^*$, and let $S$ be the equilibrium supply. If $S - Q > 0$, this difference is the amount of extra supply that would be demanded at the ceiling price and, therefore, is a measure of shortages, if there are any.

To expand this model to include different endowments of low-cost supply for different consumers is relatively simple. This case is more realistic, for example, when different gas pipelines have different endowments of old gas. In this case one can average the costs in the supply regions that provide the gas that has not been preassigned to consumers (including the shortage amount) and then calculate a rebate to the consumer that consists of averaging the unassigned gas with the preassigned and subtracting the rebate from $P^D$.

Increasing the number of supply regions is an added complexity. First, if more sources were added and the $C^*_i(S)$ were all constructed in the same way, then the supplier with the lowest transportation costs could be facing shortages while suppliers with higher transportation costs could produce more. The resolution is to truncate the supply curve and set $C^*_i(S) = \infty$ for $S > Q_i$, where $C_i(Q_i) = C^*$, except for the source with the

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highest transportation costs. For this source $C(S)$ is defined as before. If this source with high transportation costs does not supply all consumers in the tree, then there are still potential problems in assigning shortages, and ad hoc rules specific to the commodity regulated are needed to tailor the model to different priority schemes for allocating shortages.

5. CONCLUSIONS

In this paper we have provided a unified framework for representing selected price-oriented government regulations in a mathematical programming model of a market. The approach is to calculate the difference between the regulated price and marginal cost and then impose the difference as an adjustment to transportation tariffs. Conceptually, this approach is a Gauss-Seidel algorithm, alternating between the mathematical programming solution and the tariff adjustment.

There are two points to be made about the algorithm presented in this paper. First, the adjustments need not be made on the tariffs. For example, an activity can be added between each source and a distributor with, initially, a zero cost that is replaced by the adjustment. Imposing the adjustment on transportation activities is a convenience: they are already there and, typically, have constant costs. Second, if the model is represented as a nonlinear program instead of a linear program, one can make the computational procedure look more like a Newton algorithm by incorporating information on how the tariff adjustment would change with changes in quantity. Our experience with PIES showed that the tariff adjustments converged as fast as other adjustments that were necessary to achieve an equilibrium. Nevertheless, computational time was saved by having good initial guesses of the tariff adjustments.

This paper does not address ways of representing programs that are based on prohibitions or quantity regulations. Simple prohibitions, such as a complete ban on some activity, are easily represented by removing any representation of the activity from the model. More complex prohibitions, based on a series of contingencies, are more difficult to represent. An example of a complex set of regulations is the Powerplant and Industrial Fuel Use Act. This act has provisions banning electric utility construction of oil or gas-fired powerplants, yet its rules have exceptions that must be modeled. Such regulatory contingencies appear not to be representable by tariff adjustment. The implementation of this program involved a different set of procedures operating off of reduced costs in the LP. Quantity based taxes, such as income taxes, do not fall within this framework either.

There are no clean convergence rate results for the tariff adjustments as implemented in PIES because the adjustments associated with the PIES convergence process were made simultaneously with all of the tariff
adjustments after each LP solution. Doing every adjustment after each LP solution was several times faster than solving for an unregulated equilibrium, then doing a tariff adjustment and then resolving for an equilibrium. Adding the regulatory adjustments increased the number of LP solutions in the PIES algorithm by at most one or two to obtain solutions within specified tolerance. The convergence properties of PIES with simultaneous adjustments have yet to be established.

REFERENCES


