Iterative Solvers for Coupled Fluid-Solid Scattering

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Supported by the Office of Naval Research under grant N-00014-95-1-0663,
and the National Science foundation under grant DMS-007428.

University of Kentucky
January 15, 2003
The coupled scattering problem

Discretization by Finite Elements

Multigrid Method
  - Multigrid algorithms
  - Computational results in 2D

Substructuring Method
  - Substructuring by Lagrange multipliers for Helmholtz equation
  - Extension to elastic scattering
  - Extension to coupled problem
  - Computational results in 2D and 3D
Main results

- treating a coupled problem with vastly different scales
  - algorithms are invariant to scaling of physical units
  - physical units must match
  - define residual with care

- Multigrid on a properly scaled coupled problem with GMRES as smoother converges well if coarse problem fine enough

- extended FETI-H (De La Bourdonnaye, Farhat, Macedo, Magoulès, Roux, 1998, 2000; Farhat, Macedo, Tezaur, DD11, 1999) to coupled problem ⇒ coupled algorithm reduces to FETI-H in the limit for a very stiff obstacle
Coupled problem

- solid obstacle in fluid

- time harmonic fluid pressure $p(x, t) = p(x)e^{i\omega t}$
  $\implies$ Helmholtz equation in the fluid

- time harmonic solid displacement $u(x, t) = u(x)e^{i\omega t}$
  $\implies$ elastodynamic Lamé equation in the solid

- continuity of displacement & balance of normal forces
  $\implies$ wet interface conditions
Coupled Problem
Fluid Medium - the Helmholtz Equation

- isotropic, homogeneous, inviscid
- compressible, irrotational
- time-harmonic solution of the wave equation
- complex amplitude \( p(x) \): the fluid pressure at location \( x \) and time \( t \) is \( p(x)e^{i\omega t} \)

\[
-\Delta p - k^2 p = f \quad \text{in } \Omega \\
p = p_0 \quad \text{on } \Gamma_d \\
\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma_n \\
\frac{\partial p}{\partial n} + ikp = 0 \quad \text{on } \Gamma_a \text{ (radiation b.c.)}
\]

- \( k = \frac{\omega}{c_f} \), \( \omega \) is wave frequency, \( c_f \) is speed of sound in acoustic medium
- radiation boundary condition - Sommerfeld
  - does not reflect waves in the normal direction
  - first-order approximation for radiation to infinity
Coupled Problem
Elastic Medium

- isotropic, homogeneous
- small time-harmonic displacement $u(x)e^{i\omega t}$ of an elastic body
- complex displacement amplitude $u(x)$ satisfies the Lamé equations

$$\nabla \cdot \tau + \omega^2 \rho_e u = 0 \quad \text{ in } \Omega_e + \text{ boundary conditions},$$

where:

- $\rho_e$ = density of the elastic medium
- $\tau = \lambda I(\nabla \cdot u) + 2\mu e(u)$ = stress tensor
- $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ = strain tensor
- $\lambda, \mu$ = Lamé coefficients of elastic medium
Coupled Problem
Fluid-Solid Interface

1. continuity across interface

\[ n \cdot u = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial n} \]

2. balance of normal forces:

\[ n \cdot \tau \cdot n = -p \]

3. nonviscous fluid \( \rightarrow \) zero tangential tension:

\[ n \times \tau \cdot n = 0 \]
Coupled Problem
Summary of the Pressure-Displacement Formulation

Helmholtz equation in the fluid region:

$$\Delta p + k^2 p = 0 \text{ in } \Omega_f,$$

$$p = p_0 \text{ on } \Gamma_d, \quad \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma_n, \quad \frac{\partial p}{\partial n} + ikp = 0 \text{ on } \Gamma_a.$$ 

Lamé equation in the elastic region:

$$\nabla \cdot \tau + \omega^2 \rho_e u = 0 \text{ in } \Omega_e,$$

$$\tau = \lambda I(\nabla \cdot u) + 2\mu e(u), \quad e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

Wet interface conditions:

$$n \cdot u = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial n}, \quad n \cdot \tau \cdot n = -p \quad n \times \tau \cdot n = 0 \text{ on } \Gamma$$

On $\Gamma$, the value of $u$ provides load for the Helmholtz problem for $p$ and the value of $p$ provides load for the elastodynamic problem for $u$. 
Existence of solution and non-radiating modes

Solution exists \((p, u) \in H^1(\Omega_f) \times H^1(\Omega_e)^3\) and is unique up to non-radiating modes in the solid

- The boundary value problem

\[
\begin{align*}
\nabla \cdot \tau + \omega^2 \rho_e u &= 0 \quad \text{in} \quad \Omega_e \\
\tau \cdot n &= 0 \quad \text{on} \quad \Gamma \\
u \cdot n &= 0 \quad \text{on} \quad \Gamma 
\end{align*}
\]

has nonzero solution for certain frequencies and geometries - called non-radiating modes.

- For this to happen, \(\omega^2 \rho_e\) needs to be an eigenvalue of the pure traction problem and in addition \(u \cdot n = 0\).

