

# CONVERGENCE OF ALGEBRAIC MULTIGRID BASED ON SMOOTHED AGGREGATION\*

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**Abstract.** We prove an abstract convergence estimate for the Algebraic Multigrid Method with prolongator defined by a disaggregation followed by a smoothing. The method input is the problem matrix and a matrix of the zero energy modes of the same problem but with natural boundary conditions. The construction is described in the case of a general elliptic system. The condition number bound increases only as a polynomial of the number of levels, and requires only a uniform weak approximation property for the aggregation operators. This property can be a-priori verified computationally once the aggregates are known. For illustration, it is also verified here for a uniformly elliptic diffusion equations discretized by linear conforming quasiuniform finite elements. Only very weak and natural assumptions on the hierarchy of aggregates are needed.

**Key words.** Algebraic multigrid, zero energy modes, convergence theory, computational mechanics, finite elements, iterative solvers.

**1. Introduction.** This paper establishes abstract convergence bounds for an Algebraic Multigrid Method (AMG) based on smoothed aggregation. The bounds are obtained by invoking the general convergence theory of [6]. Our main result is a bound on the condition number that grows only as a power of the number of levels, and requires only a weak approximation property for the aggregates, similar to the weak approximation condition in classical AMG investigations [9, 22, 28, 37]. Our weak approximation condition can be easily verified computationally, and we show that it holds for general unstructured meshes and under natural assumptions on aggregates used to construct the coarse levels. The emphasis of this paper is on the treatment of unstructured meshes. Robustness of our form of the weak approximation condition with respect to problem coefficients, degenerated meshes, etc, will be studied elsewhere. Cf., [44] for the case of two-levels and jumps of Lamé coefficients in elasticity. The results of this paper appear to be the first bound on the condition number for an Algebraic Multigrid Method, growing only polynomially with the number of levels. Existing bounds are based on two-level bounds, cf., e.g., [9, 39], which in general result in convergence factors of the form  $1 - 2^{-m}$ , where  $m$  is the number of levels [39]. This means that for known analyses, the corresponding bound on the condition number grows exponentially with the number of levels.

Unlike classical, geometrical multigrid, where the hierarchy of meshes and the prolongation operators are defined from finite element spaces, the AMG approach strives to build the hierarchy of coarse spaces, or, equivalently, the prolongators, from matrix data only, making assumptions about the underlying differential equation and its discretization [10, 20, 36, 37] or using additional geometrical information [12, 13].

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AMG methods based on smoothed aggregation, introduced in [40, 41] and further developed in [11, 24, 42, 43, 46, 44, 45], have proved to be efficient tools for the solution of symmetric, positive definite linear algebraic systems arising from finite element discretization of elliptic boundary value problems.

In our AMG method, we build the prolongation operators by first constructing a *tentative prolongator* using an aggregation approach and the knowledge of zero energy modes of the principal part of the differential operator with natural boundary conditions (e.g., rigid body modes for elasticity), then *smoothing* its output by a carefully selected iteration. Our coarsening process is determined by the selection of aggregates, as opposed to the selection of C-points in classical AMG [10, 36, 37]

The use of zero energy modes has become a recognized way to capture the essence of the geometry, the differential operator, and the discretization, needed to build an efficient iterative method. Zero energy modes are the input of other widely used iterative methods [17, 18, 19, 25, 29, 30, 32, 33]. For common discretizations of scalar elliptic problems, zero energy modes are simply constant vectors, that is, multiples of a vector of all ones. In this case, the use of zero energy modes is an assumption about the problem rather than the use of geometrical information, and our prolongator becomes disaggregation followed by smoothing. Prolongation by disaggregation only (without smoothing) was advocated, e.g., in [3, 5].

Our bounds are based on existing general regularity-free estimates for multigrid methods. Since the first attempts to analyze AMG type methods, it was clear that the classical multigrid theory, which relies on elliptic regularity [1, 21, 31] will not apply, because this theory requires the use of properties of the underlying finite element spaces on all levels. The approach based on a strengthened Cauchy inequality [1, 4], or, equivalently, on the weak approximation property [9, 22, 23, 28], needs only assumptions that can be verified computationally, but it gives convergence estimates for two-level methods only. It is not guaranteed that the two-level convergence rate can be made arbitrarily small by increasing the number of smoothings steps [9], and simple recursive estimates result in a convergence bound that approaches 1 as a geometrical sequence [27]. This means that the bound on the condition number increases exponentially with the number of levels. A satisfactory multigrid theory based on the weak approximation property was made possible by reinterpreting multigrid as a Schwarz method [38] during the late eighties. The abstract Schwarz methods have become a recognized framework for analyzing a large class of iterative techniques in a unified manner. The early convergence results for additive variants were developed and used for domain decomposition [2, 14, 15, 16, 26], hierarchical bases [47, 48] and additive multilevel preconditioners [8]. Based on an estimate for product methods [7], the first regularity-free polynomial convergence bounds for variational multigrid were established in [6], relying on a multilevel version of the weak approximation condition and on other properties of nested finite element discretizations. The bounds of the additive variants were then improved to be independent of the number of levels by new techniques using advanced approximation theory tools [34, 35].

We use the classical multiplicative scheme of the multigrid method, including block Gauss-Seidel smoothers. In our implementation, parallelism is then achieved by coloring. Cf., [34] for theoretical and [20] for practical aspects of additive multigrid approaches.

To apply the estimates of [6] to a particular multigrid method in a straightforward manner, one needs to establish that the discrete norms in the artificially constructed coarse spaces are uniformly equivalent to appropriately scaled  $L^2$  norms, and establish

the weak approximation property in those norms. We have done this in [42] under additional (though quite reasonable) assumptions on the supports of the coarse basis functions. Essentially, we had to assume that the basis functions in the coarse space hierarchy are associated with a division of the domain into subdomains that behave much like finite elements. Verifying these assumptions is difficult because the process of building the coarse spaces is recursive and not easily predictable; all we could say was that our coarsening algorithms were designed so that they would tend to produce such a coarse space hierarchy, but this could not be guaranteed.

Our present approach to the theory is to verify the assumptions of the abstract theory from [6] by algebraic means, without reference to the  $L^2$  norm and assumptions on the supports of the coarse space shape functions. We need to assume only a weak approximation property for the tentative prolongators, rather than to work with the properties of the final prolongator operators. Thus, the weak approximation property is easy to verify once the aggregates are constructed. Our analysis requires that the mesh coarsening ratio be 3 rather than the more usual 2. However, this is inherent in the smoothed aggregation method, and leads to a method which is very efficient in practice [43].

The paper is organized as follows. The AMG algorithm is described in Sec. 2. Sec. 3 contains our principal theoretical result, a multilevel convergence proof using only a weak approximation property for aggregations. In Sec. 4, we describe the construction of a tentative prolongator from zero energy modes by aggregation, and formulate and prove the main convergence theorem. Finally, Sec. 5 contains an example showing that the assumptions of the theorem are satisfied for a finite element discretization of a second order elliptic boundary value problem.

