DISJOINT HAMILTONIAN CYCLES IN BIPARTITE GRAPHS

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Abstract. Let $G = (X, Y)$ be a bipartite graph and define $\sigma^2_2(G) = \min \{d(x) + d(y) : xy \notin E(G), \ x \in X, \ y \in Y\}$. Moon and Moser [5] showed that if $G$ is a bipartite graph on $2n$ vertices such that $\sigma^2_2(G) \geq n + 1$ then $G$ is hamiltonian, sharpening a classical result of Ore [6] for bipartite graphs. Here we prove that if $G$ is a bipartite graph on $2n$ vertices such that $\sigma^2_2(G) \geq n + 2k - 1$ then $G$ contains $k$ edge-disjoint hamiltonian cycles. This extends the result of Moon and Moser and a result of R. Faudree, et al. [3]

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1. Introduction and Terminology

For any graph $G$, let $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ denote the sets of vertices and edges of $G$ respectively. An edge between two vertices $x$ and $y$ in $V(G)$ shall be denoted $xy$. Furthermore, let $\delta(G)$ denote the minimum degree of a vertex in $G$. For a given subgraph $H$ of $G$ and vertices $x$ and $y$ in $H$, we will let $\text{dist}_H(x,y)$ denote the distance from $x$ to $y$ in $H$. Also, for convenience, given a path $P$ in $G$ and $u,v$ in $V(P)$, let $uPv$ denote the subpath of $P$ that starts at the vertex $u$ and ends at the vertex $v$. Given two disjoint sets of vertices $X$ and $Y$ in $V(G)$, we let $E_G(X,Y)$ denote the set of edges in $G$ with one endpoint in $X$ and one endpoint in $Y$. Similarly, we will let $\delta(X,Y)$ denote the minimum degree between vertices of $X$ and $Y$. A useful reference for any undefined terms is [1].

We assume that all cycles have an implicit clockwise orientation and, for convenience, given a vertex $v$ on a cycle $C$ we will let $v^+$ denote the successor of $v$ along $C$. Along the same lines, given an $x-y$ path $P$, we will let $v^+$ denote the successor of a vertex $v$ in $V(P)$ as we traverse $P$ from $x$ to $y$. Analogously, we define $v^-$ to be the predecessor of a vertex $v$ on $C$ or $P$. Given a set of vertices $S \subseteq C \subseteq P$ we let $S^+$ denote the set $\{s^+ | s \in S\}$. The set $S^-$ is defined analogously.

If $G$ is bipartite with bipartition $(X,Y)$ we will write $G = (X,Y)$. If $|X| = |Y|$ then we will say that $G$ is balanced. A proper pair in $G$ is a pair of non-adjacent vertices $(x,y)$ with $x$ in $X$ and $y$ in $Y$.

We shall denote a cycle on $t$ vertices by $C_t$. A hamiltonian cycle in a graph $G$ is a cycle that spans $V(G)$ and, if such a cycle exists, $G$ is said to be hamiltonian. Hamiltonian graphs and their

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properties have been widely studied. A good reference for recent developments and open problems is [4].

In general, we are interested in degree conditions that assure hamiltonian cycles in a graph. For an arbitrary graph $G$, define $\sigma_2(G)$ to be the minimum degree sum of non-adjacent vertices in $G$. Of interest for our work here is Ore’s Theorem [6], which uses this parameter.

**Theorem 1.1** (Ore 1960). If $G$ is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then $G$ is hamiltonian.

In a bipartite graph $G$, we are interested instead in the parameter $\sigma_2^2(G)$, defined to be the minimum degree sum of a proper pair. Moon and Moser [5] extended Ore’s theorem to bipartite graphs as follows.

**Theorem 1.2** (Moon, Moser 1960). If $G = (X, Y)$ is a balanced bipartite graph on $2n$ vertices such that $\sigma_2^2(G) \geq n + 1$, then $G$ is hamiltonian.

Faudree, Rousseau and Schelp [3] were able to give and Ore-type degree-sum conditions that assured the existence of many disjoint hamiltonian cycles in an arbitrary graph.

**Theorem 1.3** (Faudree, Rousseau, Schelp 1984). If $G$ is a graph on $n$ vertices such that $\sigma_2^2(G) \geq n + 2k - 2$ then for $n$ sufficiently large, $G$ contains $k$ edge-disjoint hamiltonian cycles.

In this paper we will extend the previous two results by proving the following.

**Theorem 1.4.** If $G = (X, Y)$ is a balanced bipartite graph of order $2n$, with $n \geq 128k^2$ such that $\sigma_2^2(G) \geq n + 2k - 1$, then $G$ contains $k$ edge-disjoint hamiltonian cycles.

### 2. Veneering Numbers and $k$-Extendibility

To prove our main theorem, we need some results on path-systems in bipartite graphs. Our strategy is to develop $k$ systems of edge-disjoint paths and show that they can be extended to $k$ edge-disjoint hamiltonian cycles. The following definitions and theorems can be found in [7].

Let $W_k(G)$ be the family of all $k$-sets $\{(w_1, z_1), \ldots, (w_k, z_k)\}$ of pairs of vertices of $G$ where $w_1, \ldots, w_k, z_1, \ldots, z_k$ are all distinct. Let $S_k(G)$ denote the collection of edge-disjoint path systems in $G$ that have exactly $k$ paths. If $W \in W_k(G)$ lists the end-points of a path system $P$ in $S_k(G)$, we say that $P$ is a $W$-linkage. A graph $G$ is said to be $k$-linked if there is a $W$-linkage for every $W \in W_k(G)$. A graph $G$ is said to be $k$-extendible if any $W$-linkage of maximal order is spanning.

In order to tailor the idea of extendible path systems to bipartite graphs, the notion of a veneering path system was introduced in [7].

**Definition 1.** A path system $P$ veneers a bipartite graph $G$ if it covers all the vertices of one of the partite sets.

