SATURATION NUMBERS FOR FAMILIES OF GRAPH SUBDIVISIONS

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ABSTRACT. For a family $\mathcal{F}$ of graphs, a graph $G$ is $\mathcal{F}$-saturated if $G$ contains no member of $\mathcal{F}$ as a subgraph, but for any edge $uv$ in $G$, $G+uv$ contains some member of $\mathcal{F}$ as a subgraph. The minimum number of edges in an $\mathcal{F}$-saturated graph of order $n$ is denoted $\text{sat}(n, \mathcal{F})$. A subdivision of a graph $H$, or an $H$-subdivision, is a graph $G$ obtained from $H$ by replacing the edges of $H$ with internally disjoint paths of arbitrary length. We let $\Delta(H)$ denote the family of $H$-subdivisions, including $H$ itself.

In this paper, we study $\text{sat}(n, \Delta(H))$ when $H$ is one of $C_4$ or $K_5$, obtaining several exact results and bounds. In particular, we determine $\text{sat}(n, \Delta(C_4))$ exactly for $3 \leq t \leq 5$ and show for $n$ sufficiently large that there exists a constant $c_3$ such that $\frac{2n}{3} \leq \text{sat}(n, \Delta(C_4)) \leq \left(\frac{5}{3} + \frac{2}{n}\right)n$. For $t \geq 36$ we show that $c_t = 8$ will suffice, and that this can be improved slightly depending on the value of $t$ (mod 8). We also give an upper bound on $\text{sat}(n, \Delta(K_5))$ for all $t$ and show that $\text{sat}(n, \Delta(K_5)) = \left\lceil \frac{3n+4}{2}\right\rceil$. This provides an interesting contrast to a 1935 result of Wagner [23], who showed that edge-maximal graphs without a $K_5$-minor have at least $\frac{114}{65}n$ edges.

1. Introduction

All graphs considered in this paper are simple, and we let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph $G$, respectively. A maximal 2-connected subgraph $B$ of a graph $G$ is a block of $G$, and if $B \supseteq H$ for some $H$, then we say that $B$ is an $H$-block of $G$. If $X$ is a cut-set of $G$, then an $X$-lobe of $G$ is a subgraph induced by $X$ and the vertices in a single component of $G - X$. An $H$-lobe of $G$ is a lobe of $G$ that is isomorphic to $H$. A pendant vertex is a vertex of degree 1.

A subdivision of a graph $H$, or an $H$-subdivision, is a graph $G$ obtained from $H$ by replacing the edges of $H$ with internally disjoint paths of arbitrary length. The vertices of $G$ corresponding to the vertices of $H$ are called the branch vertices of the subdivision, and the paths in $G$ corresponding to the edges of $H$ are called the edge-paths of the subdivision. We let $\Delta(H)$ denote the family of $H$-subdivisions, which includes $H$.

Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-saturated if no element of $\mathcal{F}$ is a subgraph of $G$, but for any edge $e$ in the complement of $G$, some element of $\mathcal{F}$ is a subgraph of $G + e$. If $\mathcal{F} = \{H\}$, then we say that $G$ is $H$-saturated. The classical extremal function $\text{ex}(n, H)$ is the maximum number of edges in an $n$-vertex $H$-saturated graph. Erdős, Hajnal and Moon [12] studied $\text{sat}(n, H)$, the minimum number of edges in an $n$-vertex $H$-saturated graph, and determined $\text{sat}(n, K_5)$. The value of $\text{sat}(n, H)$ is known precisely for very few choices of $H$, some of which we discuss in this paper. The best upper bound on $\text{sat}(H, n)$ for general $H$ appears in [18], and it remains an interesting problem to determine a non-trivial lower bound on $\text{sat}(H, n)$. For a thorough survey of results on the sat function, we refer the reader to [13].

Here, we study $\Delta(H)$-saturated graphs when $H$ is a cycle or a complete graph. In particular, we determine $\text{sat}(n, \Delta(C_4))$ exactly for $3 \leq t \leq 5$ and show that for $n$ sufficiently large there exists a constant $c_3$ such that $\frac{2n}{3} \leq \text{sat}(n, \Delta(C_4)) \leq \left(\frac{5}{3} + \frac{2}{n}\right)n$. For $t \geq 36$ we show that $c_t = 8$ suffices and that this can be improved slightly depending on the value of $t$ (mod 8). We also give an upper bound on $\text{sat}(n, \Delta(K_5))$ for all $t$ and show that $\text{sat}(n, \Delta(K_5)) = \left\lceil \frac{3n+4}{2}\right\rceil$. This provides an interesting contrast to a

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1935 result of Wagner [23], who showed that edge-maximal graphs without a $K_5$-minor have at least $\frac{11n}{5}$ edges.