**2. Description of the method.** We are interested in solving the system of linear algebraic equations

$$(2.1) \quad \mathbf{Ax} = \mathbf{b},$$

where  $A$  is a symmetric positive definite matrix. The smoothed aggregation multigrid [43] can be viewed as a standard variational multigrid method with prolongators of the form  $S_l P_{l+1}^l$ , where  $P_{l+1}^l : \mathbb{R}^{n_{l+1}} \rightarrow \mathbb{R}^{n_l}$ ,  $n_1 \equiv \text{ord}(A) > n_2 > \dots > n_L$  is the full-rank *tentative prolongator* and  $S_l : \mathbb{R}^{n_l} \rightarrow \mathbb{R}^{n_l}$  is a *prolongator smoother* derived from the matrix  $A_l$ . The hierarchy of coarse level matrices is defined by

$$(2.2) \quad A_{l+1} = (S_l P_{l+1}^l)^T A_l S_l P_{l+1}^l, \quad A_1 = A.$$

The simplest example of a tentative prolongator will be given at the end of this section. The construction of a tentative prolongator suitable for solving general elliptic problems on unstructured meshes will be the subject of Sec. 4. Although we will carry out some convergence estimates for general prolongator smoothers  $S_l : \mathbb{R}^{n_l} \rightarrow \mathbb{R}^{n_l}$ , the form of  $S_l$  we use is

$$(2.3) \quad S_l = I - \frac{4}{3\bar{\lambda}_l^M} M_l^{-1} A_l.$$

Here,  $\bar{\lambda}_l^M \geq \varrho(M_l^{-1} A_l)$ ,  $\varrho$  denotes the spectral radius, and

$$(2.4) \quad M_l = (P_l^1)^T P_l^1, \quad P_l^1 = P_2^1 \dots P_l^{l-1}, \quad P_1^1 = I.$$



the hierarchy of coarse spaces  $V_L \subset V_{L-1} \subset \dots \subset V_1$  by  $V_l = \text{Range } I_l^1$ , the norm on  $V_l$  induced by the  $\mathbb{R}^{n_l}$ -norm  $\|\mathbf{x}\|_{\mathbb{R}^{n_l}} = (\mathbf{x}^T \mathbf{x})^{1/2}$ ,

$$(3.2) \quad \|\mathbf{u}\|_l = \min\{\|\mathbf{x}\|_{\mathbb{R}^{n_l}} : \mathbf{u} = I_l^1 \mathbf{x}\},$$

and the associated inner product  $(\mathbf{u}, \mathbf{v})_l = (\mathbf{x}, \mathbf{y})_{\mathbb{R}^{n_l}}$ ,  $\mathbf{u} = I_l^1 \mathbf{x}$ ,  $\mathbf{v} = I_l^1 \mathbf{y}$ ,  $\mathbf{x}, \mathbf{y} \perp \text{Ker } I_l^1$ . If  $I_l^1$  has full rank, we have simply  $\|I_l^1 \mathbf{x}\|_l = \|\mathbf{x}\|_{\mathbb{R}^{n_l}}$ . Note that from (2.2), it follows that  $A_l = (I_l^1)^T A I_l^1$ , and

$$(3.3) \quad \|I_l^1 \mathbf{x}\|_A = \|\mathbf{x}\|_{A_l} \quad \forall \mathbf{x} \in \mathbb{R}^{n_l},$$

$$(3.4) \quad \max_{\mathbf{u} \in V_l} \left( \frac{\|\mathbf{u}\|_A}{\|\mathbf{u}\|_l} \right)^2 = \max_{\mathbf{x} \in \mathbb{R}^{n_l}} \left( \frac{\|I_l^1 \mathbf{x}\|_A}{\|\mathbf{x}\|_{\mathbb{R}^{n_l}}} \right)^2 = \varrho(A_l).$$

REMARK 3.1. The preconditioning by  $M_l^{-1}$  in (2.3) guarantees that the prolongator smoother  $S_l$  possesses the following invariance property: If  $P_l^1$  is replaced by  $P_l^1 D$ , where  $D$  is a nonsingular matrix, then  $I_l^1$  becomes  $I_l^1 D$  and  $M_l^{-1} A_l$  becomes  $D^{-1} M_l^{-1} A_l D$ , hence  $S_l$  becomes  $D^{-1} S_l D$ . Consequently, the mapping in  $V_l$  defined by the action of  $S_l$  via the isomorphism  $I_l^1$ , i.e.,  $I_l^1 \mathbf{x} \mapsto I_l^1 S_l \mathbf{x}$ , does not depend on the specific choice of  $P_l^1$ , but only on  $\text{Range } P_l^1$ .

Our estimates are based on an abstract convergence result proved in [6]. Using (3.4), it can be written in our notation as follows:

LEMMA 3.2 ([6], Theorem 1). *Assume there are linear mappings  $Q_l : V_1 \rightarrow V_l$ ,  $Q_1 = I$  and constants  $c_1, c_2 > 0$  such that*

1. *for all  $\mathbf{u} \in V_1$  and every level  $l = 1, \dots, L$*

$$(3.5) \quad \|Q_l \mathbf{u}\|_A \leq c_1 \|\mathbf{u}\|_A.$$

2. *for all  $\mathbf{u} \in V_1$  and every level  $l = 1, \dots, L-1$*

$$(3.6) \quad \|(Q_l - Q_{l+1})\mathbf{u}\|_l \leq \frac{c_2}{\sqrt{\varrho(A_l)}} \|\mathbf{u}\|_A.$$

Further assume that  $R_l$  are symmetric positive definite matrices satisfying

$$(3.7) \quad \lambda_{\min}(I - R_l A_l) \geq 0 \quad \text{and} \quad \lambda_{\min}(R_l) \geq \frac{1}{c_R^2 \varrho(A_l)}$$

with a constant  $c_R > 0$  independent of the level.

Then Algorithm 2.1 satisfies

$$\|\hat{\mathbf{x}} - MG(\mathbf{x}, \mathbf{b})\|_A \leq \left(1 - \frac{1}{c_0(L)}\right) \|\hat{\mathbf{x}} - \mathbf{x}\|_A \quad \forall \mathbf{x} \in V_1,$$

where  $\hat{\mathbf{x}}$  is the solution of (2.1), and  $c_0(L) = (1 + c_1 + c_2 c_R)^2 (L-1)$ . Moreover, the preconditioner  $P$  defined by the action of  $MG(\mathbf{0}, \cdot)$  is symmetric with respect to  $(\cdot, \cdot)_{\mathbb{R}^{n_1}}$  and  $\text{cond}(A, P) \leq c_0(L)$ .

The following lemma verifies assumptions (3.5), (3.6), of Lemma 3.2 from the properties of  $S_l$  and  $P_l^1$  rather than  $I_l^1$ . It does not assume the specific form (2.3) of the prolongator smoother.

LEMMA 3.3. *Let for every  $l = 1, \dots, L$ ,  $\bar{\lambda}_l^M \geq \varrho(M_l^{-1} A_l)$  and*

$$\tilde{Q}_l : V_1 \rightarrow \mathbb{R}^{n_l}, \quad \tilde{Q}_1 = I, \quad S_l : \mathbb{R}^{n_l} \rightarrow \mathbb{R}^{n_l}$$

be given linear operators. Assume that for some  $C_1, C_2, C_M, C_S > 0$  and all  $l = 1, \dots, L-1$ ,

$$(3.8) \quad \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_l}}^2 \leq \frac{C_1^2}{\lambda_l^M} \|\mathbf{u}\|_A^2 \quad \forall \mathbf{u} \in V_1,$$

$$(3.9) \quad \text{cond}(M_l) \leq C_M^2,$$

$$(3.10) \quad \|S_l\|_{A_l} \leq 1,$$

$$(3.11) \quad \|S_l \mathbf{x}\|_{\mathbb{R}^{n_l}}^2 \leq \lambda_{\min}^{-1}(M_l) \|P_l^1 \mathbf{x}\|_{\mathbb{R}^{n_l}}^2 \quad \forall \mathbf{x} \in \mathbb{R}^{n_l},$$

$$(3.12) \quad \|(I - S_l) \mathbf{x}\|_{\mathbb{R}^{n_l}}^2 \leq \frac{C_2^2}{\varrho(A_l)} \|\mathbf{x}\|_{A_l}^2 \quad \forall \mathbf{x} \in \mathbb{R}^{n_l},$$

$$(3.13) \quad \varrho(M_l^{-1} S_l^T A_l S_l) \leq C_S^2 \bar{\lambda}_l^M.$$

Then, for every  $\mathbf{u} \in V_1$ , the mappings  $Q_l = I_l^1 \tilde{Q}_l$  satisfy

$$(3.14) \quad \|Q_l \mathbf{u}\|_A \leq c_1(l) \|\mathbf{u}\|_A, \quad l = 1, \dots, L,$$

with  $c_1(l) = 1 + C_S C_1(l-1)$ , and

$$(3.15) \quad \|(Q_l - Q_{l+1}) \mathbf{u}\|_l \leq c_2(l) \varrho(A_l)^{-1/2} \|\mathbf{u}\|_A, \quad l = 1, \dots, L-1$$

with  $c_2(l) = C_1 C_M + C_2 \|Q_l\|_A \leq C_1 C_M + C_2 c_1(l)$ .