Let $G = (X, Y)$ be a bipartite graph. Given a $W \in W_k(G)$, we denote by $W^X$ those pairs of $W$ that are in $X^2$, by $W^Y$ those that are in $Y^2$, and by $W^1$ the set of bipartite pairs of $W$. Also, with a slight abuse of notation, we will let $W_X$ (resp. $W_Y$) be the set of vertices of $X$ (resp. $Y$) that are used in the pairs of $W$. 
Definition 2. Let $G$ be a bipartite graph and $W \in W_k(G)$. The veneering number $\vartheta^X_Y(W)$ of $W$ is defined to be
\[
\vartheta^X_Y(W) = (|X| - |Y|) - (|W^X| - |W^Y|),
\]
\[
= (|X| - |Y|) - \frac{|W^X| - |W^Y|}{2}.
\]

Note that one consequence of the definition is that $\vartheta^X_Y = -\vartheta^X_Y$. For a given path system $\mathcal{P}$, let $\partial(\mathcal{P})$ denote the set of pairs of endpoints of paths in $\mathcal{P}$ and let $\mathcal{P}$ denote $\mathcal{P} - \partial(\mathcal{P})$. We define the veneering number of such a $\mathcal{P}$ to be the veneering number of $\partial \mathcal{P}$. The veneering number of a given set of endpoints is of interest, because it represents the minimum possible number of vertices left uncovered by a path system with those endpoints.

As an example, consider $G = K_{6,7}$ and let $X$ denote the partite set of order six. Furthermore, let $(x_1, x_2)$ be a pair of distinct vertices in $X$. Clearly, any $x_1 - x_2$ path in $G$ has order at most eleven and omits at least two vertices from $Y$. Let $W = \{(x_1, x_2)\}$. Then
\[
\vartheta^X_Y(W) = (|X| - |Y|) - (|W^X| - |W^Y|)
\]
\[
= (6 - 7) - (1 - 0) = -2.
\]
This indicates that the minimum possible number of uncovered vertices in a path system with endpoints in $W$ is two. The fact that $\vartheta^X_Y(W) < 0$ indicates that those vertices would be in $Y$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two path-systems of $G$, we write $\mathcal{P}_1 \leq \mathcal{P}_2$ when every path of $\mathcal{P}_1$ is contained in a path of $\mathcal{P}_2$. The following fact will prove most useful.

Proposition 2.1. Let $G = (X \cup Y, E)$ be a bipartite graph and $\mathcal{P}_1, \mathcal{P}_2 \in S(G)$ be such that $\mathcal{P}_1 \leq \mathcal{P}_2$. Let
\[
G_1 = (X_1 \cup Y_1, E_1) = G - \mathcal{P}_1,
\]
\[
G_2 = (X_2 \cup Y_2, E_2) = G - \mathcal{P}_2,
\]
then
\[
\vartheta^X_Y(\mathcal{P}_1) = \vartheta^X_Y(\mathcal{P}_2).
\]

Proof. (sketch) Suppose that $\mathcal{P}_1$ consists entirely of the paths $\mathcal{R}_1$ and $\mathcal{R}_2$ and that $\mathcal{P}_2$ consists entirely of the path $\mathcal{S}$ with $\mathcal{R} \subset \mathcal{S}$. We consider the case where $\mathcal{R}_1$ has endpoints $y_1$ and $y_2$ in $Y$, $\mathcal{R}_2$ has endpoints $y_1'$ and $y_2'$ in $Y$ and that $\mathcal{S}$ has endpoints $x$ in $X$ and $y$ in $Y$. All of the other cases, both when the systems contain multiple paths or have different endpoints follow by a nearly identical analysis.

By definition,
\[
\vartheta^X_Y(\mathcal{P}_1) = (|X_1| - |Y_1|) - (|W^{X_1}| - |W^{Y_1}|).
\]
Since both paths in $\mathcal{R}$ have their endpoints in $Y$, $W^{X_1} = 0$ and $W^{Y_1} = 2$. Additionally, since each path in $\mathcal{R}$ has one more internal vertex in $X$ than in $Y$, we observe that $(|X_1| - |Y_1|) = -2$ and hence we conclude that $\vartheta^X_Y(\mathcal{R}) = |X| - |Y|$. Similarly,
\[
\vartheta^X_Y(\mathcal{S}) = (|X_2| - |Y_2|) - (|W^{X_2}| - |W^{Y_2}|).
\]
Since $\mathcal{S}$ has an endpoint in each partite set, it follows that $|W^{X_2}| = |W^{Y_2}| = 0$ and for some integer $\ell$, $|X_2| = |X| - \ell$ and $|Y_2| = |Y| - \ell$. Consequently, $\vartheta^X_Y(\mathcal{S}) = |X| - |Y|$. 
We are now ready to give our definition of a \( k \)-extendible bipartite graph.

**Definition 3.** Let \( G \) be a bipartite graph. Then \( G \) is said to be \( k \)-extendible if for any path-system \( \mathcal{P} \) in \( S_k(G) \) there exists some veneering path system \( \mathcal{P}' \) in \( S_k(G) \) that preserves the endpoints of \( \mathcal{P} \).

We will utilize the following in the proof of our main theorem.

**Theorem 2.2.** [7] If \( k \geq 2 \) and \( G = (X, Y) \) is a bipartite graph of order \( n \) such that \( |X|, |Y| > 3k \) and \( \sigma_2^2(G) \geq \lceil \frac{n+3k}{2} \rceil \), then \( G \) is \( k \)-extendible.

It is important to note that a maximal path system with veneering number zero is spanning. Thus, if a graph \( G \) that meets the \( \sigma_2^2 \) bound for \( k \)-extendibility has some path system \( \mathcal{P} \) in \( S_k(G) \) such that \( \vartheta(\mathcal{P}) = 0 \), then \( G \) must have a spanning path system.

We give two more results from [7] that will be very useful. The first is relatively straightforward to prove, and the second is a weaker version of a result in [5].

**Theorem 2.3.** If \( G = (X \cup Y, E) \) is a balanced bipartite graph of order \( 2n \) with \( \sigma_2(G) \geq n+2k-2 \) then for any set \( W \) in \( W_k(G) \) comprised entirely of proper pairs of \( G \), there exists a system of \( k \) edge-disjoint paths whose endpoints are exactly the pairs in \( W \).