2. Cycles

2.1. History. The $C_t$-saturated graphs of minimum size have been studied extensively. Barefoot et al. [1] obtained the first general bounds on $\text{sat}(n, C_t)$, showing that $\text{sat}(n, C_t) \geq (1 + \frac{1}{2t^2})n$ when $t \geq 5$ and $n \geq 5$. They also proved that $\text{sat}(n, C_t) \leq (1 + \frac{6}{t^2})n + O(t^2)$ when $t \geq 9$ and $t$ is odd, and that $\text{sat}(n, C_t) \leq (1 + \frac{1}{t-2})n$ when $t \geq 14$ and $t$ is even. Gould et al. [14] improved the upper bound, showing that $\text{sat}(n, C_t) \leq (1 + \frac{2}{t-3})n$ when $t \geq 17$, $t$ is odd, and $n \geq 7t$, and that $\text{sat}(n, C_t) \leq (1 + \frac{2}{t-2})n + \frac{5t^2}{4}$ when $t \geq 10$ and $t$ is even. While a number of bounds have been determined, very few precise values of $\text{sat}(n, C_t)$ are known. Prior work is summarized in Table 1.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\text{sat}(n, C_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$n - 1$ [12]</td>
</tr>
<tr>
<td>4</td>
<td>$\lceil \frac{3n-1}{2} \rceil$, when $n \geq 5$ [20, 22]</td>
</tr>
<tr>
<td>5</td>
<td>$\lceil \frac{19}{7}(n-1) \rceil$, when $n \geq 21$ [3]</td>
</tr>
<tr>
<td>6</td>
<td>$\leq \frac{3n}{4}$ [1]</td>
</tr>
<tr>
<td>7</td>
<td>$\leq \frac{19n}{14}$ [1]</td>
</tr>
<tr>
<td>8, 9, 11, 13, 15</td>
<td>$\leq \frac{3n}{4} + \frac{17}{2}$ [14]</td>
</tr>
<tr>
<td>$n$</td>
<td>$\lceil \frac{4n}{5} \rceil$, when $n \geq 20$ is even or $n \geq 17$ is odd [2, 5, 6, 7, 16]</td>
</tr>
</tbody>
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Table 1. Known values of and bounds on $\text{sat}(n, C_t)$

In this section, we study $S(C_t)$-saturated graphs of minimum size. The extremal function for this family was determined by Erdős and Gallai [11], who showed that $\text{ex}(n, S(C_t)) = (t-1)(n-1)/2$ for $3 \leq t \leq n$.

2.2. General Bounds. We determine the following general bounds on $\text{sat}(n, S(C_t))$.

**Theorem 2.1.** For $t \geq 3$ and $n \geq n(t)$, there exists an absolute constant $c$ such that $\frac{5n}{t} \leq \text{sat}(n, S(C_t)) \leq (\frac{5}{4} + \frac{c}{t})n$. In particular, if $t \geq 36$, $c = 8$ will suffice.

Comparing this result to the bounds on $\text{sat}(n, C_t)$ from [14] we observe that for large $t$ and $n$, $\text{sat}(n, C_t) \leq \text{sat}(n, S(C_t))$, which brings to light a very curious property of the saturation function. Clearly, for graphs $G \subseteq H$ and an integer $n$, $\text{ex}(n, G) \leq \text{ex}(n, H)$. It is also clear that for families $\mathcal{F} \subseteq \mathcal{R}$ of graphs, $\text{ex}(n, \mathcal{F}) \leq \text{ex}(n, \mathcal{R})$. As has been noted by several authors (for other examples, see [13, 18, 21]), these properties do not hold for for the sat function. We will demonstrate another example of this unusual behavior in Section 3.

The following lemma allows us to establish the lower bound in Theorem 2.1. A set $S$ of $k$ vertices in $G$ is a $k$-cut if $G - S$ is disconnected. Also, in the lemma and theorem that follow, we say that $G$ has property $\mathcal{P}$ if $G$ is 2-connected and every 2-cut of $G$ is complete.

**Lemma 2.2.** If $G$ is a 2-connected graph of order $n \geq 6$ such that every 2-cut of $G$ is complete, then $|E(G)| \geq \frac{3}{2}n$.

**Proof.** If $\delta(G) \geq 3$ then $|E(G)| \geq \frac{3}{2}n$. Thus we may assume that $\kappa(G) = 2$ and that $G$ contains at least one vertex of degree 2.