*Proof.* First, for any  $\mathbf{x} \in \mathbb{R}^{n_l}$ ,

$$(3.16) \quad \|S_l \mathbf{x}\|_{A_l} \leq C_S \sqrt{\bar{\lambda}_l^M} \|P_l^1 \mathbf{x}\|_{\mathbb{R}^{n_l}}.$$

Indeed,

$$\|S_l \mathbf{x}\|_{A_l}^2 = \|S_l M_l^{-1/2} M_l^{1/2} \mathbf{x}\|_{A_l}^2 \leq \varrho \left( M_l^{-1/2} S_l^T A_l S_l M_l^{-1/2} \right) \|M_l^{1/2} \mathbf{x}\|_{\mathbb{R}^{n_l}}^2,$$

where  $\varrho(M_l^{-1/2} S_l^T A_l S_l M_l^{-1/2}) = \varrho(M_l^{-1} S_l^T A_l S_l)$  is bounded from (3.13), and  $\|M_l^{1/2} \mathbf{x}\|_{\mathbb{R}^{n_l}} = \|P_l^1 \mathbf{x}\|_{\mathbb{R}^{n_l}}$ , since  $M_l = (P_l^1)^T P_l^1$ .

Let  $\mathbf{u} \in V_1$ . From the definitions of  $I_l^1$ ,  $Q_l$  and the isometry (3.3),

$$\begin{aligned} \|Q_{l+1} \mathbf{u}\|_A &= \|I_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_A = \|I_l^1 S_l P_{l+1}^l \tilde{Q}_{l+1} \mathbf{u}\|_A = \|S_l P_{l+1}^l \tilde{Q}_{l+1} \mathbf{u}\|_{A_l} \\ &\leq \|S_l (\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1}) \mathbf{u}\|_{A_l} + \|S_l \tilde{Q}_l \mathbf{u}\|_{A_l}. \end{aligned}$$

Using bound (3.16), assumptions (3.8), (3.10) and isometry (3.3), we get

$$\|Q_{l+1} \mathbf{u}\|_A \leq C_S \sqrt{\bar{\lambda}_l^M} \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^l \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_l}} + \|\tilde{Q}_l \mathbf{u}\|_{A_l} \leq C_S C_1 \|\mathbf{u}\|_A + \|Q_l \mathbf{u}\|_A.$$

Estimate (3.14) now follows by induction with  $Q_1 = I$ .

To prove (3.15), we use assumptions (3.11), (3.12) and definitions (3.2), and (3.1),

$$(3.17) \quad \begin{aligned} \|(Q_l - Q_{l+1}) \mathbf{u}\|_l &\leq \|(\tilde{Q}_l - S_l P_{l+1}^l \tilde{Q}_{l+1}) \mathbf{u}\|_{\mathbb{R}^{n_l}} \\ &= \|S_l (\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1}) \mathbf{u}\|_{\mathbb{R}^{n_l}} + \|(I - S_l) \tilde{Q}_l \mathbf{u}\|_{\mathbb{R}^{n_l}} \\ &\leq \|S_l (\tilde{Q}_l - P_{l+1}^l \tilde{Q}_{l+1}) \mathbf{u}\|_{\mathbb{R}^{n_l}} + \|(I - S_l) \tilde{Q}_l \mathbf{u}\|_{\mathbb{R}^{n_l}} \\ &\leq \lambda_{\min}^{-1/2}(M_l) \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^l \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_l}} + C_2 \varrho(A_l)^{-1/2} \|\tilde{Q}_l \mathbf{u}\|_{A_l}. \end{aligned}$$

Now, using the estimate

$$\begin{aligned}
\varrho(A_l) &= \max_{\mathbf{x} \in \mathbb{R}^{n_l}} \frac{\mathbf{x}^T M_l^{-1/2} A_l M_l^{-1/2} \mathbf{x}}{\mathbf{x}^T M_l^{-1} \mathbf{x}} \\
(3.18) \quad &\leq \max_{\mathbf{x} \in \mathbb{R}^{n_l}} \frac{\mathbf{x}^T M_l^{-1/2} A_l M_l^{-1/2} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \cdot \max_{\mathbf{x} \in \mathbb{R}^{n_l}} \frac{\mathbf{x}^T \mathbf{x}}{\mathbf{x}^T M_l^{-1} \mathbf{x}} \\
&\leq \bar{\lambda}_l^M \varrho(M_l) \leq \bar{\lambda}_l^M \lambda_{\min}(M_l) \operatorname{cond}(M_l)
\end{aligned}$$

together with isometry (3.3) and assumption (3.8), inequality (3.17) can be rewritten as

$$\begin{aligned}
\|(Q_l - Q_{l+1})\mathbf{u}\|_l &\leq \left( \frac{C_1}{\sqrt{\lambda_{\min}(M_l) \bar{\lambda}_l^M}} + \frac{C_2}{\sqrt{\varrho(A_l)}} \|Q_l\|_A \right) \|\mathbf{u}\|_A \\
&\leq \frac{C_1 \sqrt{\operatorname{cond}(M_l)} + C_2 \|Q_l\|_A}{\sqrt{\varrho(A_l)}} \|\mathbf{u}\|_A \leq \frac{C_1 C_M + C_2 \|Q_l\|_A}{\sqrt{\varrho(A_l)}} \|\mathbf{u}\|_A,
\end{aligned}$$

completing the proof of (3.15).  $\square$

The key assumption (3.8) of Lemma 3.3 is a weak approximation property for disaggregated functions. If one has the weak approximation property in the more usual form

$$\forall \mathbf{u} \in \mathbb{R}^{n_1} \exists \mathbf{u}_l \in \mathbb{R}^{n_l} : \quad \|\mathbf{u} - P_l^1 \mathbf{u}_l\|_{\mathbb{R}^{n_1}}^2 \leq \frac{\tilde{C}_1^2}{\bar{\lambda}_l^M} \|\mathbf{u}\|_A^2.$$

Then, with the choice  $\tilde{Q}_l = M_l^{-1} (P_l^1)^T$ , the mappings  $P_l^1 \tilde{Q}_l$  are orthogonal projections onto  $\operatorname{Range} P_l^1$ . Since  $\operatorname{Range} P_{l+1}^1 \subset \operatorname{Range} P_l^1$ , we obtain

$$\begin{aligned}
\|\mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 &= \|\mathbf{u} - P_l^1 \tilde{Q}_l \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 + \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 \\
(3.19) \quad &\geq \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2
\end{aligned}$$

Hence, from the minimization property of the orthogonal projection,

$$\|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 \leq \|\mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 \leq \|\mathbf{u} - P_{l+1}^1 \mathbf{u}_{l+1}\|_{\mathbb{R}^{n_1}}^2 \leq \frac{\tilde{C}_1^2}{\bar{\lambda}_{l+1}^M} \|\mathbf{u}\|_A^2,$$

and it follows that the inequality (3.8) holds with  $C_1^2 = \tilde{C}_1^2 \frac{\bar{\lambda}_l^M}{\bar{\lambda}_{l+1}^M}$ .

The prolongator smoothers enter the approximation property (3.8) only through the scaling factor  $1/\bar{\lambda}_l^M$  on its right-hand side. The spectral bound  $\bar{\lambda}_l^M$  can be interpreted as a constant in the inverse inequality on  $V_l$  and by (2.2), it depends on all prolongator smoothers  $S_k$ ,  $k < l$ . The role of the prolongator smoothers is to enforce ‘‘smoothness’’ of the coarse spaces by making the values of  $\bar{\lambda}_l^M$  small. Obviously, a smaller  $\bar{\lambda}_l^M$  allows the approximation condition (3.8) to be satisfied with a smaller constant  $C_1$ .