**Theorem 2.4.** If \( G = (X \cup Y, E) \) is a balanced bipartite graph of order \( 2n \) such that for any \( x \in X \) and any \( y \in Y \), \( d(x) + d(y) \geq n+2 \), then for any pair \( (x, y) \) of vertices of \( G \), there exists a Hamilton path between \( x \) and \( y \). The degree sum condition is the best possible.

3. **Proof of Theorem 1.4**

Suppose the theorem is not true, and let \( G \) be a counterexample of order \( 2n \) with a maximum number of edges. The maximality of \( G \) implies that for any proper pair \( (x, y) \), \( G + xy \) contains \( k \) edge-disjoint Hamilton cycles, one of these containing the edge \( xy \). Thus, with any proper pair \( (x, y) \) we will associate \( k - 1 \) edge-disjoint Hamilton cycles \( H_1, \ldots, H_{k-1} \) and an \( (x, y) \)-Hamilton path \( P = (x = z_1, z_2, \cdots, z_{2n} = y) \).

Let \( H \) denote the union of subgraphs \( H_1, \ldots, H_{k-1} \), and \( L = L(x, y) \) denote the subgraph obtained from \( G \) by removing the edges of \( H \). Before we go on proving our theorem we will state a few facts about \( L \). Throughout these proofs, we must keep in mind that

\[
n \geq 128k^2,
\]

and for any vertex \( w \) of \( G \), we have

\[
d_L(w) = d_G(w) - 2(k-1).
\]

Thus, the degree sum condition on any proper pair \( (x, y) \) of \( G \) is

\[
d_G(x) + d_G(y) \geq n + 2k - 1.
\]

This yields the following:

**Fact 1.** For any proper pair \( (x, y) \) of \( G \), we have

\[
d_L(x) + d_L(y) \geq n - 2k + 3.
\]
Fact 2. If there is a proper pair \((x, y)\) of \(G\), with
\[
d_G(x) + d_G(y) \geq n + 4k - 3,
\]
or equivalently
\[
d_L(x) + d_L(y) \geq n + 1,
\]
then \(L\) contains a Hamilton cycle.

Proof. If there were a proper pair \((x, y)\) of \(G\) such that \(d_G(x) + d_G(y) \geq n + 4k - 3\), then by (2), \(d_L(x) + d_L(y) \geq n + 1\), hence if we consider the \((x, y)\)-path \(P\) in \(L\), we see that there must be a vertex \(z \in V(P)\) such that \(z\) is in \(N(y)\) and \(z^+\). Then \(x z^+ \cup [z^+, y] \cup y z \cup [x, z]\) is a Hamilton cycle in \(L\).

\[\square\]

Note that the existence of \(P\) shows that \(L\) is connected. In fact, \(L\) must be 2-connected.

Lemma 3.1. If \(L\) has a cut-vertex, then there are \(k\) edge-disjoint Hamiltonian cycles in \(G\).

Proof. Suppose \(w\) is a cut-vertex of \(L\); we assume, without loss of generality, that \(w \in X\). Since \(L\) admits a Hamiltonian path, \(L - w\) can only have two components, one of them being balanced.

Let \(B\) be the subgraph of \(G\) induced by the balanced component of \(L - w\) and \(A = G - B\). Note that \(w \in A\), and \(E_L(A_X - w, B) = E_L(A_Y, B) = \emptyset\). Let \(a = |A_X| = |A_Y|\) and \(b = |B_X| = |B_Y|\).

Claim 1. \(a, b > \frac{n}{2k}\).

Proof: Assume \(a \leq \frac{n}{2k}\). Then \(a(2k - 2) + a < 2ak \leq n\), implying \(a(2k - 2) < n - a = b\), so \(|E_H(A_Y, B_X)| < |B_X| = b\). Thus there is a vertex \(u \in B_X\) such that \(E_H(u, A_Y) = \emptyset\), so \(E_G(u, A_Y) = \emptyset\). Take any \(v \in A_Y\). We have \(uv \notin E(G)\), so
\[
d(u) + d(v) \leq |A_X| + d_H(v, B_X) + |B_Y| + d_H(u, A_Y)
\]
\[
\leq a + 2(k - 1) + b
\]
\[
< n + 2k - 1,
\]
which contradicts the condition of our theorem. \(\square_{\text{Claim 1}}\)

The following two claims give lower bounds on the degrees of the vertices in \(L\).

Claim 2. For any \(z \in A - w\), \(d_L(z) \geq \frac{|A|}{2k} - 2k + 3\) and for any \(z \in B\), \(d_L(z) \geq \frac{|B|}{2k} - 2k + 3\).

Proof: Assume \(z \in B_Y\) (the cases \(z \in B_X, z \in A_X, z \in A_Y\) are similar). By Claim 1 and the fact that \(n \geq 128k^2\), we have \(|A_X - w| = a - 1 > \frac{n}{2k} - 1 > 2(k - 1)\), so there is a \(z' \in A_X - w\) such that \(zz' \notin E(H)\), thus \(zz' \notin E(G)\), so that \(d_L(z) + d_L(z') \geq n - 2k + 3\). Then since \(d_L(z') \leq |A_Y| = a\), we get \(d_L(z) \geq n - 2k + 3 - a = b - 2k + 3. \square_{\text{Claim 2}}\)

Claim 3. \(d_L(w) \geq \frac{n}{2k} - 2k + 3\).

Proof: If \(w\) is adjacent, in \(G\), to all the vertices of \(A_Y\), then the Claim is obviously true. If not, there is a \(v \in A_Y\) with \(uv \notin E(G)\), so that \(d_L(w) + d_L(v) \geq n - 2k + 3\). Since \(d_L(v) \leq a = n - b < n - \frac{n}{2k}\), we get
\[
d_L(w) \geq n - 2k + 3 - d_L(v)
\]
\[
> n - 2k + 3 - (n - \frac{n}{2k})
\]
\[
= n \frac{n}{2k} - 2k + 3.
\]
Finally:

**Claim 4.** \(|E_G(A_X, B_Y)|, |E_G(A_Y, B_X)| \geq 2k - 1.\)

**Proof:** If \(G[(A_X, B_Y)]\) is complete, the result is obvious. If not, there is a pair of non-adjacent vertices \(u \in A_X\) and \(v \in B_Y\), so \(d(u) + d(v) \geq n + 2k - 1\). Yet \(d(u, A_Y) \leq a\) and \(d(v, B_X) \leq b\), so
\[
d(u, B_Y) + d(v, A_X) \geq n + 2k - 1 - a - b = 2k - 1.
\]