- Bodies with certain symmetries have non-radiating modes (e.g., sphere can sustain torsional oscillations with zero radial component displacement)

- Almost all elastic bodies do not have non-radiating modes [Hargé]
Variational Formulation

Find \((p, p_\Gamma), (p - p_0, p_\Gamma) \in V_f\) and \((u, n \cdot u_\Gamma) \in V_e\) such that

\[
- \int_{\Omega_f} \nabla p \nabla \bar{q} + k^2 \int_{\Omega_f} p \bar{q} - ik \int_{\Gamma_a} p_\Gamma \bar{q}_\Gamma - \int_{\Gamma} \rho_f \omega^2 (n \cdot u_\Gamma) \bar{q}_\Gamma = 0 \\
- \int_{\Omega_e} \left( \lambda (\nabla \cdot u) (\nabla \cdot \bar{v}) + 2\mu e(u) : e(\bar{v}) \right) + \omega^2 \int_{\Omega_e} \rho_e u \cdot \bar{v} - \\
\int_{\Gamma} p_\Gamma (n \cdot \bar{v}_\Gamma) = 0
\]

\(\forall \bar{q}, \bar{q}_\Gamma \in V_f\) and \(\forall \bar{v}, n \cdot \bar{v}_\Gamma \in V_e\),

where

\[
V_f = \{ (p, p_\Gamma) \mid p \in H^1(\Omega_f), p_\Gamma \in H^1(\Gamma) \mid p = 0 \text{ on } \Gamma_d \} \\
V_e = \{ (u, n \cdot u_\Gamma) \mid u \in (H^1(\Omega_e))^3, u_\Gamma \in (H^1(\Gamma))^3 \},
\]

Same wet interface term in both equations \(\implies\) symmetric system

Industry standard (Morand and Ohayon 1995)
Finite elements $\implies$ 2 $\times$ 2 symmetric block system

$$
\begin{bmatrix}
-K_f + k^2 M_f - i k G_f & -\rho_f \omega^2 T \\
-T^* & -K_e + \omega^2 M_e
\end{bmatrix}
\begin{bmatrix}
p \\
u
\end{bmatrix} =
\begin{bmatrix}
r & 0
\end{bmatrix}
$$

$T$ is the coupling matrix: $p^*Tv = \int p(\nu \cdot v)$

$T$ is transfer of load $\implies$ lower order, weak coupling
Since $\lambda$ and $\mu$ are large $\Rightarrow$ use a scaling of the form $u = su'$ and $v = sv'$, where $s$ is a scalar, to control leading terms.

<table>
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<th>Discretization and Error Bound</th>
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<td>leading terms differ by factor of $k^2$</td>
<td>comparable leading terms</td>
</tr>
<tr>
<td>consistent with definition of norm</td>
<td>scale to $O(1)$ diagonal</td>
</tr>
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</table>

$$s = \frac{1}{c \sqrt{\rho_f \max\{\lambda, 2\mu\}}}.$$  

$$\begin{bmatrix} -S_f + k^2M_f + i k G_f & -\rho_f \omega^2 s T \\ -\rho_f \omega^2 s T^t & -\rho_f \omega^2 s^2 S_e + \rho_e \rho_f \omega^4 s^2 M_e \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix} = R$$
Finite element error estimates

If the solution \((p, u) \in H^{1+\alpha}(\Omega_f) \times H^{1+\alpha}(\Omega_e)^3\) and \(h\) is small enough then the discretization error satisfies

\[
\|p - p_h\|_{H^1(\Omega_f)}^2 + k^2\|p - p_h\|_{L^2(\Omega_f)}^2 + k^2\|u - u_h\|_{H^1(\Omega_e)}^3 + k^2\|u - u_h\|_{L^2(\Omega_e)}^3 \leq C(k)h^{2\alpha}
\]

Model Problem

- 2D channel filled with water, unit square
- excitation on the left (Dirichlet boundary condition) on $\Gamma_d$
- sound-hard sides (Neumann boundary condition) on $\Gamma_n$
- outgoing radiation on the right (absorbing boundary condition) on $\Gamma_a$
- aluminum scatterer in the middle
- discretization by uniform mesh, P1 elements
Numerical Solution
Square scatterer size 0.2 $m$ in the middle

Fluid pressure

Solid displacement
Numerical Solution
Obstacle with a gap

gap on $x$ axis
gap on $y$ axis
Multigrid for the Coupled Problem

- coarsening: standard variational bilinear interpolation, mesh ratio 2
- smoothing
  - GMRES, BICG-STAB
  - Preconditioners
    * inverse of lower triangular part of $A$
    * inverse of nodal block diagonal
  - Gauss-Seidel
- for stable smoothers can increase number of steps without causing divergence
- all smoothers work well for fluid or solid part separately
- wet interface aligned with coarsest grid, otherwise *nothing special*
Multigrid convergence, scatterer size 0.2
Decreasing $h$, increasing $k$, $k^3h^2$ constant
Adding coarse meshes
Multigrid convergence, scatterer size 0.2
Decreasing $h$, increasing $k$, $k^3h^2$ constant
Adding smoothing steps

GMRES as smoother

BICG–STAB as smoother

GMRES preconditioned by inv D as smoother

GMRES precondition by inv lower triangular part of A as smoother
Multigrid convergence, scatterer size 0.2

GMRES preconditioned by Multigrid

Multigrid with different smoothers

Decreasing $h$, $k^3h^2$ constant

$h = 1/128$, $k = 25$
Multigrid convergence, scatterer with a gap

GMRES preconditioned by Multigrid and Multigrid with smoother GMRES

Obstacle 0.2 in x–dir 0.4 in y –dir

gap on x–axis of size 40% on x 50% on y

Decreasing $h$, $k^3h^2$ constant
Summary of multigrid performance

- efficient multigrid method for the discrete coupled system

- Krylov subspace smoothers, in particular GMRES, make robust algorithms when used with multigrid.