The columns of a typical tentative prolongator  $P_{l+1}^l$  are orthogonal, as we observed in Example 2.2. By properly scaling the columns of  $P_{l+1}^l$ , we can obtain  $M_l$  equal to the identity matrix even in more general cases (see Algorithm 4.1). In such a case, (3.9) holds with  $C_M = 1$ .

Note that from (3.10), inequality (3.13) always holds with  $C_S = 1$ ; for the prolongator smoother (2.3) we will have  $C_S = 1/3$ , which gives a better bound. The remaining assumptions of Lemma 3.3 are natural algebraic requirements on the prolongator smoothers  $S_l$ , which are easily satisfied.

The next lemma shows that the prolongator smoother (2.3) satisfies the assumptions of Lemma 3.3, and justifies the choice of  $\bar{\lambda}_l^M$  in (2.5).

LEMMA 3.4. *Let  $S_l$  be given by (2.3) with  $\bar{\lambda}_l^M$  chosen as in (2.5). Then,*

$$(3.20) \quad \bar{\lambda}_l^M \geq \varrho(M_l^{-1}A_l), \quad l = 1, \dots, L,$$

inequalities (3.10), (3.11) hold, and (3.13) holds with  $C_S = 1/3$ . Further, assuming (3.9), (3.12) is satisfied with  $C_2 = (4/3)C_M$ .

*Proof.* Since  $M_1 = I$ , inequality (3.20) holds for  $l = 1$ . Assume (3.20) holds for  $l$ . Using (2.2) and the equation  $M_{l+1} = (P_{l+1}^1)^T P_{l+1}^1 = (P_l^1 P_{l+1}^l)^T P_l^1 P_{l+1}^l = (P_{l+1}^l)^T M_l P_{l+1}^l$ , we obtain

$$(3.21) \quad \begin{aligned} \varrho(M_{l+1}^{-1}A_{l+1}) &= \max_{\mathbf{x} \in \mathbb{R}^{n_{l+1}}} \frac{(P_{l+1}^l \mathbf{x})^T S_l^T A_l S_l (P_{l+1}^l \mathbf{x})}{\mathbf{x}^T M_{l+1} \mathbf{x}} \\ &= \max_{\mathbf{x} \in \mathbb{R}^{n_{l+1}}} \frac{(P_{l+1}^l \mathbf{x})^T S_l^T A_l S_l (P_{l+1}^l \mathbf{x})}{(P_{l+1}^l \mathbf{x})^T M_l (P_{l+1}^l \mathbf{x})} \\ &\leq \varrho(M_l^{-1} S_l^T A_l S_l). \end{aligned}$$

From the definition of  $S_l$  in (2.3), it follows that

$$M_l^{-1} S_l^T A_l S_l = \left( I - \frac{4}{3\bar{\lambda}_l^M} M_l^{-1} A_l \right)^2 M_l^{-1} A_l.$$

Hence, by the spectral mapping theorem,

$$\varrho(M_l^{-1} S_l^T A_l S_l) = \max_{t \in \sigma(M_l^{-1} A_l)} \left( 1 - \frac{4}{3\bar{\lambda}_l^M} t \right)^2 t \leq \max_{t \in [0, \bar{\lambda}_l^M]} \left( 1 - \frac{4}{3\bar{\lambda}_l^M} t \right)^2 t = \frac{1}{9} \bar{\lambda}_l^M.$$

This proves (3.13) with  $C_S = 1/3$ . The statement (3.20) follows from the last estimate together with (3.21).

From definition (2.3),  $S_l$  is  $A_l$ -symmetric, and, from (3.20),

$$(3.22) \quad \sigma(S_l) \subset [-1, 1],$$

which proves (3.10).

To verify (3.11), we estimate for  $\mathbf{x} \in \mathbb{R}^{n_l}$ ,

$$\begin{aligned} \|S_l \mathbf{x}\|_{\mathbb{R}^{n_l}} &= \|M_l^{-1/2} (M_l^{1/2} S_l M_l^{-1/2}) M_l^{1/2} \mathbf{x}\|_{\mathbb{R}^{n_l}} \\ &\leq \varrho(M_l^{-1/2}) \varrho(M_l^{1/2} S_l M_l^{-1/2}) \|M_l^{1/2} \mathbf{x}\|_{\mathbb{R}^{n_l}}. \end{aligned}$$

Since  $M_l = (P_l^1)^T P_l^1$ , we have  $\varrho(M_l^{-1/2}) = \lambda_{\min}^{-1/2}(M_l)$  and  $\|M_l^{1/2} \mathbf{x}\|_{\mathbb{R}^{n_l}} = \|P_l^1 \mathbf{x}\|_{\mathbb{R}^{n_1}}$ . Further, it follows from (3.22) that  $\varrho(M_l^{1/2} S_l M_l^{-1/2}) \leq 1$ . Now, (3.11) follows by direct computation.

It remains to verify (3.12). Since  $I - S_l = 4/(3\bar{\lambda}_l^M) M_l^{-1} A_l$ , (3.12) holds with

$$C_2 = \left( \frac{4}{3\bar{\lambda}_l^M} \right) \varrho(A_l)^{1/2} \max_{\mathbf{x} \in \mathbb{R}^{n_l}} \frac{\|M_l^{-1} A_l \mathbf{x}\|_{\mathbb{R}^{n_l}}}{\|\mathbf{x}\|_{A_l}} = \left( \frac{4}{3\bar{\lambda}_l^M} \right) \varrho(A_l)^{1/2} \varrho(M_l^{-1} A_l^{1/2}),$$

where

$$\varrho(M_l^{-1}A_l^{1/2}) \leq \varrho(M_l^{-1/2})\varrho(M_l^{-1/2}A_l^{1/2}) \leq \sqrt{\frac{\bar{\lambda}_l^M}{\lambda_{\min}(M_l)}}.$$

Now, from (3.18),  $C_2 \leq (4/3)\sqrt{\text{cond}(M_l)} \leq (4/3)C_M$ , concluding the proof.  $\square$

We are now ready to prove the following convergence theorem. Recall that  $\bar{\lambda}$  is a known upper bound of  $\varrho(A)$  used in (2.5).

**THEOREM 3.5.** *Let the prolongator smoothers  $S_l$  be given by (2.3) with  $\bar{\lambda}_l^M$  chosen as in (2.5). Assume that  $C_1$  and  $C_M$  are such that there are linear mappings*

$$\tilde{Q}_l : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_l}, \quad l = 1, \dots, L, \quad \tilde{Q}_1 = I,$$

such that

$$(3.23) \quad \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 \leq C_1^2 \frac{9^{l-1}}{\bar{\lambda}} \|\mathbf{u}\|_A^2 \quad \forall \mathbf{u} \in \mathbb{R}^{n_1}, \quad l = 1, \dots, L-1,$$

and

$$(3.24) \quad \text{cond}(M_l) \leq C_M^2 \quad l = 1, \dots, L.$$

Further assume that  $R_l$  are symmetric positive definite matrices satisfying (3.7) with a constant  $c_R > 0$  independent of the level.

Then,

$$\|\hat{\mathbf{x}} - MG(\mathbf{x}, \mathbf{b})\|_A \leq \left(1 - \frac{1}{c_0(L)}\right) \|\hat{\mathbf{x}} - \mathbf{x}\|_A \quad \forall \mathbf{x} \in \mathbb{R}^{n_1},$$

where  $A\hat{\mathbf{x}} = \mathbf{b}$ , and

$$c_0(L) = \left(2 + C_1 C_M c_R + \frac{4}{3} C_M c_R + \frac{1}{3} C_1 \left(1 + \frac{4}{3} C_M c_R\right) (L-1)\right)^2 (L-1)$$

In addition, if  $P : \mathbf{u} \mapsto MG(\mathbf{0}, \mathbf{u})$ , then  $P$  is a symmetric matrix and  $\text{cond}(A, P) \leq c_0(L)$ .

