Packing of Graphic $n$-tuples

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Abstract

An $n$-tuple $\pi$ (not necessarily monotone) is graphic if there is a simple graph $G$ with vertex set $\{v_1, \ldots, v_n\}$ in which the degree of $v_i$ is the $i$th entry of $\pi$. Graphic $n$-tuples $(d^{(1)}_1, \ldots, d^{(1)}_n)$ and $(d^{(2)}_1, \ldots, d^{(2)}_n)$ pack if there are edge-disjoint $n$-vertex graphs $G_1$ and $G_2$ such that $d_{G_1}(v_i) = d^{(1)}_i$ and $d_{G_2}(v_i) = d^{(2)}_i$ for all $i$. We prove that graphic $n$-tuples $\pi_1$ and $\pi_2$ pack if $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$, where $\Delta$ and $\delta$ denote the largest and smallest entries in $\pi_1 + \pi_2$ (strict inequality when $\delta = 1$); also, the bound is sharp.

Kundu and Lovász independently proved that a graphic $n$-tuple $\pi$ is realized by a graph with a $k$-factor if the $n$-tuple obtained by subtracting $k$ from each entry of $\pi$ is graphic; for even $n$ we conjecture that in fact some realization has $k$ edge-disjoint 1-factors. We prove the conjecture in the case where the largest entry of $\pi$ is at most $n/2 + 1$ and also when $k \leq 3$.

Keywords: Degree sequence, graphic sequence, graph packing, $k$-factor, 1-factor

1 Introduction

An integer $n$-tuple $\pi$ is graphic if there is a simple graph $G$ with vertex set $\{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$, where $\pi = (d_1, \ldots, d_n)$ and $d_G(v)$ denotes the degree of vertex $v$ in graph $G$. Such a graph $G$ realizes $\pi$. Two $n$-vertex graphs $G_1$ and $G_2$ pack if they can be expressed as

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edge-disjoint subgraphs of the complete graph $K_n$. We study an analogue of graph packing for graphic $n$-tuples. Let $\pi_1$ and $\pi_2$ be graphic $n$-tuples, with $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ (they need not be monotone). We say that $\pi_1$ and $\pi_2$ pack if there exist edge-disjoint graphs $G_1$ and $G_2$ with vertex set $\{v_1, \ldots, v_n\}$ such $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all $i$. In graph packing, vertices may be reordered, but in packing of graphic $n$-tuples no reordering of the indices is allowed. Graphic $n$-tuples are often called graphic sequences; we use “$n$-tuple” partly to emphasize that the order of entries matters. When not specifying the length, we use “list”.

The condition that $\pi_1 + \pi_2$ is graphic is obviously necessary for $\pi_1$ and $\pi_2$ to pack, but the following small example shows that it is not sufficient.

**Example 1.1.** Let $\pi_1 = (3, 1, 2, 2, 0, 0)$ and $\pi_2 = (1, 3, 0, 0, 2, 2)$, with sum $(4, 4, 2, 2, 2, 2)$. Both $\pi_1$ and $\pi_2$ are graphic, and the complete bipartite graph $K_{2,4}$ realizes their sum. However, in every realization of $\pi_j$, the vertex $v_j$ of degree 3 has three nonisolated neighbors. Thus $v_1$ and $v_2$ are adjacent in every realization of $\pi_1$ or $\pi_2$, and the lists do not pack. □

In fact, Dui̋r̋, Guinez, and Matamala [4] showed that determining whether two graphic $n$-tuples pack is NP-complete. Hence we focus on finding sharp sufficient conditions. In 1978, Sauer and Spencer [14] published the classical result that $n$-vertex graphs $G_1$ and $G_2$ pack if $\Delta(G_1)\Delta(G_2) < n/2$, where $\Delta(G)$ denotes the maximum vertex degree in $G$. In Section 2, we prove an analogue for $n$-tuples, showing that graphic $n$-tuples $\pi_1$ and $\pi_2$ pack if $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$, where $\Delta$ and $\delta$ denote the largest and smallest values in $\pi_1 + \pi_2$, except that strict inequality is needed when $\delta = 1$. Furthermore, the bound is sharp; we construct lists that do not pack when the maximum entry in the sum is larger by 1. We conjecture the stronger statement that two graphic $n$-tuples pack if the product of corresponding terms is always less than $n/2$; this would be a more direct analogue of the Sauer–Spencer Theorem.