We argue that if $G$ has property $\mathcal{P}$ and $x$ is a vertex of degree 2 in $G$, then $G - x$ also has property $\mathcal{P}$. If not, then $G - x$ contains an independent cut set $B$ of size at most 2. Because $B$ cannot be a cut set in $G$, the two neighbors of $x$ belong to different components of $G - x - B$. Thus the two neighbors of $x$, which form a cut set, are not adjacent, and $G$ does not have property $\mathcal{P}$.

Let $G$ be a 2-connected graph on at least six vertices with property $\mathcal{P}$. We may inductively remove vertices of degree 2 from $G$ until we have a graph $G'$ that contains no vertices of degree 2. Thus $G'$ has minimum degree 3 or is a single edge. If $G'$ has minimum degree 3, then $|E(G')| \geq \frac{3}{2}|V(G')|$, and so on.
and each vertex in $G$ that is not in $G'$ contributes exactly two edges to the edge count for $G$. Thus $|E(G)| \geq \frac{3}{2}|V(G)|$. If $G' = K_2$, then $|E(G)| = 2|V(G)| - 3$. Thus $|E(G)| \geq \frac{3}{2}|V(G)|$ if $|V(G)| \geq 6$. □

Let $G^-$ denote the graph obtained from $G$ by deleting all of the pendant vertices of $G$.

**Theorem 2.3.** Let $t$ and $n$ be integers such that $t \geq 6$ and $n$ is sufficiently large. If $G$ is a $\delta(C_t)$-saturated graph of order $n$, then $|E(G)| \geq \frac{3}{4}n$.

**Proof.** If $G$ has connectivity $\kappa \geq 3$, then $\delta(G) \geq 3$ and $|E(G)| \geq \frac{3}{4}n$. Thus we may assume that $\kappa(G) \leq 2$.

Suppose first that $\kappa(G) = 2$ and let $A = \{x, y\}$ be a 2-cut in $G$. If $x$ and $y$ are not adjacent, then $G + xy$ contains a cycle $C$ of length at least $t$ containing $xy$, and all the vertices in $C$ are in one $A$-lobe of $G$. Thus there is an $x - y$ path $P$ of length at least $t - 1$ in one $A$-lobe of $G$. Because $\kappa(G) = 2$, there is an $x - y$ path in each $A$-lobe of $G$. Joining one of these paths to $P$ completes a cycle of length at least $t$. It follows that every 2-cut in $G$ contains an edge. Consequently, if $\kappa(G) = 2$ then $G$ has property $\mathcal{P}$, and the result follows directly from Lemma 2.2.

Therefore assume that $\kappa(G) = 1$. Define a *big block* of $G$ to be a block that contains at least four vertices and a *small block* to be a block with at most three vertices. Note that two small blocks cannot be incident, as an edge joining these two blocks completes a cycle of length at most 5.

Let $\ell$ be the number of leaves in $G$ and consider the block decomposition of $G^-$. Suppose that that there are $j$ large blocks in $G^-$. By Lemma 2.2, each large block $B$ contributes at least $3|V(B)|/2$ edges to $G'$. The large blocks are either incident or are joined by small blocks, and a vertex $v$ in $G^-$ that is not in a large block must be in a $K_3$-block $B'$. Since $B'$ contains either one or two cut vertices, the vertex $v$ contributes either $3/2$ (if $B'$ has exactly one cut vertex) or $3$ (if $B'$ has two cut vertices) to the edge count of $G^-$. Thus $|E(G^-)| \geq \frac{3(n-\ell)}{2}$, and $|E(G)| \geq \frac{3(n-\ell)}{2} + \ell$. As no two small blocks can be incident to the same cut vertex it follows that $\ell \leq n/2$, so $G$ has size at least $(3/2)n - (1/4)n = (5/4)n$. □

We now show that for large enough $n$, $\text{sat}(n,\delta(C_t))$ gets arbitrarily close to the lower bound given in Theorem 2.3 as $t$ tends to infinity. Our strategy is to modify several constructions developed by Clark, Entringer, and Shapiro in [6] and [7].

Let $G$ be a 3-regular graph, let $e \in E(G)$, and let $\{v_1, v_2, \ldots, v_k\} \subseteq V(G)$. The graph $G(e)$ is obtained from $G$ via the addition of a vertex $z$ of degree 2 that is adjacent to the endpoints of $e$. A $\Delta - Y$ exchange at a vertex $v$ of degree 3 in a graph $G$ consists of deleting $v$, adding a triangle $T$ to $G$ and connecting the neighbors of $v$ in $G$ with distinct vertices on $T$. The graph $G(v_1, v_2, \ldots, v_k)$ is obtained from $G$ via the execution of a $\Delta - Y$ exchange at each $v_i$. We let $G(v_1, v_2, \ldots, v_k, e)$ denote $G(v_1, v_2, \ldots, v_k)(e)$. For examples, see Figure 1.