*Proof.* By (2.5) and Lemma 3.4,  $\varrho(M_l^{-1}A_l) \leq \bar{\lambda}_l^M = 9^{1-l}\bar{\lambda}$ . Therefore, the approximation property (3.8) in Lemma 3.3 holds with  $C_1$  from (3.23). From Lemma 3.2,  $c_0(L) = (1 + c_1(L) + c_2(L)c_R)^2(L-1)$ , where, by Lemma 3.3,  $c_1(L) = 1 + C_S C_1(L-1)$ ,  $c_2(L) = C_1 C_M + C_2 c_1(L)$ . From Lemma 3.4,  $C_S = 1/3$ ,  $C_2 = (4/3)C_M$ , and the proof is completed by a direct computation.  $\square$

**4. Choice of the tentative prolongator.** In this section we reformulate the construction of the tentative prolongators described in [43] and prove the main convergence theorem.

Our construction is based on the supernodes aggregation concept. On each level, degrees of freedom are organized in small disjoint clusters called supernodes. On the finest level, the supernodes have to be specified, e.g., as the sets of degrees of freedom associated with the finite element nodes. The coarse level supernodes are then created by our aggregation algorithm.

The input data needed for constructing the tentative prolongators are the hierarchy of aggregates and the level one matrix  $B^1$  of dimension  $n_1 \times r$  matrix

$B^1$ , where  $r$  is a positive integer. The range of  $B^1$  specifies which functions (finest level vectors) should be exactly representable on each coarse level in the sense that

$$(4.1) \quad \text{Range } B^1 \subset \text{Range } P_l^1, \quad l = 1, \dots, L-1.$$

Our main convergence result, Theorem 4.2 below, gives a convergence estimate based on assumptions on the finest-level matrix  $A_1$ , the matrix  $B^1$  and the aggregates. The key assumption of Theorem 4.2 is a weak approximation property (4.4) that is easy to verify computationally [23], thus providing a guideline for choosing aggregates and the matrix  $B^1$  needed for solving a linear system with given matrix  $A_1$ .

Both the construction of the tentative prolongators and Th. 4.2 are purely algebraic. We only need the matrices  $A^1$  and  $B^1$ , and the decomposition into aggregates.

Following the considerations in [43], we typically choose  $B^1$  to be a generator of zero energy modes. In a finite element context, this means the kernel of the stiffness matrix obtained from the finite element model with no essential boundary conditions. Zero energy modes, determined from geometry and element definition, are available in most of the existing finite element packages.

For scalar problems, the matrix  $B^1$  of zero energy modes can be often obtained without any geometric information. For Lagrange elements, the zero energy modes are simply multiples of the vector of all ones, cf., Example 2.2.

For second order systems, such as elasticity, one may apply this approach componentwise and build the matrix  $B^1$  so that its range consists of all discretized constant vector fields. Such matrix  $B^1$  can be again constructed without any geometrical information. The verification of the properties in Sect. 5 can be done for the quadratic form  $\|u\|_{H^1(\Omega)}^2$  in the place of  $a(u, u)$  and then extended to the form  $a(u, u)$  by the equivalence of norms  $a(u, u) \approx \|u\|_{H^1(\Omega)}^2$ . For elasticity, the lower bound on  $a(u, u)$  in this equivalence is Korn's inequality. The constant in Korn's inequality is, however, sensitive to domain shape and boundary conditions, and in practice one indeed observes worse convergence and loss of robustness compared with the use of zero energy modes.

The above equivalence argument is avoided for tentative prolongators constructed using all zero-energy modes, which are rigid body modes for elasticity, as in [43]. This allows one to prove the weak approximation property with a constant independent of some problem data, such as boundary conditions, shape of the computational domain, and also, under some restrictions, jumps in coefficients [23, 44].

The objective (4.1) is specified for the composite tentative prolongators  $P_l^1$ . To enforce it during the setup of  $P_{l+1}^l$ , we create simultaneously the prolongator  $P_{l+1}^l$  and the  $n_{l+1} \times r$  matrix  $B^{l+1}$  so that

$$(4.2) \quad P_{l+1}^l B^{l+1} = B^l,$$

where  $B^l$  has been constructed during the setup of  $P_l^{l-1}$  (or, is given if  $l = 1$ ).

The prolongator  $P_{l+1}^l$  is constructed from a given system of aggregates  $\{\mathcal{A}_i^l\}_{i=1}^{N_l}$  that forms a disjoint covering of level  $l$  supernodes. A simple greedy algorithm for generating aggregates based on the structure of the matrix  $A_l$  is given in [43]. The property (4.2) is enforced aggregate by aggregate; columns of  $P_{l+1}^l$  associated with the aggregate  $\mathcal{A}_i^l$  are formed by orthonormalized restrictions of the columns of  $B^l$  onto the aggregate  $\mathcal{A}_i^l$ . For each aggregate, such a construction gives rise to  $r$  degrees of freedom on the coarse level, forming a coarse level supernode.

The detailed algorithm follows. For ease of presentation, we assume that the fine level supernodes are numbered by consecutive numbers within each aggregate. This assumption can be easily avoided by renumbering.

ALGORITHM 4.1. *For the given system of aggregates  $\{\mathcal{A}_i^l\}_{i=1}^{N_l}$  and the  $n_l \times r$  matrix  $B^l$  satisfying  $P_l^1 B^l = B^1$ , we create a prolongator  $P_{l+1}^l$ , a matrix  $B^{l+1}$  satisfying (4.2) and supernodes on level  $l+1$  as follows:*

1. Let  $d_i$  denote the number of degrees of freedom associated with aggregate  $\mathcal{A}_i^l$ . Partition the  $n_l \times r$  matrix  $B^l$  into blocks  $B_i^l$  of size  $d_i \times r$ ,  $i = 1, \dots, N_l$ , each corresponding to the set of degrees of freedom on an aggregate  $\mathcal{A}_i^l$  (see Fig. 4.1).
2. Decompose  $B_i^l = Q_i^l R_i^l$ , where  $Q_i^l$  is an  $d_i \times r$  orthogonal matrix, and  $R_i^l$  is an  $r \times r$  upper triangular matrix.
3. Create the tentative prolongator  $P_{l+1}^l = \text{diag}(Q_1^l, \dots, Q_{N_l}^l)$ , cf., Fig. 4.1, and set

$$B^{l+1} = \begin{pmatrix} R_1^l \\ R_2^l \\ \vdots \\ R_{N_l}^l \end{pmatrix}.$$

4. For each aggregate  $\mathcal{A}_i^l$ , the coarsening gives rise to  $r$  degrees of freedom on the coarse level (the  $i$ -th block column of  $P_{l+1}^l$ ). These degrees of freedom define the  $i$ -th coarse level supernode.

Before formulating the convergence theorem, we introduce the *composite aggregate* and the associated norm. The composite aggregate  $\tilde{\mathcal{A}}_i^l$  is the aggregate  $\mathcal{A}_i^l$ , understood as the corresponding set of supernodes on the finest level. Formally,  $\tilde{\mathcal{A}}_i^l$  is defined by

$$(4.3) \quad \tilde{\mathcal{A}}_i^l = \mathcal{A}_i^{l,1}, \quad \text{where} \quad \mathcal{A}_i^{l,l} = \mathcal{A}_i^l, \quad \mathcal{A}_i^{l,j-1} = \bigcup_{k \in \mathcal{A}_i^{l,j}} \mathcal{A}_k^{j-1}$$

and the corresponding discrete  $l^2$ -(semi)norm of the vector  $\mathbf{x} \in \mathbb{R}^{n_1}$  by

$$\|\mathbf{x}\|_{l^2(\tilde{\mathcal{A}}_i^l)} = \left( \sum_{\text{dofs } k \text{ of } \tilde{\mathcal{A}}_i^l} x_k^2 \right)^{1/2}.$$

We are now ready to prove the main convergence theorem.