The proof is identical for \((A_Y, B_X)\). \(\square\)

By Claim 2, Claim 3, and the fact that \(n \geq 128k^2\) we have, for any pair of vertices \((u, v) \in A_X \times A_Y\)
\[
d_A(u) + d_A(v) \geq |A| - 2k + 3 + \frac{n}{2k} - 2k + 3 > |A| + 2k = 2a + 2k > a + 66k.
\]

Thus, \(A\) and by a similar computation \(B\), satisfies the conditions of Theorem 2.4. Hence take \(k\) pairs \((e_i, e'_i)\) of edges such that the \(e_i\) are distinct edges of \(E_G(A_X, B_Y)\) and the \(e'_i\) are distinct edges of \(E_G(A_Y, B_X)\). These edges exist by Claim 4.

Let \(u_i \in A_X\) and \(v_i \in B_Y\) be the end vertices of \(e_i\), and \(u'_i \in A_Y\) and \(v'_i \in B_X\) be the end vertices of \(e'_i\). Since pairs of vertices from \(A\) and \(B\) satisfy the conditions of Theorem 2.4 and removing a hamiltonian path reduces the degree sum of any pair of vertices by at most 4, there are \(k\) edge-disjoint hamiltonian paths \(U_1, \ldots, U_k\) in \(A\) such that \(u_i\) and \(u'_i\) are the end-vertices of \(U_i\), and there are \(k\) edge-disjoint hamiltonian paths \(V_1, \ldots, V_k\) in \(B\) such that \(v_i\) and \(v'_i\) are the end-vertices of \(V_i\). Together with the \(e_i\) and \(e'_i\) edges we get \(k\) edge-disjoint hamiltonian cycles in \(G\), which contradicts the assumption that no such collection of cycles exists in \(G\). Hence the lemma is proven. \(\square\)

Now we show that the 2-connectedness of \(L\) assures that \(L\) contains a relatively large cycle.

**Lemma 3.2.** If \(L\) is 2-connected, then it contains a cycle of order at least \(2n - 4k + 4\).

**Proof.** Recall that the maximality of \(G\) implies that \(L\) is traceable, so let \(P = x_1, y_1, \ldots, x_n, y_n\) be a hamiltonian path in \(L\). The path \(P\) induces a natural ordering of the vertices in \(G\), specifically, \(z < z'\) if we encounter \(z\) before \(z'\) while traversing \(P\) from \(x_1\) to \(y_n\). For convenience, we will say that a vertex \(w\) is the *minimum* (respectively *maximum*) vertex with respect to a given property if \(w < w'\) (resp. \(w' < w\)) for each other \(w'\) in \(V(L)\) satisfying this property.

Since, by assumption, \(L\) is 2-connected, each of \(x_1\) and \(y_n\) have at least two adjacencies on \(P\). Let \(x^*\) be the minimum vertex of \(N(y_n)\) and let \(y^*\) be the maximum vertex of \(N(x_1)\). We consider two cases.

**Case 1:** Suppose \(x^* \prec y^*\). Amongst all \(x_i \in N(y_n)\) and \(y_j \in N(x_1)\) such that \(x_i \prec y_j\), pick the pair, call them \(x\) and \(y\) such that \(dist_P(x, y)\) is minimum. By this choice of \(x\) and \(y\) note that there are no neighbors of \(x_1\) or \(y_n\) between \(x\) and \(y\) on \(P\). Note that the subpath \(P'\) of \(P\) that goes from
Case 2: Suppose \( y^* \prec x^* \). Since \( L \) is 2-connected, there exists a sequence of \( \ell \geq 1 \) adjacent pairs of vertices \( (u_i, v_i) \) with the following properties. First, \( u_1 \prec y^* \), \( y^* \prec v_1 \) and \( x^* \prec v_\ell \). Then, for each \( 1 \leq i \leq \ell - 1 \), \( u_i \prec u_{i+1}, v_i \prec v_{i+1} \) and \( u_{i+1} \prec v_i \). We will also choose these vertices so that \( v_{\ell-1} \prec x^* \) and \( y^* \prec u_2 \), as this will simplify things going forward.

Next, choose \( y \) in \( N(x_i) \) such that \( u_1 \prec y \) and \( dist_P(u_i, y) \) is minimum. Similarly, select \( x \) in \( N(y_n) \) such that \( x \prec v_\ell \) and \( dist_P(x, v_\ell) \) is minimum. Now we consider the cycle

\[ C' = x_1 y P u_2 y_4 u_4 \ldots u_\ell v_\ell P y_n x P v_{\ell-1} u_{\ell-1} P v_{\ell-3} \ldots v_1 u_1 P x_1. \]

In \( C' \), we omit several vertices from \( P \). Specifically, we omit \( u_1^+ P y^-, x^+ P v_\ell^- \) and segments of the form \( u_i^+ P v_{i-1}^- \) for \( 2 \leq i \leq \ell \). Note that by our choice of \( x \) and \( y \), neither \( x_1 \) nor \( y_n \) have any adjacencies in these subpaths of \( P \). Counting as above, if these subpaths contain \( 4k - 3 \) or more vertices, we violate Fact 1, while if these subpaths total \( 4k - 4 \) or fewer vertices, \( C' \) will be the desired cycle. \( \square \)

3.1. Path Systems. In order to prove an important technical lemma, we must first establish some facts about extending paths and path systems.

**Lemma 3.3.** Let \( G = (X \cup Y, E) \) be a bipartite graph, and let \( \mathcal{P} \) be a path system of \( G \). Let \( X' \) be a subset of \( (\partial \mathcal{P})_X \), and let \( Y' \) be a subset of \( Y - (\partial \mathcal{P})_Y \). Suppose that \( |X'| = s + t \), where \( s \) is the number of vertices in \( X' \) arising from paths of \( \mathcal{P} \) consisting of a single vertex. Furthermore let \( \ell \) denote the number of vertices of \( Y' \) that are endpoints of some nontrivial path in \( \mathcal{P} \). If

\[ \delta(X', Y') > \frac{t + \ell}{2} + s \]

then there exists another path system, \( \mathcal{P}' \), of \( G \) such that \( \mathcal{P} \leq \mathcal{P}' \) and \( (\partial \mathcal{P}')_X \cap X' = \emptyset \).

