Improved degree conditions for 2-factors with \( k \) cycles in hamiltonian graphs

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Abstract

In this paper, we consider conditions that ensure a hamiltonian graph has a 2-factor with exactly \( k \) cycles. Brandt et al. proved that if \( G \) is a graph on \( n \geq 4k \) vertices with minimum degree at least \( \frac{n}{2} \), then \( G \) contains a 2-factor with exactly \( k \) cycles; moreover this is best possible. Faudree et al. asked if there is some \( c < \frac{1}{2} \) such that \( \delta(G) \geq cn \) would imply the existence of a 2-factor with \( k \)-cycles under the additional hypothesis that \( G \) was hamiltonian. This question was answered in the affirmative by Sárközy, who used the regularity–blow-up method to show that there exists some \( \varepsilon > 0 \) such that for every \( k \geq 1 \) and large \( n \), \( \delta(G) \geq (\frac{1}{2} - \varepsilon) n \) suffices.

We improve on this result, giving an elementary proof that for every \( \varepsilon > 0 \) and \( k \geq 1 \), if \( G \) is a hamiltonian graph on \( n \geq \frac{12k}{\varepsilon} \) vertices with \( \delta(G) \geq (\frac{2}{5} + \varepsilon) n \), then \( G \) contains a 2-factor with \( k \) cycles.

Keywords: 2-factor, hamiltonian cycle

1 Introduction

A 2-factor of a graph \( G \) is a spanning 2-regular subgraph or, alternatively is a collection of disjoint cycles that spans \( G \). As a hamiltonian cycle is a 2-factor with precisely one component, the problem of determining when a graph \( G \) has a 2-factor is a natural and well-studied relaxation of the hamiltonian problem. A number of results concerning the existence of 2-factors, along with many related cycle and factorization problems, can be found in [1, 8, 9, 15].

It has been demonstrated a number of times that established sufficient conditions for the existence of some cycle-structural property can be relaxed if the graphs under consideration are also assumed to be hamiltonian. For instance, in [2] Amar et al. proved that the minimum degree threshold for pancyclicity in general graphs from [3] can be significantly lowered in (necessarily nonbipartite) hamiltonian graphs.

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Theorem 1. If $G$ is a hamiltonian graph of order $n$ and and $\delta(G) \geq \frac{2n+1}{5}$, then $G$ is either pancyclic or bipartite.

In [4], Brandt et al. proved the following.

Theorem 2. If $k \geq 1$ is an integer and $G$ is a graph of order $n \geq 4k$ such that $\delta(G) \geq \frac{n}{2}$, then $G$ contains a 2-factor with exactly $k$ cycles.

In fact, it was shown that the weaker Ore-type degree condition $\sigma_2(G) \geq n$ suffices in place of the given assumption on the minimum degree.

Motivated by Theorems 1 and 2, Faudree et al. proved the following result in [6].

Theorem 3. If $G$ is a hamiltonian graph of order $n \geq 6$ and $\delta(G) \geq \frac{5}{12}n + 2$, then $G$ contains a 2-factor with two components.

In [14], Pfender showed that under the additional assumption that $G$ is claw-free, the minimum degree threshold in Theorem 3 can be greatly reduced to $\delta(G) \geq 7$, which is sharp.

Theorem 3 inspired the following conjecture, which also appeared in [6].

Conjecture 1. For each integer $k \geq 2$ there exists a positive real number $c_k < \frac{1}{2}$ and integers $a_k$ and $n_k$ such that every hamiltonian graph $G$ of order $n > n_k$ with $\delta(G) \geq c_kn + a_k$ contains a 2-factor with $k$ cycles.

Conjecture 1 was affirmed by Sárközy [17] using the regularity–blow-up method.

Theorem 4. There exists a real number $\varepsilon > 0$ such that for every integer $k \geq 2$ there exists an integer $n_0 = n_0(k)$ such that every hamiltonian graph $G$ of order $n \geq n_0$ with $\delta(G) \geq (1/2 - \varepsilon)n$ has a 2-factor with $k$ components.

Sárközy’s result raises a natural question: what is the minimum $\delta = \delta(k, n)$ such that every hamiltonian graph of order $n$ with minimum degree at least $\delta$ has a 2-factor with exactly $k$ cycles. In 2012, Györi and Li announced [10] that they have shown $\delta \geq \left(\frac{5}{11} + \varepsilon\right)n$ suffices for $n$ sufficiently large. Our main result is the following improvement of Theorem 3, Theorem 4, and that result.

Theorem 5. Let $k \geq 1$ and $0 < \varepsilon < \frac{1}{10}$. If $G$ is a hamiltonian graph on $n \geq \frac{3k}{\varepsilon}$ vertices with $\delta(G) \geq \left(\frac{2}{5} + \varepsilon\right)n$, then $G$ contains a 2-factor with exactly $k$ cycles.