THEOREM 4.2. *Let the prolongator smoothers  $S_l$  be given by (2.3) with  $\bar{\lambda}_i^M$  chosen as in (2.5), and the tentative prolongators  $P_{l+1}^l$  be created by Algorithm 4.1 using the  $n_l \times r$  matrix  $B^1$  and the aggregates  $\{\mathcal{A}_i^l\}_{i=1}^{N_l}$ ,  $l = 1, \dots, L-1$ . Assume there is a constant  $C_A > 0$  such that for every  $\mathbf{u} \in \mathbb{R}^{n_1}$  and every  $l = 1, \dots, L-1$ ,*

$$(4.4) \quad \sum_{i=1}^{N_l} \min_{\mathbf{w} \in \mathbb{R}^r} \|\mathbf{u} - B^1 \mathbf{w}\|_{l^2(\tilde{\mathcal{A}}_i^l)}^2 \leq C_A \frac{9^{l-1}}{\lambda} \|\mathbf{u}\|_A^2.$$

*Further assume that  $R_l$  are symmetric positive definite matrices satisfying (3.7) with a constant  $c_R > 0$  independent of the level.*

*Then,*

$$\|\hat{\mathbf{x}} - MG(\mathbf{x}, \mathbf{b})\|_A \leq \left( 1 - \frac{1}{c_0(L)} \right) \|\hat{\mathbf{x}} - \mathbf{x}\|_A \quad \forall \mathbf{x} \in \mathbb{R}^{n_1},$$

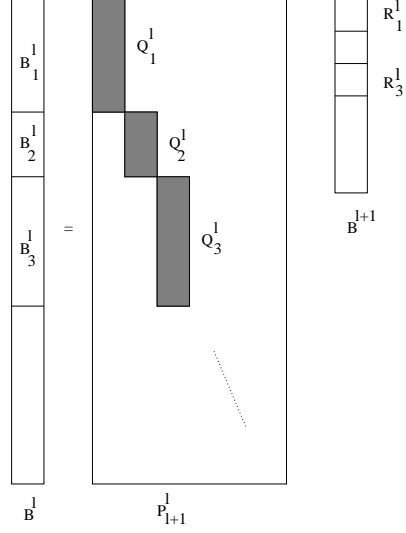


FIG. 4.1. The tentative prolongator  $P_{l+1}^l$

where  $A\hat{\mathbf{x}} = \mathbf{b}$ , and

$$c_0(L) = (2 + C_A c_R + (4/3)c_R + (1/3)C_A (1 + (4/3)c_R) (L - 1))^2 (L - 1).$$

Further, if  $P : \mathbf{u} \mapsto MG(\mathbf{0}, \mathbf{u})$ , then  $P$  is symmetric in  $(\cdot, \cdot)_{\mathbb{R}^{n_1}}$  and  $\text{cond}(A, P) \leq c_0(L)$ .

*Proof.* The proof consists of the verification of the assumptions of Theorem 3.5. The tentative prolongators  $P_{l+1}^l$  are block diagonal matrices with orthogonal blocks  $Q_i^l$ , hence orthogonal (see Step 2.) Since the product of orthogonal matrices is an orthogonal matrix,  $P_l^1$  is orthogonal and (3.24) holds with  $C_M = 1$ .

Let us show that (3.23) is satisfied with  $C_1 = C_A$ . For each supernode  $s_i^l$  on level  $l$ , define the space

$$W_i^l = \{P_l^1 \mathbf{x} \mid \mathbf{x} \in \mathbb{R}^{n_1}, x_j = 0 \forall j \notin s_i^l\}, \quad i = 1, \dots, N_{l-1}.$$

Note that the number of supernodes on level  $l$  equals the number of aggregates  $N_{l-1}$  on level  $l - 1$ . Let  $\text{dof}(\tilde{\mathcal{A}}_i^{l-1})$  be the set of degrees of freedom corresponding to the aggregate  $\tilde{\mathcal{A}}_i^{l-1}$ . From the nonzero block structure of the tentative prolongators  $P_{k+1}^k$  and the definition (4.3) of the composite aggregates  $\tilde{\mathcal{A}}_i^l$ , it follows that  $(P_l^1 \mathbf{x})_j$ ,  $j \in \text{dof}(\tilde{\mathcal{A}}_i^l)$ , depend only on  $x_k$ ,  $k \in s_i^l$ . Hence,

$$(4.5) \quad W_i^l = \left\{ \mathbf{x} \in \mathbb{R}^{n_1} \mid \exists \mathbf{y} \in \text{Range } P_l^1 : x_i = y_i \text{ if } i \in \text{dof}(\tilde{\mathcal{A}}_i^{l-1}), 0 \text{ otherwise} \right\}.$$

Since the aggregates  $\tilde{\mathcal{A}}_i^l$  form a disjoint covering of the set of the finest level supernodes, the spaces  $W_i^l$  form an orthogonal decomposition of  $\text{Range } P_l^1$  and the corresponding orthogonal projections  $T_i^l : \mathbb{R}^{n_1} \rightarrow W_i^l$ ,  $T^l : \mathbb{R}^{n_1} \rightarrow \text{Range } P_l^1$  satisfy

$$T^l = T_1^l + T_2^l + \dots + T_{N_{l-1}}^l.$$

From here and from (4.5), we get the following estimate for every  $\mathbf{u} \in \mathbb{R}^{n_1}$ ,

$$\|(I - T^l)\mathbf{u}\|_{\mathbb{R}^{n_1}}^2 = \sum_{i=1}^{N_{l-1}} \|\mathbf{u} - (T_1^l + \dots + T_{N_{l-1}}^l)\mathbf{u}\|_{l^2(\tilde{\mathcal{A}}_i^{l-1})}^2 = \sum_{i=1}^{N_{l-1}} \|\mathbf{u} - T_i^l \mathbf{u}\|_{l^2(\tilde{\mathcal{A}}_i^{l-1})}^2$$

$$(4.6) \quad = \sum_{i=1}^{N_{l-1}} \min_{\mathbf{w} \in \text{Range } P_l^1} \|\mathbf{u} - \mathbf{w}\|_{l^2(\tilde{\mathcal{A}}_i^{l-1})}^2 \leq \sum_{i=1}^{N_{l-1}} \min_{\mathbf{w} \in \mathbb{R}^r} \|\mathbf{u} - B^1 \mathbf{w}\|_{l^2(\tilde{\mathcal{A}}_i^{l-1})}^2,$$

using (4.1) in the last step.

Set  $\tilde{Q}_l = (P_l^1)^T$ . Since  $M_l = (P_l^1)^T P_l^1 = I$ , the mapping  $P_l^1 \tilde{Q}_l = P_l^1 M_l^{-1} (P_l^1)^T$  is the orthogonal projection  $T^l$  onto  $\text{Range } P_l^1$ . Then, using the equation  $P_{l+1}^1 \tilde{Q}_{l+1} = T^{l+1}$ , estimates (4.6) and (3.19), and assumption (4.4), we obtain

$$\begin{aligned} \|P_l^1 \tilde{Q}_l \mathbf{u} - P_{l+1}^1 \tilde{Q}_{l+1} \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 &\leq \|(I - P_{l+1}^1 \tilde{Q}_{l+1}) \mathbf{u}\|_{\mathbb{R}^{n_1}}^2 \\ &\leq \sum_{i=1}^{N_l} \min_{\mathbf{w} \in \mathbb{R}^r} \|\mathbf{u} - B^1 \mathbf{w}\|_{l^2(\tilde{\mathcal{A}}_i^l)}^2 \leq C_A \frac{9^{l-1}}{\lambda} \|\mathbf{u}\|_A^2, \end{aligned}$$

proving (3.23) with  $C_1 = C_A$ . Now, the proof follows from  $C_M = 1$ ,  $C_1 = C_A$ , using Theorem 3.5.  $\square$

**5. Model Problem.** The goal of this section is to verify the key assumption (4.4) of Theorem 4.2 on a simple example. The weak approximation property for problems of linear elasticity has been investigated in [44]. For the verification of the smoothing condition (3.7) for commonly used smoothers we refer to [6]. Note that for the Richardson iteration given by  $R_l = \varrho(A_l)^{-1} I$ , (3.7) holds with  $c_R = 1$ .