- large number of smoothing steps - GMRES preconditioned by multigrid gives comparable results to multigrid with smoother GMRES

- more smoothing steps have a stabilizing effect on the solution.

- adding coarse meshes - Gauss Seidel as smoother diverges, Krylov smoothers are robust

- multigrid methods are found to diverge, or give very poor convergence for high wave numbers, totally different methods are required for this purpose.

- in some cases, multigrid with GMRES as smoother gives slightly better convergence rates than GMRES preconditioned by one multigrid cycle, otherwise about same.

- multigrid with smoother GMRES and multigrid with smoother GMRES preconditioned by inverse of lower triangular part of $A$ were most robust and gave best results, however they require the most memory to store all fine grid vectors.
GMRES with $2 \times 2$ block diagonal ILU preconditioning
scatterer size 0.2 in the middle

- in fluid let $L_f \cdot R_f$ be approximate factorization of $-K_f + k^2M_f - ikG_f$
- in solid let $L_e \cdot R_e$ be approximate factorization of $-\rho_f\omega^2s^2K_e + \rho_f\rho_e\omega^4s^2M_e$

GMRES preconditioned by the block diagonal matrix

$$
\begin{pmatrix}
(L_f \cdot R_f)^{-1} & 0 \\
0 & (L_e \cdot R_e)^{-1}
\end{pmatrix}
$$

For some frequencies, the matrix $-K_e + \omega^2M_e$ will be singular - scatterer is at resonance.
FETI-H for coupled problem

- FETI-H for Helmholtz equation (De La Bourdonnaye, Farhat, Macedo, Magoulès, F. Roux 1998; Farhat, Macedo Lesoinne 2000)
- Extension to solid
- Coupled system
FETI-H for two subdomains (continuous form)

\[ -\Delta p^1 - k^2 p^1 = 0 \quad \text{in } \Omega_1 + \text{b.c.} \]
\[ -\Delta p^2 - k^2 p^2 = 0 \quad \text{in } \Omega_2 + \text{b.c.} \]
\[ p^1 = p^2 \quad \text{on } \Gamma_{12} \]
\[ \frac{\partial p^1}{\partial \nu^1} = -\frac{\partial p^2}{\partial \nu^2} \quad \text{on } \Gamma_{12} \]

FETI idea: enforce \( p^1 = p^2 \) by Lagrange multiplier, eliminate \( p^1, p^2 \), iterate on the system for the Lagrange multiplier. But here \( p^1, p^2 \) may not be determined due to (near) resonance. Hence \( \frac{\partial p^1}{\partial \nu^1} = -\frac{\partial p^2}{\partial \nu^2} \) replaced by a linear combination

\[ \frac{\partial p^1}{\partial \nu^1} - ikp^1 = -\frac{\partial p^2}{\partial \nu^2} - ikp^2 \quad \text{on } \Gamma_{12} \]
FETI-H for two subdomains (discrete form)

Discrete equations: \( Kp = f \), where \( K = S - k^2M + ikM_S \)

Decomposed, continuity \( p^1 = p^2 \) on \( \Gamma_{12} \) to be enforced by Lagrange multipliers:

- Subassembly in \( \Omega_1 \): \( K^1p^1 + B^{1T}\lambda = f^1 \)
- Subassembly in \( \Omega_2 \): \( K^2p^2 + B^{2T}\lambda = f^2 \)
- Continuity across \( \Gamma_{12} \): \( B^1p^1 + B^2p^2 = 0 \)

After replacement of interface condition by a linear combination becomes

\[
(K^1 + R^1)p^1 = f^1 - B^{1T}\lambda \\
(K^2 + R^2)p^2 = f^2 - B^{2T}\lambda \\
B^1p^1 + B^2p^2 = 0
\]

Upon assembly the contributions of \( R^1 \) and \( R^2 \) to the global system would cancel \( \implies \) equivalent system

The artificial radiation condition gives \( R^1 = ikM_{12}, R^2 = -ikM_{12} \) (the interface mass matrix or its diagonal approximation, with natural dof mapping)
FETI-H: multiple subdomains