Additionally, our proof uses only elementary techniques, and hence requires a significantly smaller bound on $n$ than that required by Theorem 4.

One vexing aspect of Conjecture 1 and the related work described here is that it is possible that a sublinear, or even constant, minimum degree would suffice to ensure a hamiltonian graph has a 2-factor of the desired type. Specifically, while the 4-regular graph obtained by replacing every vertex of a cycle with a copy of $K_5 - e$ is an example of a family of hamiltonian graphs that do not contain a 2-factor with two cycles, currently we know of no example that has $\delta \geq 5$. 
2 Proof of Theorem 5

A key ingredient in our proof is the following lemma of Nash-Williams [12]. For completeness, we
provide a brief sketch of the proof using modern terminology.

**Lemma 1.** Let $G$ be a 2-connected graph on $n$ vertices. If $\delta(G) = d \geq \frac{n+2}{3}$, then either $G$ has a
hamiltonian cycle or $\alpha(G) \geq d + 1$.

**Proof (sketch).** Choose a longest cycle $C$ in $G$, which has length at least $\min\{2\delta(G), |G|\}$ by a
classical result of Dirac [5]. If some component of $G - C$ is a vertex $v$, then the successors of the
neighbors of $v$ on $C$ along with $v$ form an independent set of size at least $d + 1$. Otherwise, let $P$
be a longest path in $G - C$ and suppose that $P$ has length at least 1. Consider the endpoints of $P$
say $x_1$ and $x_p$; all of their neighbors are in $V(P) \cup V(C)$. If $v_i \in V(C)$ is a neighbor of $x_1$, then $x_p$
cannot be adjacent to any of the $p$ vertices preceding $v_i$ or the $p$ vertices succeeding $v_i$. Counting
edges between $\{x_1, x_p\}$ and $C$ in two ways gives a contradiction. \qed

We also need the following well known result of Pósa [16].

**Theorem 6.** Let $G$ be a graph on $n$ vertices and for all $i \in [n]$ let $S_i = \{v \in V(G) : \deg(v) \leq i\}$. If for all $1 \leq i < \frac{n-1}{2}$, $|S_i| < i$ and $|S_{(n-1)/2}| \leq \frac{n-1}{2}$, then $G$ has a hamiltonian cycle.

Finally, a **convex geometric graph** $G$ is a graph whose vertices are embedded in the plane in
convex position, and whose edges (subject to the embedding of $V(G)$) are straight line segments.
We will utilize the following extremal result of Kupitz [11] (see [7], Theorem 1.9).

**Theorem 7.** Let $G$ be a convex geometric graph of order $n \geq 2k + 1$. If $|E(G)| > (k - 1)n$, then
$G$ contains $k$ pairwise noncrossing edges.

We are now ready to prove Theorem 5.

**Proof.** Let $k \geq 2$, $0 < \varepsilon < \frac{1}{10}$, and $n \geq \frac{3k}{\varepsilon^2}$. Note that this implies $k \leq \frac{1}{3}\varepsilon n$. We consider two cases,
based on the connectivity of $G$.

**Case 1:** $\kappa(G) = \varepsilon n$.

Let $S$ be a minimum cut-set of $G$. As $G$ is hamiltonian, we have $2 \leq |S| < \varepsilon n$ and further, since
$\delta(G - S) \geq (\frac{2}{3} + \varepsilon) n - |S| > \frac{2}{3}n$, $G - S$ has exactly two components. Let $X$ and $Y$ be those two
components, and suppose without loss of generality that $|X| \geq |Y|$. Note then that

$$\frac{2}{5}n < |Y| \leq \frac{n}{2} \leq |X| < \frac{3}{5}n. \quad (1)$$

Consequently, each pair of adjacent vertices $u$ and $v$ in $Y$ satisfies $|(N(u) \cap N(v)) \cap Y| \geq \frac{4}{5}n - |Y| \geq
\frac{4}{5}(2|Y|) - |Y| = \frac{3}{5}|Y|$. So we may remove a family $T$ of $k - 2$ vertex-disjoint triangles from $Y$ and
set $Y' = Y \setminus \bigcup_{T \in T} V(T)$.

Now, let $X^* = \{v \in S : \deg(v, X) > \varepsilon n\}$, $Y^* = S \setminus X^*$, $G_1 = G[X \cup X^*]$, and $G_2 = G[Y \cup Y^*]$.
We now verify that $G_1$ and $G_2$ satisfy the conditions of Theorem 6, in which case the hamiltonian
cycles in $G_1$ and $G_2$ along with $T$ give us the desired 2-factor with $k$ cycles.
First note that by (1), $|X \cup X'| \geq \frac{n}{2}$ so $|Y' \cup Y'| \leq \frac{n}{2}$. Now for all $y \in Y'$,

$$\deg(y, G_2) \geq \left(\frac{2}{5} + \epsilon\right)n - \deg(y, X) - |S| - |Y' \setminus Y'| \geq \left(\frac{2}{5} - 2\epsilon\right)n \geq \frac{n}{5} > |Y'|,$$

and for all $y \in Y'$,

$$\deg(y, G_2) \geq \left(\frac{2}{5} + \epsilon\right)n - |X'| \geq \frac{2}{5}n \geq \frac{2}{5}(2|Y' \cup Y'|) \geq \frac{4}{5}|Y' \cup Y'|.$$

These calculations show that $S_i = \emptyset$ for all $i < \frac{n}{2}$ and $|S_i| < \frac{n}{2}$ for all $\frac{n}{2} \leq i \leq \frac{|Y' \cup Y'|}{2}$.