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  be a bounded domain,  $\tau_h$  a quasiuniform finite element mesh on  $\Omega$ , and  $V_h$  a P1 or Q1 finite element space associated with  $\tau_h$ . At some of the boundary vertices, zero Dirichlet boundary condition is imposed for functions in  $V_h$ . We assume the standard scaling of the finite element basis,  $\|\varphi_i\|_{L^\infty} = 1$  and solve a second order scalar elliptic problem

$$(5.1) \quad \text{find } u \in V_h \text{ such that } a(u, v) = f(v) \text{ for every } v \in V_h,$$

where  $f \in H^{-1}(\Omega)$  and  $a(\cdot, \cdot)$  is a coercive and bounded bilinear form on  $H^1(\Omega)$ .

For solving the resulting linear system (2.1), we use Algorithm 2.1, where the prolongator smoothers are defined by (2.3) and (2.5) and the tentative prolongators are created by Algorithm 4.1. In order to do so, we need to specify the supernodes on the finest level, the supernode aggregates  $\{\mathcal{A}_i^l\}_{i=1}^{N_l}$  on each level  $l < L$ , and the matrix  $B^1$ .

On level 1, each supernode consists of the degree of freedom associated with one finite element vertex with no essential boundary condition imposed.

We assume that on every level  $l < L$ , for each aggregate  $\mathcal{A}_i^l$  there is a ball  $U_i^l \subset \mathbb{R}^d$  such that

1. all finite element vertices of the corresponding composite aggregate  $\tilde{\mathcal{A}}_i^l$  are located within  $U_i^l$ ,
2.  $\text{diam}(U_i^l) \leq C3^l h$ , where  $h$  is the characteristic meshsize of  $\tau_h$  and  $C$  is a positive constant independent of the level,
3. there is an integer constant  $N$  independent of the level such that every point  $\mathbf{x} \in \Omega$  belongs to at most  $N$  balls  $U_i^l$ . (Overlaps of the balls are bounded.)

The heuristic greedy algorithm described in [43] tends to generate aggregates satisfying the above assumption.

In order to satisfy assumption (4.4), we need to choose  $B^1$  so that on each aggregate,  $\min_{\mathbf{w} \in \mathbb{R}^r} \|\mathbf{u} - B^1 \mathbf{w}\|$  is small compared to the energy norm of  $\mathbf{u}$ . Therefore,

with the Poincaré inequality in mind, we choose  $B^1$  to be the discrete representation of the unit function, the vector of ones.

Let  $\mathbf{u} = (u_1, \dots, u_{n_1})^T$  be a given vector and  $u = u_1\varphi_1 + \dots + u_{n_1}\varphi_{n_1}$  the corresponding finite element function. In what follows,  $C$  is a generic constant independent of  $u$ ,  $\mathbf{u}$ , the meshsize  $h$  and the level  $l$ . We introduce a domain  $\Omega' \subset \Omega$  consisting of all elements of the mesh  $\tau_h$ , that are not adjacent to a finite element vertex with prescribed Dirichlet boundary condition. Then,  $\varphi_1 + \dots + \varphi_i = 1$  on  $\Omega'$  and, as all active degrees of freedom are located in  $\bar{\Omega}'$ , the equivalence of discrete and continuous  $L^2$ -norms gives

$$(5.2) \quad h^d \|\mathbf{u} - B^1 p\|_{l^2(\bar{\mathcal{A}}_i^l)}^2 \leq C \|u - p\|_{L^2(U_i^l \cap \Omega')}^2 \leq C \|Eu - p\|_{L^2(U_i^l)}^2, \quad p \in \mathbb{R}^1.$$

Here,  $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^d)/\mathbb{R}^1 \equiv \{v : |v|_{H^1(\mathbb{R}^d)} < \infty\}$  is the extension operator satisfying  $Eu = u$  on  $\Omega$  and  $|Eu|_{H^1(\mathbb{R}^d)} \leq C|u|_{H^1(\Omega)}$ .

To verify (4.4), we need to estimate the minimum of the expression on the left-hand side of (5.2) with respect to  $p \in \mathbb{R}^1$ . This can be done using the scaled Poincaré inequality applied to the right-hand side of (5.2): for each ball  $U_i^l$ , there is a number  $p_i^l = p_i^l(Eu)$  such that  $\|Eu - p_i^l\|_{L^2(U_i^l)} \leq C \text{diam}(U_i^l) |Eu|_{H^1(U_i^l)}$ . Here,  $C$  is a Poincaré constant on the unit ball. Hence, for all balls  $U_i^l$  it holds that

$$(5.3) \quad \min_{p \in \mathbb{R}^1} \|\mathbf{u} - B^1 p_i^l\|_{l^2(\bar{\mathcal{A}}_i^l)}^2 \leq \|\mathbf{u} - B^1 p_i^l\|_{l^2(\bar{\mathcal{A}}_i^l)}^2 \leq Ch^{-d} \text{diam}(U_i^l)^2 |Eu|_{H^1(U_i^l)}^2.$$

From the assumption that  $\text{diam}(U_i^l) \leq C3^l h$ , the property  $|Eu|_{H^1(\mathbb{R}^d)} \leq C|u|_{H^1(\Omega)}$ , estimate (5.3), the bounded overlaps of the balls  $U_i^l$ , the well-known estimate  $\varrho(A) \leq Ch^{d-2}$ , and the  $H^1$ -equivalence of  $a(\cdot, \cdot)$  we get

$$(5.4) \quad \sum_{i=1}^{N_l} \min_{\mathbf{w} \in \mathbb{R}^1} \|\mathbf{u} - B^1 \mathbf{w}\|_{l^2(\bar{\mathcal{A}}_i^l)}^2 \leq C \frac{9^{l-1}}{h^{d-2}} |Eu|_{H^1(\mathbb{R}^d)}^2 \leq C \frac{9^{l-1}}{\varrho(A)} \|\mathbf{u}\|_A^2,$$

completing the verification of (4.4).

Note the very weak dependence of our estimate on the actual shape of the aggregates; the constant  $C$  in the estimate above depends on the shape of the aggregates only through the intersection parameter  $N$ . Also, the estimate is independent of the essential boundary conditions.

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#### REFERENCES

- [1] R. E. BANK AND T. DUPONT, *An optimal order process for solving elliptic finite element equations*, Math. Comp., 36 (1981), pp. 35–51.
- [2] P. E. BJØRSTAD AND J. MANDEL, *Spectra of sums of orthogonal projections and applications to parallel computing*, BIT, 31 (1991), pp. 76–88.
- [3] R. BLAHETA, *A multilevel method with overcorrection by aggregation for solving discrete elliptic problems*, J. Comput. Appl. Math., 24 (1988), pp. 227–239.
- [4] D. BRAESS, *The contraction number of a multigrid method for solving the Poisson equation*, Numer. Math., 37 (1981), pp. 387–404.
- [5] ———, *Towards algebraic multigrid for elliptic problems of second order*, Computing, 55 (1995), pp. 379–393.
- [6] J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp., 57 (1991), pp. 23–45.