- for each subdomain decide sign of the artificial radiation condition
- for each interface between subdomains, add artificial radiation matrix to subdomain matrix if not in conflict between neighbors
- subdomain problems guaranteed solvable (Farhat 1998)
- eliminate original dofs, get system for Lagrange multipliers
  \[ \mathbf{F}_l \lambda = \mathbf{d} \]
- reduced problems has the size of the interface
- symmetric, but not Hermitian \( \rightarrow \) GCR(GMRES)
- Numerical experiments indicate scalability with respect to \( h \)
Enforce an optional constraint $Q^T r^k = 0$ for the residuals $r^k = d - F_I \lambda^k$ by using the splitting

$$\lambda = \bar{P} \bar{\lambda} + \lambda^0$$

The original interface problem is transformed into the modified problem

$$P^T F_I P \lambda = F_I P \lambda = P^T d$$

where

$$P = I - Q (Q^T F_I Q)^{-1} Q^T F_I$$

and

$$\lambda^0 = Q (Q^T F_I Q)^{-1} Q^T d$$
The coarse space basis is

\[ Q = \begin{bmatrix} B^1Q^1 & \cdots & B^sQ^s & \cdots & B^NQ^N \end{bmatrix} \]

where \( Q^s_j \) is the discrete representation of the plane wave \( e^{ikd_j^T \mathbf{x}} \), with the directions \( d_j = [\cos \theta_j, \sin \theta_j] \), where \( \theta_j = (j - 1)\frac{2\pi}{N_\theta}, \ j = 1, \ldots, N_\theta \).
Extension of FETI-H to elastic scattering, two subdomains

\[ \nabla \cdot \tau(u^1) + \omega^2 \rho_e u^1 = 0 \quad \text{in } \Omega_1 + \text{b.c.} \\
\nabla \cdot \tau(u^2) + \omega^2 \rho_e u^2 = 0 \quad \text{in } \Omega_2 + \text{b.c.} \]

\[ u^1 = u^2 \quad \text{on } \Gamma_{12} \quad \text{continuity of displacement} \]

\[ \tau(u^1) \cdot \nu^1 = -\tau(u^2) \cdot \nu^2 \quad \text{on } \Gamma_{12} \quad \text{continuity of traction} \]

FETI idea: enforce \( u^1 = u^2 \) by Lagrange multipliers, eliminate \( u^1, u^2 \), iterate on the system for the Lagrange multiplier. But here \( u^1, u^2 \) may not be determined due to (near) resonance. Hence \( \tau(u^1) \cdot \nu^1 = -\tau(u^2) \cdot \nu^2 \) replaced by linear combination:

\[ \tau(u^1) \cdot \nu^1 + i\omega a u^1(\nu^1 \cdot u^1) = -\tau(u^2) \cdot \nu^2 + i\omega a u^2(\nu^2 \cdot u^2) \quad \text{on } \Gamma_{12}. \]
**Elastic scattering, two subdomains, discrete form**

Discrete equations: \( Ku = f \) where \( K = S - \omega^2 M \)

Decomposed and \( u^1 = u^2 \) on \( \Gamma_{12} \) enforced by Lagrange multipliers:

- **subassembly in** \( \Omega_1 \): \( K^1 u^1 + B^1 \lambda = f^1 \)
- **subassembly in** \( \Omega_2 \): \( K^2 u^2 + B^2 \lambda = f^2 \)
- **continuity across** \( \Gamma_{12} \): \( B^1 u^1 + B^2 p^2 = 0 \)

Equivalent to:

\[
(K^1 + R^1)u^1 = f^1 - B^1 \lambda \\
(K^2 + R^2)u^2 = f^2 - B^2 \lambda \\
B^1 u^1 + B^2 u^2 = 0
\]

Upon assembly the contributions of \( R^1 \) and \( R^2 \) to the global system would cancel \( \implies \) equivalent system
Substructuring for the coupled problem: Overview

- FETI-H:
  - decomposition of solid and fluid domains
  - artificial radiation condition to ensure solvability on subdomains
  - Lagrange multipliers on subdomain interfaces
  - eliminate primary variables $\Rightarrow$ uncoupled local problems

- here: local problems coupled across wet interface

- solution: DUPLICATE variables on wet interface, eliminate original primary variables, leave copies in the system iterated on

- $\Rightarrow$ numerically almost triangular system containing FETI-H for fluid and solid as diagonal blocks

Jan Mandel, Iterative Substructuring with Lagrange Multipliers for Coupled Fluid-Solid Scattering, DD14, 2002


http://www-math.cudenver.edu/~jmandel/papers
Model coupled problem

- 2D channel filled with water
- excitation on the left (Dirichlet boundary condition) on $\Gamma_d$
- sound-hard sides (Neumann boundary condition) on $\Gamma_n$
- outgoing radiation on the right (absorbing boundary condition) on $\Gamma_a$
- aluminum scatterer in the middle
Model 2D Problem Decomposed in 5x5 Fluid and 2x2 Solid Subdomains
Decomposition and subassembly

Non-overlapping subdomains: \( \bar{\Omega}_f = \bigcup_{s=1}^{N_f} \Omega^s_e, \quad \bar{\Omega}_e = \bigcup_{s=1}^{N_e} \Omega^s_e. \)

Subdomain local vectors and matrices:

\[
\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p}^1 \\ \vdots \\ \mathbf{p}^{N_f} \end{bmatrix}, \quad \hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^{N_e} \end{bmatrix},
\]

\[
\hat{\mathbf{K}}_f = \begin{bmatrix} \mathbf{K}^1_f & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{K}^{N_f}_f \end{bmatrix}, \quad \mathbf{p}^s \mathbf{K}^s_f \mathbf{q} = \int_{\bar{\Omega}^s_f} \nabla p \nabla q,
\]