Likewise, by (1), $|Y| > \frac{2n}{5}$ and thus $|X \cup X'| < \frac{3}{5}n$. Now for all $x \in X$,

$$\deg(x, G_1) > \epsilon n > |X'|$$

and for all $x \in X$,

$$\deg(x, G_1) \geq \left(\frac{2}{5} + \epsilon\right)n - |X'| \geq \frac{2}{5}n \geq \frac{2}{5}\left(\frac{5}{3}|X \cup X'\right) = \frac{2}{3}|X \cup X'|.$$

These calculations show that $S_i = \emptyset$ for all $i < \epsilon n$ and $|S_i| < \epsilon n \leq i$ for all $\epsilon n \leq i \leq \frac{|X' \cup X'|}{2}$.

**Case 2:** $\kappa(G) \geq \epsilon n$.

Suppose first that $\alpha(G) < \frac{2n}{5}$, and let $x_1, \ldots, x_{k-1}$ be distinct vertices in $V(G)$. Since $\left(\frac{2}{5} + \epsilon\right)n - 3(k-1) > \frac{2n}{5} > \alpha(G)$, we can iteratively construct a family of $k - 1$ vertex disjoint triangles each consisting of $x_i$ and some edge in $N(x_i)$. Let $G'$ be the graph obtained by deleting these $k - 1$ triangles. As

$$\delta(G') \geq \left(\frac{2}{5} + \epsilon\right)n - 3(k-1) > \frac{2n}{5} > \alpha(G) \geq \alpha(G'),$$

and $\kappa(G') \geq 2$, Lemma 1 implies that $G'$ is hamiltonian, completing the desired 2-factor with $k$ cycles.

Therefore, we may assume that $\alpha(G) \geq \frac{2n}{5}$. Let $H$ be a hamiltonian cycle in $G$ and let $X = \{x_1, x_2, \ldots, x_t\}$ be a largest independent set in $G$. Assuming that $H$ has an implicit clockwise orientation, for each $i \in [t]$, let $y_i$ be the successor of $x_i$ along $H$ and let $Y = \{y_1, y_2, \ldots, y_t\}$. Note that since $X$ is independent, $X \cap Y = \emptyset$.

Let $D$ be an auxiliary directed graph with vertex set $V(D) = \{(x_1, y_1), \ldots, (x_t, y_t)\}$ such that $((x_i, y_i), (x_j, y_j)) \in E(D)$ if and only if $x_i y_j$ is an edge in $G$ (see (a) and (b) in Figure 1). As each vertex $v$ in $G$ has at least $\delta(G) - (n - |X| - |Y|) \geq \left(\frac{2}{5} + \epsilon\right)n - (n - 2t)$ neighbors in $X \cup Y$, and $n \leq \frac{5}{2} t$, we have that

$$\delta^+(D) \geq \left(\frac{2}{5} + \epsilon\right)n - (n - 2t) = 2t - \left(\frac{3}{5} - \epsilon\right)n \geq 2t - \left(\frac{3}{2} - \epsilon\right)t = \left(\frac{1}{2} + \epsilon\right)t. \quad (2)$$

Thus $e(D) \geq \left(\frac{1}{2} + \epsilon\right)t^2$ and consequently the number of 2-cycles in $D$ is at least $\epsilon t^2$.

Finally, let $\Gamma$ be a convex geometric graph with $V(\Gamma) = V(D)$ such that $\{(x_i, y_i), (x_j, y_j)\} \in E(\Gamma)$ if and only if $\{(x_i, y_i), (x_j, y_j)\}$ induces a 2-cycle in $D$. By (2), $e(\Gamma) \geq \epsilon t^2$, so Theorem 7 implies that $\Gamma$ contains at least $\epsilon t$ pairwise disjoint edges. These $\epsilon t \geq \frac{2}{5} n \geq k$ pairwise disjoint edges correspond to non-overlapping pairs of consecutive chords on $H$ (as in part (a) of Figure 1), which allow us to split $H$ into a 2-factor with $k$ cycles, completing the proof. \qed
The hamiltonian cycle $H$ with vertices from $X$ colored white and vertices from $Y$ colored black

(b) The auxiliary directed graph $D$

c) The auxiliary convex geometric graph $\Gamma$

Figure 1

References