- [7] ———, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.
- [8] J. H. BRAMBLE, J. E. PASCIAK, AND J. XU, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.
- [9] A. BRANDT, *Algebraic multigrid theory: The symmetric case*, Appl. Math. Comput., 19 (1986), pp. 23–56.
- [10] A. BRANDT, S. F. MCCORMICK, AND J. W. RUGE, *Algebraic multigrid (AMG) for sparse matrix equations*, in Sparsity and Its Applications, D. J. Evans, ed., Cambridge University Press, Cambridge, 1984.
- [11] M. BREZINA AND P. VANĚK, *A black-box iterative solver based on a two-level Schwarz method*, Computing, 64 (1999), pp. 233–263.
- [12] T. F. CHAN AND B. F. SMITH, *Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes*, in Domain Decomposition Methods in Scientific and Engineering Computing: Proceedings of the Seventh International Conference on Domain Decomposition, D. E. Keyes and J. Xu, eds., vol. 180 of Contemporary Mathematics, Providence, Rhode Island, 1994, American Mathematical Society, pp. 175–189.
- [13] T. F. CHAN, J. XU, AND L. ZIKATANOV, *An agglomeration multigrid method for unstructured grids*, in Tenth international conference on Domain Decomposition methods, X.-C. C. J Mandel, Chabel Farhat, ed., vol. 218 of Contemporary Mathematics, American Mathematical Society, 1998, pp. 67–81.
- [14] M. DRYJA, *A method of domain decomposition for 3-D finite element problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM, pp. 43–61.
- [15] M. DRYJA, B. F. SMITH, AND O. B. WIDLUND, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal., 31 (1994).
- [16] M. DRYJA AND O. B. WIDLUND, *Some domain decomposition algorithms for elliptic problems*, in Iterative Methods for Large Linear Systems, New York, 1989, Academic Press, pp. 273–291.
- [17] C. FARHAT, P.-S. CHEN, J. MANDEL, AND F.-X. ROUX, *The two-level FETI method - Part II: Extension to shell problems, parallel implementation and performance results*, Comp. Meth. Appl. Mech. Engrg., 155 (1998), pp. 153–180.
- [18] C. FARHAT AND J. MANDEL, *The two-level FETI method for static and dynamic plate problems - Part I: An optimal iterative solver for biharmonic systems*, Comp. Meth. Appl. Mech. Engrg., 155 (1998), pp. 129–152.
- [19] C. FARHAT AND F.-X. ROUX, *An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 379–396.
- [20] T. GRAUSCHOPF, M. GRIEBEL, AND H. REGLER, *Additive multilevel-preconditioners based on bilinear interpolation, matrix dependent geometric coarsening and algebraic multigrid coarsening for second order elliptic PDEs*, Appl. Numer. Math., 23 (1997), pp. 63–96.
- [21] W. HACKBUSCH, *On the multigrid method applied to difference equations*, Computing, 20 (1978), pp. 291–306.
- [22] ———, *Multigrid Methods and Applications*, vol. 4 of Computational Mathematics, Springer-Verlag, Berlin, 1985.
- [23] M. KOČVARA AND J. MANDEL, *A multigrid method for three-dimensional elasticity and algebraic convergence estimates*, Appl. Math. Comput., 23 (1987), pp. 121–135.
- [24] J. KŘÍŽKOVÁ AND P. VANĚK, *Two-level preconditioner with small coarse grid appropriate for unstructured meshes*, Numer. Linear Algebra Appl., 3 (1996), pp. 255–274.
- [25] P. LE TALLEC, J. MANDEL, AND M. VIDRASCU, *A Neumann-Neumann domain decomposition algorithm for solving plate and shell problems*, SIAM J. Numer. Anal., 35 (1998), pp. 836–867.
- [26] P.-L. LIONS, *On the Schwarz alternating method I*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM, pp. 1–42.
- [27] J. F. MAITRE AND F. MUSY, *Multigrid methods: convergence theory in a variational framework*, SIAM J. Numer. Anal., 21 (1984), pp. 657–671.
- [28] J. MANDEL, *Étude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre*, Comptes Rendus Acad. Sci. Paris, Sér. I, 298 (1984), pp. 469–472.
- [29] ———, *Iterative solvers by substructuring for the p-version finite element method*, Comput. Methods Appl. Mech. Engrg., 80 (1990), pp. 117–128. International Conference on Spectral and High Order Methods for Partial Differential Equations, Como, Italy, June 1989.

- [30] ———, *Balancing domain decomposition*, Comm. in Numerical Methods in Engrg., 9 (1993), pp. 233–241.
- [31] J. MANDEL, S. F. MCCORMICK, AND R. E. BANK, *Variational multigrid theory*, in Multigrid Methods, S. F. McCormick, ed., vol. 3 of Frontiers in Applied Mathematics, SIAM Books, Philadelphia, 1987, pp. 131–177.
- [32] J. MANDEL AND R. TEZAUER, *Convergence of a substructuring method with Lagrange multipliers*, Numerische Mathematik, 73 (1996), pp. 473–487.
- [33] J. MANDEL, R. TEZAUER, AND C. FARHAT, *A scalable substructuring method by Lagrange multipliers for plate bending problems*, SIAM J. Numer. Anal., 36 (1999), pp. 1370–1391.
- [34] P. OSWALD, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, in Constructive Theory of Functions, Proc. Int. Conf. Varna 1991, Sofia, 1992, Bulg. Acad. Sci., pp. 203–214.
- [35] P. OSWALD, *Multilevel Finite Element Approximation, Theory and Applications*, Teubner Skripten zur Numerik, Teubner Verlag, 1994.
- [36] J. W. RUGE AND K. STÜBEN, *Efficient solution of finite difference and finite element equations by algebraic multigrid (AMG)*, in Multigrid Methods for Integral and Differential Equations, D. J. Paddon and H. Holstein, eds., The Institute of Mathematics and its Applications Conference Series, Clarendon Press, Oxford, 1985, pp. 169–212.
- [37] ———, *Algebraic multigrid (AMG)*, in Multigrid Methods, S. F. McCormick, ed., vol. 3 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1987, pp. 73–130.
- [38] H. A. SCHWARZ, *Über einige Abbildungsaufgaben*, Ges. Math. Abh., 11 (1869), pp. 65–83.
- [39] K. STÜBEN, *Algebraic multigrid (AMG): An introduction with applications*, Tech. Rep. 53, GMD, St. Augustin, Germany, April 1999.
- [40] P. VANĚK, *Acceleration of convergence of a two level algorithm by smoothing transfer operators*, Appl. Math., 37 (1992), pp. 265–274.
- [41] ———, *Fast multigrid solver*, Appl. Math., 40 (1995), pp. 1–20.
- [42] P. VANĚK, J. MANDEL, AND M. BREZINA, *Algebraic multigrid on unstructured meshes*, UCD/CCM Report 34, Center for Computational Mathematics, University of Colorado at Denver, December 1994. <http://www-math.cudenver.edu/ccmreports/rep34.ps.gz>.
- [43] P. VANĚK, J. MANDEL, AND M. BREZINA, *Algebraic multigrid based on smoothed aggregation for second and fourth order problems*, Computing, 56 (1996), pp. 179–196.
- [44] P. VANĚK, M. BREZINA, AND R. TEZAUER, *Two-grid method for linear elasticity on unstructured meshes*, SIAM J. Sci. Comp., 21 (1999), pp. 900–923.
- [45] P. VANĚK, J. MANDEL, AND M. BREZINA, *Two-level algebraic multigrid for the Helmholtz problem*, in Tenth International Conference on Domain Decomposition, J. Mandel, C. Farhat, and X.-C. Cai, eds., vol. 218 of Contemporary Mathematics, Providence, RI, 1998, American Mathematical Society, pp. 349–356.
- [46] P. VANĚK, R. TEZAUER, M. BREZINA, AND J. KRÍŽKOVÁ, *Two-level method with coarse space size independent convergence*, in Domain Decomposition Methods in Sciences and Engineering, 8th International Conference, Beijing, P.R. China, 1996, R. Glowinski, J. Périaux, Z. Shi, and O. Widlund, eds., Wiley, 1997, pp. 233–240.
- [47] H. YSERENTANT, *On the multi-level splitting of finite element spaces*, Numer. Math., 49 (1986), pp. 379–412.
- [48] ———, *Two preconditioners based on the multi-level splitting of finite element spaces*, Numer. Math., 58 (1990), pp. 163–184.