\( (\hat{\mathbf{K}}_e \hat{\mathbf{M}}_f \hat{\mathbf{M}}_e \text{ defined similarly}) \)

\[
\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{T}^{11} & \cdots & \mathbf{T}^{1,N_e} \\ \vdots & \ddots & \vdots \\ \mathbf{T}^{N_f,1} & \cdots & \mathbf{T}^{N_f,N_e} \end{bmatrix}, \quad \mathbf{p}^r \mathbf{T}^{rs} \mathbf{v}^s = \int_{\partial \Omega^r_f \cap \partial \Omega^s_e} p (\nu \cdot \mathbf{v})
\]
Intersubdomain continuity

Local to global maps $N_f$ and $N_e$:

$$
K_f = N_f^* \hat{K}_f N_f, \quad K_e = N_e^* \hat{K}_e N_e
$$

$$
\hat{\mathbf{p}} = N_f \mathbf{p}, \quad \hat{\mathbf{u}} = N_e \mathbf{u}.
$$

To enforce same values between subdomains:

$$
B_f = [B_f^1, \ldots, B_f^{N_f}], \quad B_e = [B_e^1, \ldots, B_e^{N_e}]
$$

such that

$$
B_f \hat{\mathbf{p}} = 0 \iff \hat{\mathbf{p}} = N_f \mathbf{p} \quad \text{for some } \mathbf{p}
$$

$$
B_e \hat{\mathbf{u}} = 0 \iff \hat{\mathbf{u}} = N_e \mathbf{u} \quad \text{for some } \mathbf{u}
$$
Decomposed system

Introduce Lagrange multipliers to enforce intersubdomain continuity:

\[
\begin{bmatrix}
-\hat{K}_f + k^2\hat{M}_f & -\omega^2 \rho_f \hat{T} & B_f^* & 0 \\
-\hat{T}^* & -\hat{K}_e + \omega^2 \hat{M}_e & 0 & B_e^* \\
B_f & 0 & 0 & 0 \\
0 & B_e & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{p} \\
\hat{u} \\
\lambda_f \\
\lambda_e
\end{bmatrix}
= \begin{bmatrix}
\hat{r} \\
0 \\
0 \\
0
\end{bmatrix}
\]

The system for \((\hat{p}, \hat{u}, \lambda_f, \lambda_e)\) is equivalent to the original system via the local to global maps: \(\hat{p} = N_f p\) and \(\hat{u} = N_e u\).

But \(-\hat{K}_f + k^2\hat{M}_f\) and \(-\hat{K}_e + \omega^2 \hat{M}_e\) are typically (close to) singular due to (near) resonance . . .
Regularized system

\[
\begin{bmatrix}
\hat{A}_f & -\omega^2 \rho_f \hat{T} & B_f^* & 0 \\
-\hat{T}^* & \hat{A}_e & 0 & B_e^* \\
B_f & 0 & 0 & 0 \\
0 & B_e & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{p} \\
\hat{u} \\
\lambda_f \\
\lambda_e
\end{bmatrix}
= \begin{bmatrix}
\hat{f} \\
0 \\
0 \\
0
\end{bmatrix}
\]

where with \( \sigma_{st} = \pm 1 \), \( \sigma_{st} = -\sigma_{st} \),

\[
\hat{A}_f = -\hat{K}_f + k^2 \hat{M}_f + \hat{R}_f \\
\hat{A}_e = -\hat{K}_e + \omega^2 \hat{M}_e + \hat{R}_e
\]

\[
\hat{R}_f = (R_{fs})_{rs}, \quad p^s R_f q^s = i \alpha_0 k \sum_{t \neq s} \sigma_{st} \int_{\partial \Omega_f \cap \partial \Omega_f} pq,
\]

\[
\hat{R}_e = (R_{es})_{rs}, \quad u^s R_e v^s = i \alpha_0 \omega \sqrt{\lambda + 2 \mu} \rho_e \sum_{t \neq s} \sigma_{st} \int_{\partial \Omega_e \cap \partial \Omega_e} (n \cdot u)(n \cdot v),
\]

\( \hat{N}_f^* R_f \hat{N}_f = 0, \hat{N}_e^* R_e \hat{N}_e = 0 \) \Rightarrow equivalent to the decomposed system
Choice of the artificial radiation matrices

If $\sigma_{st}$ does not change sign on $\Omega^s_j$, then $-\hat{K}^s_j + k^2\hat{M}^s_j + \hat{R}'_f$ is regular (Farhat, Macedo, Lesoinne 2000). Similarly for solid subdomains.

In computational tests, we assigned $\sigma_{st}$ by counting, did not try to avoid change of sign.

If $\alpha_0 = \pm 1$, the artificial intersubdomain conditions are satisfied by wave in the normal direction (pressure wave in solid).