**Figure 1.** A graph $G$ and the modifications $G(e)$ and $G(v, e)$.

Barefoot et al [1] constructed a family of $(C_{4m})$-saturated graphs for odd $m$ utilizing Isaacs’ family of snarks [15], defined as follows. Let $m \geq 3$ be an odd integer and take all subscripts that follow modulo $4m$. Let $J_m$ denote the graph with

$$V(J_m) = \{v_i : 0 \leq i \leq 4m - 1\},$$

and

$$E(J_m) = E_0 \cup E_1 \cup E_2 \cup E_3$$

where

$$E_0 = \{v_{4j}v_{4j+1}, v_{4j}v_{4j+2}, v_{4j}v_{4j+3} : 0 \leq j \leq m - 1\},$$

$$E_1 = \{v_{4j+1}v_{4j+7} : 0 \leq j \leq m - 1\},$$

$$E_2 = \{v_{4j}v_{4j+6} : 0 \leq j \leq m - 1\},$$

$$E_3 = \{v_{4j+2}v_{4j+4} : 0 \leq j \leq m - 1\}.$$
shown in Figure 2. Barefoot et. al showed that $H$ from Theorems 2.4 and 2.5. A graph $u$ and $C$ that sat($H$) from identifying multiple copies of $H$ at $v$ is also $G$-saturated. For a family of graphs $\mathcal{F}$, define an $\mathcal{F}$-builder similarly. This notion was used to prove the following.

**Theorem 2.4** (Barefoot et. al [1]). For $n \geq 4m \geq 12$, $m$ odd, sat($n$, $C_{4m}$) $\leq \frac{4m-1}{8m-2}n + 2m$.

Let $H_{4m}$ be a copy of $J_m$ with the addition of pendant vertices adjacent to every vertex save $v_0$, as shown in Figure 2. Barefoot et. al showed that $H_{4m}$ is a $C_{4m}$-builder with distinguished vertex $v_0$.

![Figure 2](image)

The graph $J_3$ and a $C_{12}$-builder based on it.

This construction, at first, seems less than optimal in light of the results in [1, 14], which show that sat($C_t$, $n$) approaches $n$ as $n$ and $t$ grow large. However, of interest here is the observation that any graph $G$ constructed by identifying copies of the builders $H_{4m}$ at the distinguished vertex is also $S(C_{4m})$-saturated, as each block in $G$ has circumference at most $4m - 1$. This observation leads to the next sequence of results, which utilizes modifications of Issacs’ snarks to construct $S(C_t)$-saturated graphs.

A graph $G$ of order $n$ is **maximally non-hamiltonian** if $G$ is not hamiltonian, but for any pair of nonadjacent vertices $x$ and $y$ in $G$, $G + xy$ is hamiltonian.

**Theorem 2.5** (Clark, Entringer, and Shapiro [7]). Let $e = \{v_0, v_2\}$.

1. For $m \geq 5$, the graph $J_m(e)$ is maximally nonhamiltonian.
2. For $m \geq 5$, $J_m(v_2)$ is maximally nonhamiltonian.
3. For $m \geq 7$, $J_m(v_2, v_{14})$ is maximally nonhamiltonian.
4. For $m \geq 9$, $J_m(v_2, v_{14}, v_{26})$ is maximally nonhamiltonian.
5. For $m \geq 9$, $J_m(v_{14}, e), J_m(v_{14}, v_{26}, e)$, and $J_m(v_{14}, v_{26}, v_{38}, e)$ are maximally nonhamiltonian.

We provide an upper bound for sat($n$, $S(C_t)$) for all sufficiently large $t$ by combining the constructions from Theorems 2.4 and 2.5. A graph $G$ is **hamiltonian-connected** if for every pair of vertices $u, v \in G$ there is a hamiltonian path from $u$ to $v$ and $G$ is **maximally nonhamiltonian-connected** if $G$ is not hamiltonian-connected but for any pair of nonadjacent vertices $x$ and $y$ in $G$, $G + xy$ is hamiltonian-connected. The following theorem from will be useful.

**Theorem 2.6** (Kalinowski and Skupien [17]). For any odd $m \geq 7$, $J_m$ is maximally nonhamiltonian-connected.

For ease of reference we will group the modified snarks in the following way. As above, and for the remainder of this section, let $e = v_0v_2$, and let $z$ be the vertex of degree 2 in in $J_m(e)$.

$$E_2 = \{v_{4j+2}v_{4j+6} : 0 \leq j \leq m - 1\},$$

and

$$E_3 = \{v_{4j+3}v_{4j+5} : 0 \leq j \leq m - 1\}.$$

A graph $H$ with a distinguished vertex $v$ is a $G$-builder if $H$ is $G$-saturated and the graph resulting from identifying multiple copies of $H$ at $v$ is also $G$-saturated. For a family of graphs $\mathcal{F}$, define an $\mathcal{F}$-builder similarly. This notion was used to prove the following.