Kundu’s Theorem [9], published in 1973 and proved independently by Lovász [10] at about the same time, characterizes when a graphic $n$-tuple has a realization containing a spanning subgraph that is “almost” $k$-regular. In the language of packing, the result states that if $\pi_1$ is graphic and each term in $\pi_2$ is $k$ or $k - 1$, then $\pi_1$ and $\pi_2$ pack if $\pi_1 + \pi_2$ is graphic.

In Section 3, we consider extensions of the $k$-factor case of Kundu’s Theorem, where a $k$-factor of a graph is a spanning $k$-regular subgraph. Kundu’s Theorem implies that a graphic $n$-tuple $\pi$ is realizable by a graph having a $k$-factor if the list obtained by subtracting $k$ from each entry is graphic. We conjecture the stronger statement that in fact when $n$ is even there is a realization containing $k$ edge-disjoint 1-factors (that is, a $k$-edge-colorable $k$-factor). We prove the conjecture when the largest entry is at most $n/2 + 1$. We also prove the more
difficult result that the conjecture holds when $k \leq 3$, by proving in general that there is a realization containing a $k$-factor that has two edge-disjoint 1-factors.

2 An Analogue of the Sauer-Spencer Theorem

The Sauer–Spencer Theorem immediately implies that $n$-vertex graphs $G_1$ and $G_2$ pack when their maximum degrees sum to less than $\sqrt{2n}$. Chen [2] gave a short proof of Kundu’s Theorem; we use a similar technique to prove our result for packing of graphic $n$-tuples. When the least entry in the sum is 1, the maximum allowed by the hypothesis is the same as in the Sauer–Spencer Theorem. Note that when we prove directly that $\pi_1$ and $\pi_2$ pack, it follows immediately that $\pi_1 + \pi_2$ is graphic.

Again let $\Delta$ and $\delta$ denote the largest and smallest entries in $\pi_1 + \pi_2$. Before proving that our condition is sufficient for $\pi_1$ and $\pi_2$ to pack, we present a simple construction that proves sharpness when $\delta = 1$. We later obtain sharpness for $\delta \geq 2$ via a slight modification of this construction.

Example 2.1. For $\delta, m \in \mathbb{N}$ with $m > 1$, let $n = 2\delta m^2$. We construct graphic $n$-tuples $\pi_1$ and $\pi_2$ with $\Delta = \sqrt{2\delta n}$ that do not pack. Let

$$\pi_1 = (\delta m, \delta m, (2\delta m)^{\delta(m-1)}, 0^{\delta(m-1)}, (\delta m)^{\delta-1}, 0^{\delta-1}, \delta^{\delta(m^2-m)}, 0^{\delta(m^2-m)})$$

and

$$\pi_2 = (\delta m, \delta m, 0^{\delta(m-1)}, (2\delta m)^{\delta(m-1)}, 0^{\delta-1}, (\delta m)^{\delta-1}, 0^{\delta(m^2-m)}, \delta^{\delta(m^2-m)}),$$

where the exponents denote multiplicity (lengths of constant sublists). The lists have length $2\delta m^2$, as desired. Also, the largest and smallest entries in $\pi_1 + \pi_2$ are $2\delta m$ and $\delta$, respectively, so $\Delta = \sqrt{2\delta n}$. (The Erdős–Gallai conditions [6] readily imply that $\pi_1 + \pi_2$ is graphic, but this is not important). It remains to show that $\pi_1$ and $\pi_2$ are graphic but do not pack.

To show that $\pi_i$ is graphic, start with $K_{\delta m + 1}$, split its vertices into sets $V_1, \ldots, V_{m-1}$ of size $\delta$ plus $\delta + 1$ leftover vertices, for each $i$ make the vertices of $V_i$ adjacent to a set $X_i$ of $\delta m$ new vertices, and add to these $\delta m^2 + 1$ vertices a set of $\delta m^2 - 1$ isolated vertices.

Given any realization of $\pi_1$, let $S$ be the set of $\delta m + 1$ vertices with degree exceeding $\delta$. Their degree-sum is $2\delta^2 m^2 - \delta m(\delta - 1)$, which equals $2(\delta m + 1) + \delta^2(m^2 - m)$. To reach this total, $S$ must induce a complete graph, and all other edges must join $S$ to vertices of degree $\delta$. Thus $v_1$ and $v_2$ are adjacent in every realization of $\pi_1$. The same argument applies to $\pi_2$; again $v_1$ and $v_2$ are adjacent in every realization. Since $v_1$ and $v_2$ are adjacent in all realizations of both lists, $\pi_1$ and $\pi_2$ do not pack.