But we will see numerically that $\alpha_0 = 0$ is OK except when very close to resonance.
**Augmented system**

**Key idea:** \( \hat{T}\hat{u}, \hat{T}^*\hat{p} \) depend on the values of \( \hat{u}, \hat{p} \) on the wet interface \( \Gamma \) only. Define \( \hat{J}_f, \hat{J}_e \) as expanding vector on \( \Gamma \) by zero entries, then

\[
\hat{T}\hat{u} = \hat{T}\hat{J}_e\hat{u}_\Gamma, \quad \hat{u}_\Gamma = \hat{J}_e^*\hat{u},
\]

\[
\hat{T}^*\hat{p} = \hat{T}^*\hat{J}_f\hat{p}_\Gamma, \quad \hat{p}_\Gamma = \hat{J}_f^*\hat{p}.
\]

Get the augmented system

\[
\begin{bmatrix}
\hat{A}_f & 0 & B_e^* & 0 & 0 & -\omega^2 \rho_f \hat{T}\hat{J}_e \\
0 & \hat{A}_e & 0 & B_e^* & -\hat{T}^*\hat{J}_f & 0 \\
B_f & 0 & 0 & 0 & 0 & 0 \\
0 & B_e & 0 & 0 & 0 & 0 \\
-J^*_f & 0 & 0 & 0 & I & 0 \\
0 & -J^*_e & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
\hat{p} \\
\hat{u} \\
\hat{p}_\Gamma \\
\hat{u}_\Gamma \\
\lambda_f \\
\lambda_e
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\mathbf{r}} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
Primal-dual method for the coupled problem

- scale the augmented system to symmetrize the coupling terms and so that the fluid and the elastic equations have the same order of magnitude

- eliminate the subdomain degrees of freedom \( \hat{p} \) and \( \hat{u} \) from the augmented system

- solve the resulting reduced system for the intersubdomain Lagrange multipliers \( \lambda_f, \lambda_e \) and the wet interface degrees of freedom \( \hat{p}_\Gamma, \hat{u}_\Gamma \) by Generalized Conjugate Residuals

- precondition by projection on a coarse space
• multiply the second equation by $\omega^2 \rho_f$

• symmetric diagonal scaling

\[
\begin{bmatrix}
\tilde{A}_f & 0 & \tilde{B}_f^* & 0 & 0 & -\tilde{TJ}_e \\
0 & \tilde{A}_e & 0 & \tilde{B}_e^* & -\tilde{T^*J}_f & 0 \\
\tilde{B}_f & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{B}_e & 0 & 0 & 0 & 0 \\
-\tilde{J}_f^* & 0 & 0 & 0 & I & 0 \\
0 & -\tilde{J}_e^* & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
\tilde{p} \\
\tilde{u} \\
\tilde{\lambda}_f \\
\tilde{\lambda}_e \\
\tilde{p}_r \\
\tilde{u}_r
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{r} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

where

\[
\tilde{A}_f = D_f \hat{A}_f D_f, \quad \tilde{A}_e = \omega^2 \rho_f D_e \hat{A}_e D_e, \quad \tilde{T} = \omega^2 \rho_f D_f \hat{T} D_e, \\
\tilde{B}_f = E_f B_f D_f, \quad \tilde{B}_e = E_e B_e D_e, \quad \tilde{r} = D_f \hat{r}, \\
\hat{p} = D_f \tilde{p}, \quad \hat{u} = D_e \tilde{u}, \quad \lambda_f = D_f \tilde{\lambda}_f, \quad \lambda_e = \omega^2 \rho_f D_f \tilde{\lambda}_f.
\]
Reduced system

Eliminating

\[
\begin{align*}
\tilde{p} &= \tilde{A}_f^{-1}(\tilde{r} - \tilde{B}_f^*\tilde{\lambda}_f + \tilde{T}_e\tilde{u}_\Gamma) \\
\tilde{u} &= \tilde{A}_e^{-1}(-\tilde{B}_e^*\tilde{\lambda}_e + \tilde{T}^*J_f\tilde{p}_\Gamma)
\end{align*}
\]

gives the final reduced system

\[
Fx = b,
\]

where

\[
F = \begin{bmatrix}
\tilde{B}_f\tilde{A}_f^{-1}\tilde{B}_f^* & 0 & 0 & -\tilde{B}_f\tilde{A}_f^{-1}\tilde{T}J_e \\
0 & \tilde{B}_e\tilde{A}_e^{-1}\tilde{B}_e & -\tilde{B}_e\tilde{A}_e^{-1}\tilde{T}^*J_f & 0 \\
J_f^*\tilde{A}_f^{-1}\tilde{B}_f^* & 0 & I & J_f^*\tilde{A}_f^{-1}\tilde{T}J_e \\
0 & J_e\tilde{A}_e^{-1}\tilde{B}_e & -J_e\tilde{A}_e^{-1}\tilde{T}^*J_f & I
\end{bmatrix},
\]

and \(\tilde{T}\) is small for small \(kh\) or stiff scatterer. Here,

\[
\begin{bmatrix}
\lambda_f \\
\lambda_e \\
\tilde{p}_\Gamma \\
\tilde{u}_\Gamma
\end{bmatrix}, \quad
\begin{bmatrix}
\tilde{B}_f\tilde{A}_f^{-1}\tilde{r} \\
0 \\
-J_f^*\tilde{A}_f^{-1}\tilde{r} \\
0
\end{bmatrix}.
\]
Decoupling of fluid and elastic equations