Let $J_m$ be a copy of $J_m$ with the addition of pendant vertices adjacent to every vertex save $v_0$, as shown in Figure 2. Barefoot et. al showed that $H_{4m}$ is a $C_{4m}$-builder with distinguished vertex $v_0$.
Lemma 2.7. If $G$ is a 3-regular maximally nonhamiltonian graph and $xy$ is an edge such that either $N(x) - y$ or $N(y) - x$ is independent, then $xy$ is on a cycle of length $|V(G)| - 1$ in $G$. If, in addition, there is an edge $e \neq xy$ in $G$ such that $G(e)$ is maximally nonhamiltonian, then $xy$ is on a cycle of length $|V(G(e))| - 1$ in $G(e)$.

Proof. Suppose without loss of generality $xy$ is an edge in $G$ with $N(y) = \{x, y_1, y_2\}$ and $y_1$ not adjacent to $y_2$. Since $G$ is maximally nonhamiltonian there is a path $P$ of length $|V(G)| - 1$ joining $y_1$ and $y_2$. Since $y$ has degree 3 and lies on $P$, and $y_1y_2$ is not a subpath of $P$, the edge $xy$ must be on $P$. Therefore, either $P = y_1xyP_1y_2$ or $P = y_1P_1xyy_2$ for some path $P_1$ in $G$ (see Figure 3). In the former case, $yxP_1yP$ is a $|V(G)| - 1$ cycle, and in the latter $y_1P_1xyy_2$ is a $|V(G)| - 1$ cycle. If $G(e)$, with $xw \neq e$, is maximally nonhamiltonian, then we can utilize a hamiltonian path in $G(e)$ between $y_1$ and $y_2$ to construct a $|V(G(e))| - 1$ cycle containing $xy$. □

![Figure 3](image_url) Every edge in a modified snark is on a long cycle.

Lemma 2.8. Let $m \geq 9$ be an odd integer and let $J$ be a graph in $\mathcal{J}_m$. If $u$ and $w$ are (not necessarily distinct) vertices in $J$ then there exists a $u - v_0$ path $P_1$ and a $w - v_0$ path $P_2$ in $J$ such that the total length of $P_1$ and $P_2$ is at least $|V(J)|$.

Proof. If $u$ is not adjacent to $v_0$, then by Theorems 2.5 and 2.6 there is a hamiltonian path joining $u$ and $v_0$. Any $w - v_0$ path then yields the desired paths. Thus we may assume that both $u$ and $w$ are adjacent to $v_0$. If $J \in \mathcal{J}_m^0$, then $N(v_0)$ is independent, and we may apply Lemma 2.7, so that $wv_0$ and $wv_1$ are in cycles of length $|V(J)| - 1$. This yields two paths with combined length $2|V(J)| - 4 > |V(J)|$. If $J \in \mathcal{J}_m^1$ and $u$ and $w$ are not $z$, then $N(v_0)$ is independent in $J - z$ so we may apply Lemma 2.7 to $J - z$. Thus $wv_0$ and $wv_1$ are in cycles of length $|V(J)| - 2$, yielding two paths with combined length $2|V(J)| - 6 > |V(J)|$. If $u$ or $w$ is $z$, then we extend the $v_2 - v_0$ path of length $|V(J)| - 3$ to a $z - v_0$ path of length $|V(J)| - 2$ as necessary. □

We now construct a family of $\mathcal{S}(C_1)$-saturated graphs.

Lemma 2.9. Let $m \geq 9$ be an odd integer and let $J \in \mathcal{J}_m^1$. Joining at most one pendant vertex to all vertices of $J$ other than $v_0$ and $v_2$ produces an $\mathcal{S}(C_{|V(J)|})$-saturated graph. Similarly, if $H \in \mathcal{J}_m^0$, then joining at most one pendant vertex to all vertices of $H$ except $v_0$ results in an $\mathcal{S}(C_{|V(H)|})$-saturated graph.

Proof. Let $J \in \mathcal{J}_m^1$. It suffices to prove that the graph $J'$ obtained by joining a pendant vertex to each vertex in $V(J) - \{v_0, v_2\}$ is $\mathcal{S}(C_{|V(J')|})$-saturated.