3
Given a graph \( G \) and a set \( S \subseteq V(G) \), let \( G[S] \) denote the induced subgraph of \( G \) with vertex set \( S \), and let \( N_G(S) \) be the set of vertices having a neighbor in \( S \). A clique is a pairwise adjacent set of vertices.

**Theorem 2.2.** Let \( \pi_1 \) and \( \pi_2 \) be graphic \( n \)-tuples. If

\[
\Delta \leq \sqrt{2\delta n} - (\delta - 1),
\]

where \( \Delta \) and \( \delta \) denote the maximum and minimum values in \( \pi_1 + \pi_2 \), then \( \pi_1 \) and \( \pi_2 \) pack, except that strict inequality is required when \( \delta = 1 \).

**Proof.** Let \( \pi_1 \) and \( \pi_2 \) be graphic \( n \)-tuples. If \( \delta = 0 \), then \( \Delta \leq \sqrt{2\delta n} - (\delta - 1) \) implies that realizations are edgeless or consist of matchings on disjoint vertex sets, so \( \pi_1 \) and \( \pi_2 \) pack. Therefore, we may assume \( \delta \geq 1 \). We prove that if \( \pi_1 \) and \( \pi_2 \) fail to pack, then \( \Delta \geq \sqrt{2\delta n} - (\delta - 1) \), with strict inequality when \( \delta > 1 \).

Among realizations of \( \pi_1 \) and \( \pi_2 \) on vertices \( v_1, \ldots, v_n \) that have the required degrees at each vertex, choose \( G_1 \) and \( G_2 \) to minimize the number of edges that appear in both graphs. Since \( \pi_1 \) and \( \pi_2 \) do not pack, we may consider an edge \( xy \) in \( E(G_1) \cap E(G_2) \).

Let \( G = G_1 \cup G_2 \), and let \( I = V(G) - (N_G(x) \cup N_G(y)) \). With \( \delta \geq 1 \), we have \( \Delta < \sqrt{n} \), so \( I \neq \emptyset \). Let \( Q = N_G(I) \). Suppose that \( G_1 \) or \( G_2 \) has an edge \( uv \) such that \( u \in I \) and \( \{x, y\} \not\subseteq N_G(v) \); by symmetry, we may assume \( yv \notin E(G) \). Replacing \( \{xy, uv\} \) with \( \{xu, yv\} \) in that graph reduces the number of shared edges without changing vertex degrees, contradicting the choice of \( G_1 \) and \( G_2 \) (see Figure 1a).

For \( j \in \{1, 2\} \), let \( Q_j = N_{G_j}(I) \); we claim that \( Q_j \) is a clique in \( G \). Otherwise, choose \( w, w' \in Q_j \) with \( ww' \notin E(G) \). Let \( z \) and \( z' \) be (not necessarily distinct) vertices in \( I \) such that \( zw, z'w' \in E(G_j) \). Since \( ww' \notin E(G_j) \), replacing \( \{z'w', wz, xy\} \) with \( \{w'w, zx, yz'\} \) in \( E(G_j) \) reduces the number of shared edges without changing vertex degrees (see Figure 1b).

![Diagram](image-url)
Since \( Q = Q_1 \cup Q_2 \), and \( Q_1 \) and \( Q_2 \) are cliques in \( G \), the complement of \( G[Q] \) is bipartite. Letting \( r \) be the number of edges in \( G[Q] \), we obtain

\[
r \geq \left( \frac{|Q|}{2} \right) - \frac{|Q|^2}{4} = \frac{|Q|^2}{4} - \frac{|Q|}{2}.
\]

(1)

Next, note that \( |I| = n - |N_G(x) \cup N_G(y)| = n - |N_G(x)| - |N_G(y)| + |N_G(x) \cap N_G(y)| \). Since \( xy \) is a shared edge, \( |N_G(x)| \) and \( |N_G(y)| \) are at most \( \Delta - 1 \). With \( Q \subseteq N_G(x) \cap N_G(y) \),

\[
|I| \geq n - 2\Delta + 2 + |Q|.
\]

(2)

Each vertex \( v \in I \) has at least \( \delta \) incident edges in \( G_1 \) and \( G_2 \) together, and each neighbor is in \( Q \). Since \( Q \subseteq N_G(x) \cap N_G(y) \), at most \( (\Delta - 2)|Q| - 2r \) edges of \( G_1 \) and \( G_2 \) together have endpoints in \( I \) and \( Q \). Therefore,

\[
|I| \leq \frac{(\Delta - 2)|Q| - 2r}{\delta}.
\]