**Proof.** We will first show that \( s \) may be assumed to be 0. If \( s > 0 \), let \( P_1, P_2, \ldots, P_s \) be the trivial paths of \( \mathcal{P} \) contained in \( X' \). Now, for every \( i \in [s] \), replace \( P_i = \{ x_i \} \) with a path \( P'_i \) on three vertices such that the endvertices of \( P'_i \) are new vertices added to \( X' \) and the middle vertex of \( P'_i \) is a new vertex added to \( Y \). In addition, let the endvertices of \( P'_i \) be adjacent to the neighbors of \( x_i \). Let \( \mathcal{P}_1 \) be the new path system, and let \( X'_1 \), consisting of \( X' \) and the vertices added to \( X' \), be the new set of endvertices we wish to eliminate.

The new system \( \mathcal{P}_1 \) now contains no trivial paths, and \( |X'_1| = t + 2s \). Thus, if our lemma were true for systems with no trivial paths, then the condition

\[ \delta(X', Y') > \frac{t + 2s}{2} = \frac{t}{2} + s \]

ensures the existence of a path system \( \mathcal{P}'_1 \) such that \( \mathcal{P}_1 \leq \mathcal{P}'_1 \) and \( (\partial \mathcal{P}'_1)_X \cap X' = \emptyset \). By replacing every \( P'_i \) by \( P_i \) within the appropriate paths of \( \mathcal{P}'_1 \), we obtain the desired path system of \( G \).
So assume that $X' = \{x_1, \ldots, x_t\}$. Note that the result clearly holds if $t = 1$, so assume that $t \geq 2$. Our goal is to find edges from each $x_i$ to vertices in $Y'$, allowing us to create a new path system in which no $x_i$ is an endpoint.

Given some $x_i$ in $X'$, let $P_i$ be the path in $\mathcal{P}$ containing $x_i$, let $w_i$ be the other endpoint of $P_i$. Our goal is to select an element $y_k$ in $NY'(x_i)$ that will allow us to extend $P_i$ to a path with one fewer endpoint in $X'$. We will extend the $P_i$, $1 \leq i \leq t$, in order and at the time we consider $x_i$, let $Z_i$ denote the set of internal vertices in the current (updated) path system. It remains to show that $N_{Y'}(x_i) - w_i - Z_i$ is nonempty.

Initially, no vertex of $Y'$ was interior to a path in $\mathcal{P}$. Each vertex in $Y'$ that was already an endpoint of some nontrivial path in $\mathcal{P}$ can be selected once to extend a path and each other vertex in $Y'$ can be selected twice. If exactly $j$ vertices in $Y' \cap Z_i$ were endpoints of some nontrivial path in $\mathcal{P}$, then

$$|Z_i| \leq \max\{0, \frac{i - j - 2}{2} + j\} \leq \frac{t - j - 2}{2} + j \leq \frac{t + \ell}{2} + 1.$$  

This implies that

$$|N_{Y'}(x_i) - w_i - Z_i| \geq \delta(X', Y') - 1 - |Z_i| > \frac{t + \ell}{2} - 1 - |Z_i| > 0.$$  

The following corollary is obtained from Lemma 3.3 by induction on $k$:

**Corollary 3.4.** Let $G = (X \cup Y, E)$ be a bipartite graph, let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be $k$ edge-disjoint path systems, and let $Y' \subset Y - \bigcup_{i=1}^k \text{int}(\mathcal{P}_i)_Y$. For all $i \in [k]$ let $X_i \subset (\partial \mathcal{P}_i)_X$ and $|X_i| = s_i + t_i$, where $s_i$ is the number of vertices of $X_i$ arising from paths of $\mathcal{P}_i$ consisting of a single vertex. Furthermore let $\ell_i$ denote the number of vertices of $Y'_i$ that are endpoints of some nontrivial path in $\mathcal{P}_i$. If for all $i \in [k]$,  

$$\delta(X_i, Y') > \frac{t_i + \ell_i}{2} + s_i + 2(k - 1)$$

then there exist $k$ edge-disjoint path systems $\mathcal{P}_1', \ldots, \mathcal{P}_k'$ such that for all $i \in [k]$, $\mathcal{P}_i \leq \mathcal{P}_i'$ and $(\partial \mathcal{P}_i')_{X_i} = \emptyset$.

### 3.2. The Degree-Product Lemma

The remainder of the proof of Theorem 1.4 relies on a result pertaining to degree products as opposed to degree sums. We feel it would be interesting to investigate similar results.

**Lemma 3.5.** If $G$ has no proper pair $(u, v)$ such that $d_{L}(u)d_{L}(v) \geq 12k(n - 12k)$ then $G$ has $k$ edge-disjoint hamiltonian cycles.

**Proof:** Suppose $G$ has no such vertices. Let $A$ be the subgraph of $G$ generated by the vertices of degree less than $16k$, and $B$ the subgraph generated by the vertices of degree greater or equal to $16k$. By (3) and (1) no bipartite pairs $(u, v)$ of $A$ are proper.

Next we show that no bipartite pairs $(u, v)$ of $B$ can be proper. Suppose that $(u, v)$ was a proper bipartite pair and without loss of generality, assume that $d_{L}(u) \geq d_{L}(v)$. Since $v$ has degree at least $16k$ in $G$, we have that $d_{L}(v) \geq 14k + 2$ and by Fact 1 we know that $d_{L}(u) \geq \frac{n - 2k + 3}{2}$. If $d_{L}(u) \geq \frac{6n}{7} - 2k + 3$ then since $n$ is at least $128k^2$, $d_{L}(u) > \frac{12k(n - 12k)}{14k + 2}$ then $d_{L}(u)d_{L}(v) > 12k(n - 12k)$, a contradiction. If, otherwise, $d_{L}(u) < \frac{6n}{7} - 2n + 3$ then Fact 1 implies that $d_{L}(v) > \frac{9}{7}$, so that

$$d_{L}(u)d_{L}(v) > \frac{n}{7}n - 2k + 3.$$
which exceeds $12k(n - 12k)$ since $n$ is at least $128k^2$.