- the first diagonal block $\tilde{B}_f \tilde{A}_f^{-1} \tilde{B}_f^*$ is the FETI-H operator

- the second diagonal block $\tilde{B}_e \tilde{A}_e^{-1} \tilde{B}_e$ is the same for elasticity

- as the material is more rigid $\lambda, \mu \to \infty$, the scaling matrix $D_e \to 0$
  $\Rightarrow$ the coupling term $\tilde{T} = \omega^2 \rho_f D_f \hat{T} D_e \to 0$
  $\Rightarrow$ the fluid and elastic equations are uncoupled in the limit
  $\Rightarrow$ iterations “should” behave much like for fluid or elasticity alone
  (can anything be said in general about the convergence of GMRES for a block diagonal matrix from convergence on the diagonal blocks... let alone block triangular...?)
Estimate of the coupling matrix

Since \( \|D_e\| \approx (\omega^2 \rho f \mu h^{n-2})^{-1/2} \), it follows that

\[
\|\tilde{T}\| = \omega^2 \rho f \|D_f \hat{T} D_e\| \leq \omega^2 \rho f \|D_f\| \|\tilde{T}\| \|D_e\|
\approx \omega^2 \rho f (h^{n-2})^{-1/2} h^{n-1} (\omega^2 \rho f \mu h^{n-2})^{-1/2}
= \omega h \rho_f^{1/2} \mu^{-1/2} = k h \rho_f^{1/2} \mu^{-1/2}.
\]

Consequently, the system will become \textbf{numerically decoupled} if

\[
k h \rho_f^{1/2} \mu^{-1/2} \ll 1
\]
**Iterative solution**

Enforce the residual condition

\[ Q^*(Fx - b) = 0 \]

throughout the iterations. For given \( v \) use the initial approximation

\[ x^{(0)} = v + Qw, \]

\( w \) obtained by solving the residual correction equation,

\[ Q^*(F(v + Qw) - b) = 0. \]

\( Fx = b \) solved by GCR with left preconditioning by the projection

\[ P = I - Q(Q^*FQ)^{-1}Q^*F \]

and initial iterate \( x^{(0)} \). Equivalently, GCR applied to

\[ PFx = Pb. \]

Iterations run in a subspace:

\[ Q^*F(x^{(n)} - x^{(0)}) = 0 \]
Selection of coarse space

\[
Q = \begin{bmatrix}
D_f B_f Y_f & 0 & 0 & 0 \\
0 & D_e B_e Y_e & 0 & 0 \\
0 & 0 & D_f J_f^* Z_f & 0 \\
0 & 0 & 0 & D_f J_f^* Z_e \\
\end{bmatrix}
\]

Coarse space selection for multipliers (same/similar to FETI-H):

\( Y_f = \text{diag}(Y^s_f) \), columns of \( Y^s_f \) are discrete representations of plane waves in a small number of equally distributed directions, or discrete representation of the constant function.

\( Y_e = \text{diag}(Y^s_e) \), columns of \( Y^s_e \) are discrete representations of elastic plane waves (both pressure and shear) in a small number of equally distributed directions, or discrete representation of the rigid body motions.

Coarse space selection for wet interface (New):

The matrices \( Z^s_f \) and \( Z^s_e \) are chosen in the same way as \( Y^s_f \) and \( Y^s_e \), with possibly different selection of the number of directions and selection of constant or rigid body modes.

Some of the matrices \( Y^s_f, Y^s_e, Z^s_e \), or \( Y^s_e \) may be void.
Artificial radiation condition on wet interface

\[
\begin{bmatrix}
  -K_f + k^2 M_f - ik G_f & -\rho_f \omega^2 T \\
  -T^* & -K_e + \omega^2 M_e
\end{bmatrix}
\begin{bmatrix}
  p \\
  u
\end{bmatrix}
= \begin{bmatrix}
  r \\
  0
\end{bmatrix}
\]

But if there is only one solid subdomain \( K_e - \omega^2 M_e \) may be (nearly) singular ⇒ no convergence of iterations on reduced system with \( u \) eliminated

Solution: simulate radiation of the solid into the fluid (physics: solid damped by the radiation condition in the fluid via the fluid): add the first equation multiplied by \( i\alpha T' \) to the second equation

The diagonal block becomes

\[-K_e + \omega^2 M_e + i\beta \rho_f \omega^2 T'T\]

choose \( \beta = \beta_0 \omega \sqrt{\rho_e (\lambda + 2\mu) / \|T\|_1} \) so that the added term approximates the radiation condition satisfied by pressure waves in normal direction and physical units match ⇒ proceed as before, but now iterations converge

Generalization to multiple fluid subdomains straightforward: add decomposed equations for fluid subdomains, multiplied by the appropriate submatrix of \( T \), to the equation for the elastic domain

Numerical results: loss of convergence when \( \omega^2 \) very close to eigenvalue, but all is fine with multiple solid subdomains
Computational complexity

Convergence improves with increasing coarse space size but complexity increases.

Assume: complexity of LU decomposition of a block sparse matrix of order $nm$, where $m$ is the order of dense blocks, is about $n^\alpha m^3$, complexity of one solution is $n^\beta m^2$.