Since $J$ does not contain a subcubic $\langle V(J) \rangle$-cycle, neither does $J'$. By Theorem 2.5, $J$ is maximally nonhamiltonian and therefore is $\mathcal{S}(C_{|V(J)|})$-saturated. To prove that $J'$ is $\mathcal{S}(C_{|V(J)|})$-saturated we need only consider the addition of edges to $J'$ that have at least one endpoint in $V(J') - V(J)$. Let $r$ be a pendant vertex in $J'$ with neighbor $x$ and let $y$ be a neighbor of $x$ distinct from $r$. First, consider the addition of the edge $ry$ to $J'$. By Lemma 2.7, the edge $xy$ lies on an $|V(J')| - 1$ cycle in $J'$, so replacing $xy$ with the path $xry$ in $J' + ry$ yields a $|V(J')|$ cycle.

Next, consider $J' + rw$ where $w \in V(J)$ is not adjacent to $x$. By Theorem 2.5, $xw$ completes a $\langle V(J') \rangle$-cycle $C$ in $J'$ that does not contain $r$. Thus, adding $rw$ produces a cycle of length $|V(J)| + 1$ by replacing $xw$ in $C$ with $xrw$. Similarly, if $r_1$ is a pendant vertex in $J'$ whose neighbor $w$ is not adjacent to $x$, then the addition of the edge $r_1y$ yields a $\langle V(J) \rangle + 2$-cycle that is an extension of the $\langle V(J) \rangle$-cycle in $J + xw$. Finally, if $r_2$ is a pendant vertex adjacent to $y \in N(x)$, then the cycle of length $|V(J)| - 1$ in $J'$ that includes $xy$ (assured by Lemma 2.7) can be extended to a cycle of length $|V(J)| + 1$ by replacing the edge $xy$ with the path $xrr_1y$. Therefore, $J'$ is $\mathcal{S}(C_{|V(J')|})$-saturated.

In a similar manner, we can create $\mathcal{S}(C_{|V(H)|})$-saturated graphs by joining at most one pendant vertex to all vertices of $H \in \mathcal{J}_m^0$ other than $v_0$. Due to the similarities, we omit the details here. □
Note that for $J \in \mathcal{F}_m$, we cannot join pendant vertices to either $v_0$ or $v_2$. Because $J$ is nonhamiltonian, a longest cycle $C$ in $J$ containing $v_0v_2$ has at most $|V(J)| - 2$ vertices (it misses both $z$ and some other vertex). If a pendant vertex $r$ is adjacent to $v_0$, then a longest cycle in $J + rv_2$ containing $rv_2$ contains one more vertex than $C$ and therefore has at most $|V(J)| - 1$ vertices. By symmetry, the same is true for a pendant vertex joined to $v_2$.

Lemmas 2.7, 2.8, and 2.9 yield the following.

**Lemma 2.10.** Every graph $G \in \mathcal{J}$ is an $S(C_{|V(G)|})$-builder.

We now determine upper bounds on $\text{sat}(n, S(C_t))$ for sufficiently large $t$. For each $t$ and $n$ we construct an $S(C_t)$-saturated graph $G(t, n)$.

**Theorem 2.11.** Let $m \geq 9$ be an odd integer.

\[
\text{sat}(n, S(C_{4m})) \leq \left( \frac{10m - 1}{8m - 2} \right) n + 2m
\]

(1)

\[
\text{sat}(n, S(C_{4m+1})) \leq \left( \frac{10m + 1}{8m - 1} \right) n + 2m + 1
\]

(2)

\[
\text{sat}(n, S(C_{4m+2})) \leq \left( \frac{10m + 3}{8m + 2} \right) n + 2m
\]

(3)

\[
\text{sat}(n, S(C_{4m+3})) \leq \left( \frac{10m + 6}{8m + 3} \right) n + 2m + 2
\]

(4)

\[
\text{sat}(n, S(C_{4m+4})) \leq \left( \frac{10m + 9}{8m + 6} \right) n + 2m + 2
\]

(5)

\[
\text{sat}(n, S(C_{4m+5})) \leq \left( \frac{10m + 11}{8m + 7} \right) n + 2m + 3
\]

(6)

\[
\text{sat}(n, S(C_{4m+6})) \leq \left( \frac{10m + 14}{8m + 10} \right) n + 2m + 3
\]

(7)

\[
\text{sat}(n, S(C_{4m+7})) \leq \left( \frac{10m + 16}{8m + 11} \right) n + 2m + 4
\]

(8)

*Proof.* The proof of (1) is from [1], and we reproduce it here to demonstrate how each of (1) - (8) is proved. Let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m$ other than $v_0$; note that $|V(G - v_0)| = 8m - 2$. Construct the graph $G(4m, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m - 2} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m - 2) - n$ pendants. By Lemma 2.10, $G(4m, n)$ is $S(C_{4m})$-saturated and has order $n = 1 + s(8m - 2) - r$ and size $s(10m - 1) - r = \frac{10m - 1}{8m - 2}(n - 1) + \frac{2m + 1}{8m - 2} r < \frac{10m - 1}{8m - 2} n + 2m$.