Thus $A$ and $B$ induce complete bipartite graphs. Assume without loss of generality, that $|A_X| \geq |A_Y|$, and set $\lambda = |A_X| - |A_Y| = |B_Y| - |B_X|$. We can assume $\lambda < 4k - 3$ since otherwise we could find a proper non-adjacent pair $(x, y) \in V(B_X) \times V(A_Y)$ with $d_G(x) + d_G(y) \geq |B_Y| + \lambda + |A_X| + \lambda = n + \lambda \geq n + 4k - 3$, and Fact 2 would imply a Hamilton cycle in $L$, hence $k$ edge-disjoint Hamilton cycles in $G$.

**Claim 5.** We have $\delta(A_X, B_Y) \geq \lambda + 2k - 1$ and $\delta(A_Y, B_X) \geq 2k - 1 - \lambda$

Let $x \in A_X$ such that $d(x, B_Y) = \delta(A_X, B_Y)$. By (3), every vertex $y \in B_Y - N(x, B_Y)$ must verify

$$d_G(y) \geq n + 2k - 1 - d_G(x)$$

$$= n + 2k - 1 - |A_Y| - d(x, B_Y)$$

$$= |B_Y| + 2k - 1 - \delta(A_X, B_Y),$$

so

$$d_G(y, A_X) \geq |B_Y| + 2k - 1 - \delta(A_X, B_Y) - |B_X|$$

$$= \lambda + 2k - 1 - \delta(A_X, B_Y)$$

This implies that

$$d_G(A_X - x, B_Y - N(x, B_Y)) \geq (|B_Y| - \delta(A_X, B_Y))(\lambda + 2k - 1 - \delta(A_X, B_Y))$$

yet, since the vertices of $A_X$ can be adjacent to no more than $\lambda + 4k - 1$ vertices of $B_Y$ (by Fact 2), we see that

$$d_G(A_X - x, B_Y - N(x, B_Y)) \leq (|A_X| - 1)(\lambda + 4k - 1).$$

Thus if $\lambda + 2k - 1 - \delta(A_X, B_Y) > 0$, (5) and (6) imply

$$|B_Y| \leq \frac{(|A_X| - 1)(\lambda + 4k - 1)}{\lambda + 2k - 1 - \delta(A_X, B_Y)} + \delta(A_X, B_Y)$$

$$\leq (16k)(8k - 4) + 2k - 2$$

which contradicts the fact that $n \geq 128k^2$, hence $\delta(A_X, B_Y) \geq \lambda + 2k - 1$.

The proof of $\delta(A_Y, B_X) \geq 2k - 1 - \lambda$ is similar. $\square_{\text{Claim 5}}$

We distinguish two cases, according to the size of $A_X$:

**Case 1:** Suppose $1 \leq |A_Y| \leq 2k - 1$. Then Claim 5 and the completeness of $A$ imply

$$\delta(A_Y) \geq |A_X| + 2k - 1 - \lambda$$

$$= |A_Y| + 2k - 1$$

$$> |A_Y| + 2(k - 1).$$

Now, we apply Corollary 3.4 with $P_i = X_i = A_Y$ for all $i$, and let $Y' = X$. This implies, in the language of the corollary, that $\delta(X_i, Y') = \delta(A_Y)$. Thus, we find that there are $k$ edge-disjoint systems $P_1, \ldots, P_k$ whose paths have all order three and whose endvertices are all in $X$.

Further, since $A$ is a complete bipartite graph, we may choose these path-systems so that they cover a subset $A_X'$ of $\min(|A_X|, 2|A_Y|)$ vertices of $A_X$. That is to say, if $|A_X| \leq 2|A_Y|$, $A_X' = A_X$. 


so these systems each cover \( A \) entirely, and if \(|A_X| > 2|A_Y|\), we require that they each cover the same proper subset \( A'_X \) of \( A_X \) having order \( 2|A_Y| \).

For all \( i \in [k] \) we let \( \mathcal{P}'_i = \mathcal{P}_i \) when \(|A_X| \leq 2|A_Y|\), and \( \mathcal{P}'_i = \mathcal{P}_i \cup (A_X - A'_X) \) when \(|A_X| > 2|A_Y|\).

In either case, we now have \( k \) edge-disjoint path systems which cover \( A \).

Again we wish to apply Corollary 3.4 to the \( \mathcal{P}'_i \) with \( X_i = (\partial \mathcal{P}'_i)_{A_X} \), to extend to a family of \( k \) edge-disjoint systems \( \mathcal{P}'_1', \ldots, \mathcal{P}'_k' \) such that every path in each of these systems has both endvertices in \( B \).

We may do so since if \(|A_X| \leq 2|A_Y|\) then all \( t_i = |A_X| \) vertices of \( X_i \) come from non-trivial paths, and if \(|A_X| > 2|A_Y|\) then \( t_i = 2|A_Y| \) vertices of \( X_i \) also come from non-trivial paths, and \( s_i = |A_X| - 2|A_Y| \) of them come from paths consisting of exactly one vertex, so by Claim 5,

\[
d(A_X, B_Y) \geq \lambda + 2k - 1
\]

\[
> \frac{t_i}{2} + s_i + 2(k - 1).
\]

Consider some matching \( M_1 \) that contains exactly one edge from each non-empty path in \( \mathcal{P}'_i \).

Clearly, \( \partial_Y^X(M_1) = 0 \), and therefore by Proposition 2.1 we have that

\[
\partial(\partial(\mathcal{P}'_i/)) = 0
\]

in \( G - \hat{\mathcal{P}}_i \). Thus, as \( \partial(\mathcal{P}'_i/ \subset B \), and \( B \) induces a complete bipartite graph, we can link the endpoints of the paths in \( \mathcal{P}'_i \) to form a Hamiltonian cycle in \( G \).