(3)

Together, (2) and (3) yield

\[
(\Delta - 2)|Q| - 2r \geq \delta(n - 2\Delta + 2 + |Q|).
\]

(4)

Using (1) to substitute for \( r \), letting \( q = |Q| \), and simplifying brings us to

\[
q(\Delta - 1 - \delta - q/2) \geq \delta(n - 2\Delta + 2).
\]

(5)

The left side is maximized when \( q = \Delta - 1 - \delta \). Since the inequality must hold there, \( (\Delta - 1 - \delta)^2 \geq 2\delta(n - 2\Delta + 2) \). Adding \( 4\delta(\Delta - 1) \) to both sides yields \( (\Delta - 1 + \delta)^2 \geq 2\delta n \), or

\[
\Delta \geq \sqrt{2\delta n} - (\delta - 1).
\]

(6)

To complete the sufficiency proof, we show that equality cannot hold in (6) when \( \delta \geq 2 \). Equality in (6) requires equality in the inequalities that produced it. Equality holds in (5) only when \( q = \Delta - 1 - \delta \). Equality in (4) (equivalent to (5)) requires equality in (3) and (2). Thus \( \delta|I| \) equals both sides of (4), and also \( Q = N_G(x) \cap N_G(y) \) and \( |N_G(x)| = |N_G(y)| = \Delta - 1 \). By this last equality, \( G_1 \) and \( G_2 \) share no edges incident to \( x \) or \( y \) except \( xy \).

Equality in (3) requires \( N_G(w) = Q \) whenever \( w \in I \). Since exactly \( (\Delta - 2)|Q| - 2r \) edges have endpoints in \( Q \) and \( I \), and by definition \( G[Q] \) has \( r \) edges, the edges joining \( Q \) to \( I \cup \{x, y\} \) and within \( Q \) exhaust the total degree sum available to vertices of \( Q \). We conclude that in \( G \) each vertex of \( Q \) has degree \( \Delta \) and has no neighbor in \( N_G(x) \cup N_G(y) \) outside \( Q \).
Let \( X = N_G(x) - N_G(y) - \{y\} \), and let \( Y = N_G(y) - N_G(x) - \{x\} \) (see Figure 2). Since \(|N_G(x)| = |N_G(y)| = \Delta - 1\) and \(|N_G(x) \cap N_G(y)| = q = \Delta - 1 - \delta\), we have \(|X| = |Y| = \delta - 1\). If \( G_j \) has edges within both \( X \) and \( Y \), say \( uu' \in G_j[X] \) and \( vv' \in G_j[Y] \), then consider whether \( uv \in E(G_j) \). If so, then replacing \( \{xy, uv\} \) with \( \{yu, vx\} \) in \( G_j \) reduces the number of shared edges; if not, then replacing \( \{vv', xy, u'u\} \) with \( \{v'x, yu, uv\} \) does so. Hence by symmetry we may assume that edges of \( G[X] \) lie only in \( G_1 \) and edges of \( G[Y] \) lie only in \( G_2 \). Now vertices of \( X \) are isolated in \( G_2 \) and have at most \( \delta - 1 \) neighbors in \( G_1 \) (including \( x \)). If \( X \) is nonempty, then this contradicts the definition of \( \delta \). Hence equality in (6) requires \( X = \emptyset \) and \( \delta = 1 \).

**Theorem 2.3.** The result of Theorem 2.2 is sharp: for \( \delta, m \in \mathbb{N} \) with \( m \geq \delta \geq 2 \), there exist \( \pi_1 \) and \( \pi_2 \) with \( n = 2\delta m^2 \) such that \( \Delta = \sqrt{2\delta n} - (\delta - 2) \) but \( \pi_1 \) and \( \pi_2 \) do not pack.

**Proof.** We consider only \( \delta \geq 2 \) since the construction in Example 2.1 proves sharpness for \( \delta = 1 \). Choose \( m \in \mathbb{N} \) with \( m \geq \delta \), and let \( n = 2\delta m^2 \). Let \( G \) be the construction using these parameters in Example 2.1. We modify \( G \) to reduce the maximum degree by \( \delta - 1 \). This will also reduce \( \Delta \) by \( \delta - 1 \) in the sum of two specified orderings of the vertex degrees.