Denote:
- $N$ the total number of degrees of freedom
- $N_s$ the number of subdomains
- $N_i$ the total number of degrees of freedom on subdomain interfaces,
- $N_c$ the number of coarse space basis vectors per subdomain
- $N_t$ interfaces between the subdomains

Then the dominant cost of the method is

\[ T_{\text{setup}} \approx N_s (N/N_s)^\alpha + N_c N_t (N/N_s)^\beta + N_s^\alpha N_c^3 + N_c^2 N_i, \]

\[ T_{\text{iteration}} \approx N_s (N/N_s)^\beta + N_s^\beta N_c^2. \]
### Numerical results: scalability with number of solid subdomains

**Problem description**
- $h$: mesh size
- $k$: number of parallel processes
- $f$: number of subdomains
- $e$: number of wet interfaces

<table>
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<th>$k$</th>
<th>Subdomains</th>
<th>Coarse directions</th>
<th>Number of degrees of freedom</th>
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Numerical results: scalability with number of fluid and solid subdomains

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<th>Number of degrees of freedom</th>
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### Numerical results: Decreasing $h$, $k^3h^2$ constant, constant number of elements per subdomain

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Numerical results: Decreasing $h$, $k^3 h^2$ constant, same subdomains

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Varying strength of artificial radiation conditions

h=1/40, 4x4 subdomains in fluid and 4x4 in solid, 8 coarse directions in fluid, 4 pressure waves and 4 shear waves in solid, scatterer size 0.4x0.4
Varying strength of artificial radiation conditions

$h=1/40$, 4x4 subdomains in fluid and 4x4 in solid, 8 coarse directions in fluid, 4 pressure waves and 4 shear waves in solid, scatterer size 0.4x0.4, selection of basis instead of orthogonalization of the rows of $B$ and the columns of $Q$
Varying strength of artificial radiation conditions

\( h = 1/40, \) 4x4 subdomains in fluid and 1x1 in solid, 8 coarse directions in fluid, 4 pressure waves and 4 shear waves in solid, scatterer size 0.2x0.2
Spectrum of preconditioned operator for FETI-H

Fast convergence of FETI-H is due to clustering of the spectrum of the preconditioned operator at a point different from zero.

h=1/40, k=10, 4x4 subdomains, 4 coarse directions
The spectrum of the method for the coupled problem is very similar to the spectrum of FETI-H. Fast convergence is again due to clustering of the spectrum of the preconditioned operator at a point different from zero.

$h=1/40$, $k=10$, 4x4 and 2x2 fluid subdomains, 4 fluid coarse directions
Computational test for unstructured 3D grids

- Numbers reported here from Matlab runs on imported data
- Coarse problem from eigenvectors of subdomain matrices
- Artificial radiation strength $a_0 = 0.1$
- Cube scatterer $0.2 \times 0.2 \times 0.2$ in cube waveguide $1 \times 1 \times 1$, $k=10$
  3362 fluid dofs and 1203 solid dofs on subdomain interfaces, reduced
  system (augmented by wet dofs) size 7114
  unstructured grid, 16 fluid and 4 solid subdomains

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Most of the slowdown in a coupled problem comes from solving the solid and fluid at the same time... not from the coupling
Convergence for unstructured meshes
2D, coarse by directions

2D, 1x1 square discretized regularly by 400x400 Q1 elements. For the coupled case, there is a solid square 0.4x0.4 in the middle.

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Convergence for unstructured meshes
2D, coarse by directions

2D, 1x1 square discretized regularly by 400x400 Q1 elements. For the coupled case, there is a solid square 0.4x0.4 in the middle.

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Convergence for unstructured meshes
3D, coarse by directions

3D, 1x1x1 cube discretized by P1 elements, for the coupled case, there is a solid cube 0.4x0.4x0.4 in the middle, 22396 nodes, 27548 dofs

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Convergence for unstructured meshes
3D, coarse by directions

3D, 1x1x1 cube discretized by P1 elements, for the coupled case, there is a solid cube 0.4x0.4x0.4 in the middle, 158514 nodes, 188073 dofs

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Spectrum of cube, fluid block only, no coarse
Spectrum of solid block operator, no coarse, 3D example
• Spectrum of coupled problem is close to the union of the spectra of the fluid and the solid blocks

• Coarse space tends to focus the spectrum along the positive real half-axis
For good convergence of GMRES or other Krylov space methods, spectrum should be concentrated away from the origin.
**Conclusion**

- FETI-H: good performance, numerically/computationally scalable
- coupled method reduces in the limit of rigid scatterer to FETI-H for solid and fluid at the same time (triangular system)
- coupled method takes often about the same number of iterations as FETI-H for rigid scatterer, sometimes the sum of numbers of iterations for fluid and solid separately
- coupled method prototype in Matlab, 3D parallel implementation Radek Tezaur based on FETI code by Charbel Farhat and Michel Lesoinne
- MG: 2D Matlab prototype, very promising
We are looking for PhD students!
Research and Teaching Assistantships available
http://www-math.cudenver.edu/graduate
Applications requesting Teaching Assistantships due February 15, 2003

http://www-math.cudenver.edu/IMACS03
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