Suppose then that we have extended \( \mathcal{P}'_1', \ldots, \mathcal{P}'_{t-1} \) \((t \leq k)\) to the disjoint Hamiltonian cycles \( H_1, \ldots, H_{t-1} \). As above, Proposition 2.1 implies that

\[
\partial(\partial(\mathcal{P}'_i/)) = 0
\]

in \( G - \hat{\mathcal{P}}_i \). Assume that \( \mathcal{P}'_i \) has exactly \( j \) paths, and let \( \{x_1, y_1\}, \ldots, \{x_j, y_j\} \) denote the pairs of endpoints of these paths. Additionally, let the set \( W = \{\{y_1, x_2\}, \{y_2, x_3\}, \ldots, \{y_j, x_1\}\} \). As \( B \) induces a complete bipartite graph with each partite set having size at least \( n - |A_Y| - \lambda \geq n - 6k \), it is simple to see that there is a \( W \)-linkage in \( G_i := G - \mathcal{P}'_i - \bigcup_{j=1}^{t-1} E(H_j) \). Note that there are at most \( j \leq |A_Y| < 2k \) paths in \( \mathcal{P}'_i \), so if we are able to show that \( G_i \) is 2\( k \)-extendible we will be done.

By Corollary 2.2, it suffices to show that

\[
s^2(G_i) > \frac{|V(G_i)| + 6k}{2} \geq \frac{2n - 2k}{2} \geq n - k.
\]

In \( G \), the minimum degree of a vertex in the subgraph induced by \( B \) is \( n - (|A_Y| + \lambda) \geq n - 6k \). In removing the edges from the \( t - 1 \) other hamiltonian cycles, each vertex loses \( 2t - 2 < 2k - 2 \) adjacencies. Thus, it is clear that \( s^2(G_i) \) certainly exceeds \( n - k \), completing this case.

**Case 2:** Suppose \(|A_Y| \geq 2k\). Let \( A'_X \) be a subset of \(|A_Y| \) vertices of \( A_X \). As \( A \) is a complete bipartite graph, there are \( k \) edge-disjoint hamiltonian cycles in \((A'_X \times A_Y)_{G} \), and we let \( x_1, y_1, \ldots, x_k, y_k \) be independent edges of \((A'_X \times A_Y)_{G} \) such that \( x_i, y_i \) is an edge of the \( i^{th} \) hamiltonian cycle.

Using Claim 5 we get that \( \delta(A'_X, B_Y) \geq 2k - 1 \) and \( \delta(A_Y, B_X) \geq 2k - 1 - \lambda \) so

\[
\delta(A_Y, B'_X) \geq |A_X - A'_X| + \delta(A_Y, B_X) \geq 2k - 1.
\]
Let \( B' = G - A_X' - A_Y \). We have
\[
\sigma_2(B') \geq \delta(A_X - A_X', B_Y) + |B_X| \\
\geq |B_X| + \lambda + 2k - 1 \\
= |B'| + 2k - 1
\]

One may then use the edges of \( E(A_X', B_Y) \) and \( E(A_Y, X - A_X') \) along with Theorem 2.3 to find \( k \) edge-disjoint Hamiltonian cycles in \( G \). \( \square \)

Before we proceed to prove the main theorem, we give one final technical lemma.

**Lemma 3.6.** Let \( G \) be a graph containing a Hamiltonian cycle \( C \) and let \( S \) and \( R \) be nonempty disjoint subsets of \( V(G) \). If \( |S| \leq |E(R, S)| - |R| \) then there are four distinct vertices \( c_1, c_2, c_3, c_4 \), encountered in that order on \( C \), such that one of the following holds:

(a) \( c_1, c_3 \in R, c_2, c_4 \in S, c_1c_2 \in E(G), \) and \( c_3c_4 \in E(G) \), or  
(b) \( c_1, c_2 \in R, c_3, c_4 \in S, c_1c_3 \in E(G), \) and \( c_2c_4 \in E(G) \), or  
(c) \( c_1, c_4 \in S, c_2, c_3 \in R, c_1c_3 \in E(G), \) and \( c_2c_4 \in E(G) \).

**Proof:** First, note that if \( R' = \{ r \in R : d(r, S) > 0 \} \) and \( S' = \{ s \in S : d(s, R) > 0 \} \), then
\[
|R'| + |S'| \leq |R| + |S| \leq |E(R, S)| = |E(R', S')|
\]
so we may assume that every vertex of \( R \) is adjacent to at least one vertex of \( S \), and vice versa. Further, observe that the inequality in the statement of the lemma cannot hold if \( |R| = 1 \) or \( |S| = 1 \). Thus, both \( R \) and \( S \) have at least two vertices.

If \( |R| = |S| = 2 \), then \( |E(R, S)| = 4 \), and one of (a), (b), or (c) must occur. So assume without loss of generality that \( |R| \geq 3 \), and let \( R = \{ u_1, \ldots, u_r \} \), where the labels on the vertices of \( R \) are determined by a chosen orientation of \( C \). Suppose the theorem is not true. Then we claim that \( C \) can be traversed such that all of the vertices of \( R \) are encountered before all of the vertices of \( S \). Let \( P \) and \( P' \) be the two \([u_1, u_r]\) paths on \( C \), with \( P \) being the path containing all of the \( u_i \) for \( 1 \leq i \leq r \).

To avoid (a), all of \( u_j \)'s neighbors in \( S \) and all of \( u_j \)'s neighbors in \( S \) must lie either entirely in \( P \) or entirely in \( P' \). If \( (N(u_1) \cup N(u_r)) \cap S \subset P \) no vertex of \( S \) can lie in \( P \), for then the edge between this vertex and any of its neighbors in \( R \) would cause (a), (b), or (c) to occur. But this means that the claim is proven for this case.