Recall that the construction of \( G \) begins with a complete graph \( K_{\delta m+1} \) whose vertex set is composed of sets \( V_1, \ldots, V_{m-1} \) of size \( \delta \) plus \( \delta + 1 \) additional vertices. For each \( i \) the set \( V_i \) is adjacent to a set \( X_i \) of \( \delta m \) new vertices, and there are \( \delta m^2 - 1 \) additional isolated vertices. Each vertex in \( \bigcup_i V_i \) has degree \( 2\delta m \), each extra vertex in the clique has degree \( \delta m \), and \( \delta m(m-1) \) vertices outside the clique have degree \( \delta \).

Modify \( G \) by removing \( \delta - 1 \) of the extra vertices from the clique, reducing the degrees of the other vertices by \( \delta - 1 \). For \( 1 \leq i \leq \delta - 1 \), put one of the removed vertices into \( X_i \). Hence the number of vertices remains \( 2\delta m^2 \), the vertices of \( V_1, \ldots, V_{\delta-1} \) have degree \( 2\delta m - \delta + 2 \), those of \( V_\delta, \ldots, V_{m-1} \) have degree \( 2\delta m - \delta + 1 \), the two unmoved extra vertices have degree \( \delta m - \delta + 1 \), and the remaining vertices have degree \( \delta \). The new graph \( G' \) realizes the \( n \)-tuples.
\( \pi_1' \) and \( \pi_2' \) given by
\[
((\delta m - \delta + 1)^2, (2\delta m - \delta + 2)^{\delta(\delta - 1)}, 0^{\delta(\delta - 1)}, (2\delta m - \delta + 1)^{\delta(m - \delta)}, 0^{\delta(m - \delta)}, \delta^{\delta(m^2 - m) + \delta - 1}, 0^{\delta(m^2 - m) + \delta - 1})
\]
and
\[
((\delta m - \delta + 1)^2, 0^{\delta(\delta - 1)}, (2\delta m - \delta + 2)^{\delta(\delta - 1)}, 0^{\delta(m - \delta)}, (2\delta m - \delta + 1)^{\delta(m - \delta)}, 0^{\delta(m^2 - m) + \delta - 1}, \delta^{\delta(m^2 - m) + \delta - 1}).
\]

By construction, \( \pi_1' \) and \( \pi_2' \) are graphic. To show that they do not pack, we argue as in Example 2.1. In any realization of \( \pi_1' \), let \( S \) be the set of \( \delta m - \delta + 2 \) vertices with degrees exceeding \( \delta \). Their degrees sum to
\[
2\delta m \delta(m - 1) - (\delta - 1)\delta(m - 1) + \delta(\delta - 1) + 2\delta m - 2(\delta - 1),
\]
which equals \( 2\left(\frac{\delta m - \delta + 2}{2}\right) + \delta^2(m^2 - m) + \delta(\delta - 1) \). To achieve this total, again \( S \) must be a clique. As in Example 2.1, \( v_1 \) and \( v_2 \) must be adjacent in all realizations of both graphs; hence \( \pi_1 \) and \( \pi_2 \) do not pack. \( \square \)

If \( a + b < \sqrt{2n} \), then also \( ab < n/2 \). Hence the conjecture below would strengthen Theorem 2.2 when \( \delta = 1 \) and provide a more direct analogue to the Sauer-Spencer Theorem.

**Conjecture 2.4.** Let \( \pi_1 \) and \( \pi_2 \) be graphic \( n \)-tuples, with \( \delta \) the least entry in \( \pi_1 + \pi_2 \). If \( \delta \geq 1 \) and the product of corresponding entries in \( \pi_1 \) and \( \pi_2 \) is always less than \( n/2 \), then \( \pi_1 \) and \( \pi_2 \) pack.

For fixed \( \delta \), a suitable bound on the product of corresponding entries to guarantee packing may be something like \( \delta n/2 - O(\delta \sqrt{\delta n}) \).

### 3 Extensions of Kundu’s Theorem

Let \( D_k(\pi) \) denote the \( n \)-tuple obtained from an \( n \)-tuple \( \pi \) by subtracting \( k \) from each entry. The “regular” case of Kundu’s Theorem states that if \( \pi \) and \( D_k(\pi) \) are graphic, then some realization of \( \pi \) has a \( k \)-factor. To extend the theorem, one could try to guarantee that some realization of \( \pi \) has edge-disjoint regular factors of degrees \( k_1, \ldots, k_t \), where \( \sum_{i=1}^t k_i = k \).