Therefore $\text{sat}(n, S(C_{4m})) \leq \frac{10m - 1}{8m - 2} n + 2m$.

The proofs of each of (2) - (8) are identical (1), so we give only the constructions of the relevant graphs and leave the algebraic calculations to the reader.

For (2), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(e)$ other than $v_0$ and $v_2$. Construct $G(4m + 1, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m - 1} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m - 1) - n$ pendants.

For (3), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_2)$ other than $v_0$. Construct $G(4m + 2, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m + 2} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 2) - n$ pendants.

For (4), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_1, e)$ other than $v_0$ and $v_2$. Construct $G(4m + 3, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m + 3} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 3) - n$ pendants.

For (5), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_2, v_1)$ other than $v_0$. Construct $G(4m + 4, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m + 6} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 6) - n$ pendants.

For (6), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_2, v_{26}, e)$ other than $v_0$ and $v_2$. Construct $G(4m + 5, n)$ by identifying $s = \left\lceil \frac{n - 1}{8m + 7} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 7) - n$ pendants.
For (7), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_2, v_{14}, v_{26})$ other than $v_0$. Construct $G(4m + 6, n)$ by identifying $s = \left\lceil \frac{n - 1}{2m + 10} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 10) - n$ pendants.

For (8), let $G$ be the graph obtained by joining a pendant vertex to every vertex in $J_m(v_{14}, v_{26}, v_{38}, e)$ other than $v_0$ and $v_2$. Construct $G(4m + 7, n)$ by identifying $s = \left\lceil \frac{n - 1}{2m + 11} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 11) - n$ pendants. \hfill $\square$

The reader should note that (5), and (7) may be improved slightly by joining a single vertex pendant to every vertex in $J_m(v_{14}, v_{26}, e)$ other than $v_0$ and $v_2$. Construct $G(2m + 7, n)$ by identifying $s = \left\lceil \frac{n - 1}{2m + 11} \right\rceil$ copies of $G$ at $v_0$ and deleting $r = 1 + s(8m + 11) - n$ pendants.

2.3. Short Cycles. Any cycle is a subdivision of $C_3$, and therefore all $S(C_3)$-saturated graphs are trees; thus $\text{sat}(n, S(C_3)) = n - 1$ for all $n$. We now characterize all $S(C_4)$-saturated graphs and then determine $\text{sat}(n, S(C_4))$.

**Proposition 2.12.** A connected graph $G$ with at least two vertices is $S(C_4)$-saturated if and only if

1. every block of $G$ is isomorphic to either $K_2$ or $K_3$, and
2. no two $K_2$-blocks of $G$ share a vertex.

**Proof.** The necessity of (1) follows from the elementary fact that a 2-connected graph with at least four vertices contains a cycle with length at least 4. If $xy$ and $xz$ are $K_2$-blocks of $G$ that share a vertex, then the edge $yz$ completes only one cycle, and it has length 3. Thus Property (2) is also necessary.

For sufficiency, (1) and (2) imply that when $u$ and $v$ are not adjacent there is a $u - v$ path of length at least 3 in $T$. Therefore $T + uv$ has circumference at least 4. \hfill $\square$

**Theorem 2.13.** For $n \geq 1$, $\text{sat}(n, S(C_4)) = n + \left\lceil \frac{n-3}{2} \right\rceil$.

**Proof.** Let $G$ be an $n$-vertex $S(C_4)$-saturated graph of minimum size. By Lemma 2.12, we know that every block in $G$ is isomorphic to $K_2$ or $K_3$, and that no two $K_2$-blocks share a vertex. Therefore, deleting all of the edges contained in $K_2$-blocks of $G$ yields a graph $G'$ consisting of a matching and independent vertices. As $G$ is connected and deleting the edge set of a $K_3$-block preserves the parity of the number of components in a graph, we conclude that $G'$ has an odd number of components. If $G'$ has $k$ components, then the matching in $G'$ has $n - k$ edges and $G$ contains $\lfloor k/2 \rfloor$ $K_3$-blocks. Thus $|E(G)| = n - k + 3\lfloor k/2 \rfloor$, which is minimized when $k$ is minimized. Since the components of $G'$ contain at most two vertices, $k \geq \lceil n/2 \rceil$. As $k$ is also odd, we conclude that $k = 2\lceil \frac{n+1}{4} \rceil + 1$, which yields $|E(G)| = n + \left\lceil \frac{n-3}{2} \right\rceil$. \hfill $\square$

We now turn our attention to $S(C_5)$-saturated graphs. The book is the graph $B_t \cong K_2 \vee \overline{K_t}$ and the copy of $K_2$ in the construction of $B_t$ is the spine of the book. The $t$ vertices in the independent set are the pages of the book.