So suppose that \( (N(u_1) \cup N(u_r)) \cap S \subset P \). Also, define \( v_i \) to be the vertex with highest index \( i \) such that \( v_i \in N(u_1) \cap S \), and let \( v_j \) be the vertex with lowest index \( j \) such that \( v_j \in N(u_r) \cap S \). Then \( i \neq j \), or else (b) occurs. No vertex of \( R \) lies between \( v_i \) and \( v_j \), or else (a), (b), or (c) would occur. Then \( u_1, \ldots, u_k \) lie along the path \([u_1, v_i] \), and \( u_{k+1}, \ldots, u_r \) lie along the path \([v_j, u_r] \) for some \( k \) between \( 1 \) and \( r - 1 \). All vertices of \( S \) on the path \([u_1, v_i] \), must lie on the path \([u_{k+1}, v_i] \), or else (a), (b) or (c) will occur. Similarly, all vertices of \( S \) on the path \([v_j, u_r] \), must lie on the path \([v_j, u_{k+1}] \). But this implies that the claim holds. If necessary, relabel the vertices of \( R \) such that \( P = [u_1, u_r] \) contains no elements of \( S \). Since (b) or (c) will be violated if two chords from \( R \) to \( S \) cross, a simple count reveals that \( |S| \geq |E(R, S)| - (|R| - 1) \), a contradiction. \( \square \)

### 3.3. Proof of Theorem 1.4.

**Proof:** Let \( C \) be a cycle of \( L \) of maximal order which minimizes \( d_L(T, C) \), where \( T = L - C \). By Lemma 3.2
\[
t = \frac{|T|}{2} \leq 2k - 2 \tag{10}
\]
Let $u \in T_X$ and $v \in T_Y$ such that $d_L(u, C) + d_L(v, C)$ is maximal. Let $\alpha = d_L(u, C)$ and $\beta = d_L(v, C)$. We assume, without loss of generality, that $\alpha \leq \beta$.

We may assume that
\[
\alpha \geq 2k + 4. \quad (11)
\]

Indeed, by Fact 1, every vertex of $Y - N_G(u)$ has degree greater or equal to $n - 2k + 3 - t - \alpha$ in $L$. If $\alpha \leq 2k + 3$, this would yield that there are at least $n - t - (2k + 3) - 2(k - 1) \geq n - 6k$ vertices that have degree at least $n - 2k + 3 - t - (2k + 3) \geq n - 6k$ in $L$. Let $S \subseteq Y$ denote this set of vertices.

Let the vertices $x$ and $y$, in $X$ and $Y$ respectively, be such that $(x, y)$ is a proper pair in $G$. Assume first that there is some vertex $s$ in $S$ such that $(x, s)$ is a proper pair in $G$. Then since $d_L(s) \geq n - 6k$, Fact 2 implies that $d_L(x) < 6k + 1$. Therefore $d_L(x)d_L(y) < (6k + 1)n$.

Suppose then that $x$ is adjacent to every vertex in $S$. Then $d_G(x) \geq |S| \geq n - 6k$ and hence $d_L(x) \geq n - 8k - 1$. By Fact 2, it follows that $d_L(y) < 8k + 1$ and hence $d_L(x)d_L(y) < (8k + 1)n$. Since $n \geq 128k^2$, both $(6k + 1)n$ and $(8k + 1)n$ are strictly less than $12k(n - 12k)$. Therefore, if $\alpha \leq 2k + 3$, $G$ contains $k$ disjoint hamiltonian cycles by Lemma 3.5 and hence we may assume that $\alpha \geq 2k + 4$.

Note that
\[
\alpha + \beta \leq n - t + 1 \leq n - 2k + 3
\]
or else $C$ could be extended.

We must have $|N_L(u, C)^+ \cap N_L(v, C)| \leq 1$ and $|N_L(u, C) \cap N_L(v, C)^+| \leq 1$. Let $R = N_L(v, C)^+ - N_G(u, C)$. Then
\[
|R| \geq d_L(v, C) - d_H(u, C) - |N_L(u, C) \cap N_L(v, C)^+|
\]
\[
\geq \beta - 2(k - 1) - 1
\]
\[
= \beta - 2k + 1 \quad (12)
\]

For every $r \in R$ $ru \notin E(G)$, so by Fact 1,
\[
d_L(r) + d_L(u) = d_L(r, T) + d_L(r, C) + d_L(u) \geq n - 2k + 3,
\]
hence
\[
d_L(r, C) \geq n - 2k + 3 - d_L(u, C) - d_L(u, T) - d_L(r, T)
\]
\[
\geq n - 2k + 3 - \alpha - t - t
\]
Together with the fact that $\sum_{r \in R} d_L(r, T) \leq t - 1$ (since otherwise, we could extend $C$), we get
\[
d_L(R, C) = \sum_{r \in R} d_L(r, C)
\]
\[
\geq \sum_{r \in R} (n - 2k + 3 - d_L(u, C) - d_L(u, T) - d_L(r, T))
\]
\[
= |R|(n - 2k + 3) - |R|(\alpha + t) - \sum_{r \in R} d_L(r, T)
\]
\[
\geq |R|(n - 2k + 3 - \alpha - t) - t + 1. \quad (16)
\]
Let $S = N_L(u, C)$. We have
\[
d_L(R, S) \geq d_L(R, C) - |C_X - S|
\geq |R|(n - 2k + 3 - \alpha - t) - t + 1 - (n - t) + |S|
= |R|(n - 2k + 3 - \alpha - t) + |S| + 1 - n
\]

If Lemma 3.6 with $G = C$, $R = R$, and $S = S^+$ were to hold, then we could extend $C$. Therefore, the assumption of Lemma 3.6 fails, and we have
\[
|S| - (d_L(R, S) - |R| + 1) \geq 0
|S| - ((|R|(n - 2k + 3 - \alpha - t) + |S| + 1 - n) - |R| + 1) \geq 0
n - 2 - |R|(n - 2k + 2 - \alpha - t) \geq 0
\tag{17}
\]

By (12) and (11), we have $|R| \geq \alpha - 2k + 1 \geq 3$, so (17) yields
\[
n - 2 - 3(n - 2k + 2 - \alpha - t) \geq 0
3\alpha \geq 2n - 2k + 9
\tag{18}
\alpha \geq \frac{2}{3}n - \frac{2}{3}k - 3t + 3.
\tag{19}
\]

Yet, as $\alpha \leq \beta$, $t \leq 2k - 1$, and $n \geq 128k^2 \geq 46k$, this would imply
\[
\alpha + \beta \geq \frac{4}{3}n - \frac{4}{3}k - 6(2k - 1) + 6 > n + 2k
\]
contradicting (3.3). \qed

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**References**