When \( n \) is odd, no regular \( n \)-vertex graph has odd degree, so existence requires all \( k_1, \ldots, k_t \) even. In that case, existence then follows immediately from Kundu’s Theorem and Petersen’s 2-Factor Theorem [12]; the latter states that every 2r-regular graph decomposes into 2-factors. It remains to consider even \( n \).
Conjecture 3.1. Let \( n \) be an even integer. If \( \pi \) is a graphic \( n \)-tuple such that \( D_k(\pi) \) is also graphic, and \( k_1, \ldots, k_t \) are positive integers with sum \( k \), then some realization of \( \pi \) has edge-disjoint regular factors with degrees \( k_1, \ldots, k_t \).

Conjecture 3.1 is immediately equivalent to the following conjecture.

Conjecture 3.2. Let \( n \) be an even integer. If \( \pi \) is a graphic \( n \)-tuple such that \( D_k(\pi) \) is also graphic, then some realization of \( \pi \) has \( k \) edge-disjoint 1-factors.

Our main result (Theorem 3.9) toward Conjecture 3.2 combines with Petersen’s Theorem to yield Conjecture 3.1 when \( k \) is even and at most two of \( k_1, \ldots, k_t \) are odd, and when \( k \) is odd and at most one of \( k_1, \ldots, k_t \) is odd.

We have proved several special cases of Conjecture 3.2. The first uses a lemma proved by A.R. Rao and S.B. Rao [13] in their study of what was called the “\( k \)-Factor Conjecture” before it became Kundu’s Theorem.

Lemma 3.3. Fix \( k \in \mathbb{N} \), and let \( \pi \) be a graphic \( n \)-tuple such that \( D_k(\pi) \) is also graphic. If \( r \) is a positive integer such that \( r \leq k \) and \( rn \) is even, then \( D_r(\pi) \) is also graphic.

Let \( \Delta(G) \) and \( \delta(G) \) denote the largest and smallest vertex degrees in a graph \( G \).

Theorem 3.4. Fix \( k, n \in \mathbb{N} \) with \( n \) even, and let \( \pi \) be a graphic \( n \)-tuple such that \( D_k(\pi) \) is also graphic. If every entry in \( \pi \) is at most \( n/2 + 1 \), then some realization of \( \pi \) has \( k \) edge-disjoint 1-factors.

Proof. The proof is by induction on \( k \). For \( k = 0 \), the statement is vacuous, and the case \( k = 1 \) is a special case of Kundu’s Theorem. Suppose then that \( k \geq 2 \) and that \( D_k(\pi) \) is graphic. By Lemma 3.3, \( D_2(\pi) \) is graphic, and since \( D_k(\pi) \) is graphic the induction hypothesis implies that there is a realization \( G \) of \( D_2(\pi) \) having \( k - 2 \) disjoint 1-factors.

The hypothesis on \( \pi \) yields \( \Delta(G) \leq n/2 - 1 \), so \( \delta(G) \geq n/2 \). Dirac’s Theorem [3] now implies that \( G \) has a spanning cycle \( C \). Since \( n \) is even, \( C \) decomposes into two edge-disjoint 1-factors. Therefore, \( G \cup C \) is a realization of \( \pi \) having \( k \) edge-disjoint 1-factors.

We also obtain Conjecture 3.2 in those cases where every entry in \( \pi \) is large, by applying Theorem 3.4 to the \( n \)-tuple obtained by subtracting every entry of \( D_k(\pi) \) from \( n - 1 \).

Corollary 3.5. Fix \( k, n \in \mathbb{N} \) with \( n \) even, and let \( \pi \) be a graphic \( n \)-tuple such that \( D_k(\pi) \) is also graphic. If every entry in \( \pi \) is at least \( n/2 + k - 2 \), then some realization of \( \pi \) has \( k \) edge-disjoint 1-factors.
Our main result in this section is that, under the conditions of Conjecture 3.2, there is a realization of $\pi$ having edge-disjoint factors $M_1, M_2, F$ that are regular of degrees 1, 1, and $k - 2$. This implies Conjecture 3.2 for $k \leq 3$; for Conjecture 3.1, it allows one or two of $k_1, \ldots, k_t$ to be odd when $k$ is odd or even, respectively.

We use a well-known description of the maximum matchings in a graph. Say that a matching $M$ avoids a vertex $x$ if $M$ has no edge incident to $x$. The *Gallai–Edmonds decomposition* of a graph $G$ is a partition of $V(G)$ into three sets defined as follows (the presentation by Lovász and Plummer [11] uses $(D, A, C)$ instead of our $(A, B, C)$):

\[
A = \{x \in V(G) : \text{some maximum matching avoids } x\},
\]
\[
B = \{x \in V(G) - A : x \text{ has a neighbor in } A\},
\]
\[
C = V(G) - (A \cup B).
\]

A *near-perfect* matching in $G$ is a matching that avoids exactly one vertex. A graph is *factor-critical* if each vertex is avoided by some near-perfect matching. The *deficiency* $\text{def}(G)$ of a graph $G$ is defined to be $\max_{X \subseteq V(G)} (o(G - X) - |X|)$, where $o(H)$ is the number of odd components (odd number of vertices) in $H$. It is immediate that every matching in $G$ avoids at least $\text{def}(G)$ vertices, and the Berge–Tutte Formula [1] states that equality holds for a maximum matching.

The Gallai–Edmonds Structure Theorem [5, 7, 8] describes the maximum matchings in a graph in terms of its Gallai–Edmonds Decomposition. We state only the parts we need.

**Theorem 3.6.** If $(A, B, C)$ is the Gallai–Edmonds Decomposition of a graph $G$, then (a) the components of $G[A]$ are factor-critical, and (b) every maximum matching in $G$ consists of a near-perfect matching in each component of $G[A]$, a perfect matching in $G[C]$, and a matching of $B$ into vertices in distinct components of $G[A]$.

Consider the decomposition $(A, B, C)$ of a graph $G$ having an even number of vertices but no 1-factor. Say that a component of $G[A]$ is *missed* by a matching $M$ if it has no vertex matched with a vertex of $B$ in $M$. By Theorem 3.6, a maximum matching in $G$ misses at least two components of $G[A]$. Our structural lemma, which may be of independent interest, is that when $G$ is regular we can ensure that two such components will be nontrivial, where a graph is *nontrivial* if it has at least one edge.

**Lemma 3.7.** Let $(A, B, C)$ be the Gallai–Edmonds decomposition of a regular graph $F$ with an even number of vertices. If $F$ does not have a 1-factor, then some maximum matching in $F$ misses two nontrivial components of $F[A]$. 

\[9\]
Proof. When $F$ has no 1-factor, the set $A$ is nonempty. Let $S$ be the set of isolated vertices in $F[A]$. By Theorem 3.6, every maximum matching in $F$ pairs $B$ with vertices of distinct components of $F[A]$. Since the number of vertices is even, every maximum matching misses at least two components of $F[A]$.

Among the maximum matchings in $F$, choose $M$ to miss the most nontrivial components of $F[A]$. If $M$ does not miss two such components, then $|S| \geq 1$ and $M$ misses at least one vertex $a$ of $S$. Let $B'_S = B - B_S$.

Let $R$ be the set of vertices reachable from $a$ by $M$-alternating paths in $F$. Since $M$ matches $B$ into $A$, such paths move from $A$ to $B$ by edges not in $M$ and return to $A$ via $M$. If $a, \ldots, b, a'$ are the vertices of such a path with $b \in B'_S$, then exchanging membership between $M$ and $E(F) - M$ along the path produces a new matching $M'$ that misses one more nontrivial component of $F[A]$ than $M$. The choice of $M$ thus implies $R \subseteq S \cup B_S$.

As we explore $M$-augmenting paths from $a$, reaching a vertex in $B_S$ also immediately adds a new vertex of $S$. Thus $|R \cap S| = |R \cap B_S| + 1$. This contradicts $k$-regularity, since $N(R \cap S) \subseteq R \cap B_S$. We conclude that a maximum matching missing the most nontrivial components must miss at least two.

Our second lemma concerns an auxiliary graph used in the proof of the theorem.

Lemma 3.8. Let $l$ and $m$ be positive odd integers. Let $H$ be the graph with vertices $v_{i,j}$ for $i \in \mathbb{Z}_l$ and $j \in \mathbb{Z}_m$ such that each $v_{i,j}$ is adjacent to the four vertices of the form $v_{i \pm 1,j \pm 1}$. Let $S$ be an independent set in $H$. If the first coordinates of the vertices in $S$ are distinct, and the second coordinates of the vertices in $S$ are distinct, then $H - S$ contains an odd cycle.

Proof. When we arrange the vertices in the natural $l$-by-$m$ grid, the condition on $S$ implies each row and column has at most one vertex of $S$. It suffices to find an odd closed walk avoiding $S$. The vertices $v_{1,1}, \ldots, v_{lm,lm}$ form an odd closed walk; it suffices unless $v_{r,r} \in S$ for some $r$. Since $S$ is independent, $v_{r-1,r+1} \notin S$. Also, $v_{r-2,r}, v_{r,r+2} \notin S$. Replacing $v_{r,r}$ with $v_{r-2,r}, v_{r-1,r+1}, v_{r,r+2}$ increases the length of the walk by 2 but decreases the number of vertices of $S$ on it by 1. Doing this independently for each vertex of $S$ on it yields an odd closed walk avoiding $S$.

We can now prove the main result of this section.

Theorem 3.9. Fix $n, k \in \mathbb{N}$ with $n$ even and $k \geq 2$. If $\pi$ is a graphic $n$-tuple such that $D_k(\pi)$ is also graphic, then some realization of $\pi$ has a $k$-factor with two edge-disjoint 1-factors.
Proof. Since $D_k(\pi)$ is graphic, Kundu’s Theorem provides a realization of $\pi$ with a $k$-factor. Among all such realizations, choose a realization $G$ and $k$-factor $F$ in it to lexicographically maximize $(r, s)$, where $r$ is the maximum number of edge-disjoint 1-factors in $F$ and $s$ is the maximum size of a maximum matching in the graph $\hat{F}$ left by deleting $r$ 1-factors from $F$.

If $r \geq 2$, then the claim holds. Otherwise, $r \leq 1$ and $0 < s < n/2$. If $r = 1$, let $\hat{M}$ be the specified matching; if $r = 0$, then $\hat{M} = \emptyset$. View $\overline{G}$, $\hat{M}$, $\hat{F}$, and $G - E(F)$ as a decomposition of $K_n$ into edge-disjoint subgraphs.

Let $(A, B, C)$ be the Gallai–Edmonds decomposition of $\hat{F}$. By Lemma 3.7, $\hat{F}$ has a maximum matching $M$ that misses two nontrivial components of $\hat{F}[A]$; call them $Q$ and $Q'$. Since $Q$ and $Q'$ are components of $\hat{F}[A]$, each edge of $K_n$ joining them is not in $\hat{F}$.

If edges $xy$ in $Q$ and $x'y'$ in $Q'$ exist such that $xx'$ and $yy'$ lie in the same graph among $\{\overline{G}, \hat{M}, G - E(F)\}$, then switching $\{xy, x'y'\}$ into it and $\{xx', yy'\}$ into $\hat{F}$ yields a realization $G'$ of $\pi$ with a $k$-factor $F'$ (having a 1-factor if $r = 1$). Since $Q$ and $Q'$ are factor-critical (by Theorem 3.6), $Q - x$ and $Q' - x'$ have 1-factors. Since $M$ misses $Q$ and $Q'$, replacing the edges of $M$ in $Q$ and $Q'$ with $xx'$ and 1-factors of $Q - x$ and $Q' - x'$ yields a matching $M'$ in $F'$ that is larger than $M$. By the choice of $G$ and $F$, no such $xx'$ and $yy'$ exist.

Being factor-critical and nontrivial, $Q$ and $Q'$ are nonbipartite; hence each contains an odd cycle. Let $\{u_1, \ldots, u_l\}$ and $\{w_1, \ldots, w_m\}$ be the vertices along odd cycles chosen in $Q$ and $Q'$, respectively. Form the auxiliary graph $H$ of Lemma 3.8, with vertices $v_{i,j}$ for $i \in \mathbb{Z}_l$ and $j \in \mathbb{Z}_m$. Let $S$ be the subset of $V(H)$ corresponding to edges of the form $u_iw_j$ that belong to $\hat{M}$. If $r = 0$, then $S$ is empty; if $r = 1$, then $S$ has at most one vertex in each row and column, because $\hat{M}$ is a matching.

The vertices of $H - S$ correspond to other edges $u_iw_j$ in $K_n$, each belonging to $\overline{G}$ or to $G - E(F)$. By Lemma 3.8, $H - S$ contains an odd cycle, and hence two adjacent vertices in $H - S$ correspond to edges from the same subgraph. These edges have the form $xx'$ and $yy'$ previously forbidden. We conclude that $r \geq 2$, as desired.

\[ \square \]

Corollary 3.10. Conjecture 3.2 is true for $k \leq 3$.

We believe that the conclusion of Lemma 3.8 remains true when two such independent sets $S$ and $S'$ are deleted. This would improve Theorem 3.9 to produce a realization having a $k$-factor with three edge-disjoint 1-factors, yielding Conjecture 3.2 for $k \leq 4$ and Conjecture 3.1 with one more odd value in $k_1, \ldots, k_t$ than allowed by Theorem 3.9. The method cannot extend beyond that, because when $l = m = 3$ there may be three independent sets of size 1 in $H$ that together occupy one column, and then what remains is bipartite.
References