**Lemma 2.14.** If $G$ is a 2-connected $S(C_5)$-saturated graph of order $n$ that is not complete, then $n \geq 5$ and $G \cong B_{n-2}$.

**Proof.** Since $G$ is not complete, $G$ must have order at least 5. Suppose that $G$ is 3-connected. If $G$ has order strictly greater than 5, then Menger’s Theorem implies that $G$ contains a cycle of length at least 5. Hence we conclude that $G$ has order exactly 5, and since $\delta(G) \geq 3$, it follows that $G$ is hamiltonian, again a contradiction.

Thus we may assume that $\kappa(G) = 2$. Consider a minimum cut set $X = \{x, y\}$ in $G$. Each $X$-lobe of $G$ contains an $x - y$ path of length at least 2. If any of these paths has length greater than 2, then $G$ contains a cycle of length at least 5. Since $G$ contains no cut-vertices, we conclude that each $X$-lobe has order exactly 3, and hence $G$ is isomorphic to either $B_{n-2}$ or $K_{2, n-2}$. Since adding the edge $xy$ in the latter case does not create a cycle of length at least 5, we conclude that $G \cong B_{n-2}$. \hfill $\square$

Lemma 2.14 implies that every block of a $S(C_5)$-saturated graph $G$ is either a complete graph of order at most 4 or a book with at least three pages. Next, we consider the feasible placement of $K_2$-blocks within a $S(C_5)$-saturated graph.
Lemma 2.15. Let $G$ be an $S(C_5)$-saturated graph and let $B = xy$ be a $K_2$-block of $G$. If $x$ is a cut-vertex of $G$, and $B' \neq B$ is a block of $G$ that contains $x$, then either $B' \cong K_4$ or $B' \cong B_1$ and $x$ does not lie on the spine of $B'$.

Proof. Let $z$ be a vertex in $B'$ that is distinct from $x$. If $B'$ is isomorphic to either $K_2$ or $K_3$, then the longest cycle completed by the edge $yz$ contains either three or four vertices, contradicting the assumption that $G$ is $S(C_5)$-saturated.

Suppose that $B'$ is isomorphic to $B_t$, and let $w \neq x$ be in the spine of $B'$. If $wx$ is the spine of $B'$, then the longest cycle in $G + wy$ has four vertices. By Lemma 2.14, it follows that $B'$ is either $K_4$ or $B_t$, and in the latter case, $x$ is not in the spine. □

The following Proposition follows easily from Lemmas 2.14 and 2.15.

Proposition 2.16 (Characterization of $S(C_5)$-Saturated Graphs). A graph $G$ is $S(C_5)$-saturated if and only if

1. every block of $G$ is either a complete graph of order at most 4 or a book with at least three pages, and
2. for any $K_2$-block $B$ and block $B' \neq B$ such that $B \cap B' \neq \emptyset$, either $B'$ is a $K_4$-block or $B'$ is a $B_t$-block with $t \geq 3$ such that $B \cap B'$ is a page of $B'$.

For $n \geq 21$, Chen [3] showed that sat($n, C_5$) = $\lceil \frac{10(n-1)}{7} \rceil$. We show that sat($n, S(C_5)$) behaves similarly, although the proof of our next result and the corresponding result from [3] differ significantly.

Theorem 2.17. For $n \geq 5$,
\[ \text{sat}(n, S(C_5)) = f(n) = \left\lceil \frac{10(n-1)}{7} \right\rceil. \]

Proof. Two triangles identified at a vertex is a 5-vertex graph with $f(5)$ edges, and $K_4$ with a pendant attached to two vertices is a 6-vertex graph with $f(6)$ edges. Using Proposition 2.16 it is straightforward to verify that these graphs are $S(C_5)$-saturated and have the minimum number of edges.

For $n \geq 7$, we establish the upper bound by using the graphs $H_0, \ldots, H_6$ pictured in Figure 4 to construct an $S(C_5)$-saturated graph $G_n$ of order $n$ with $f(n)$ edges.

![Figure 4. $S(C_5)$-builders (subscripts indicate value of $n \pmod{7}$)](image)

For $n \geq 7$ construct $G_n$ by identifying $\lfloor n/7 \rfloor - 1$ copies of $H_1$ and one copy of $H_n \pmod{7}$ at the vertices that are marked with open circles. By Proposition 2.16, $G_n$ is $S(C_5)$-saturated for each $n \geq 7$.

To establish the lower bound for $n \geq 7$, we proceed by induction on $n$. Using Proposition 2.16 it is easy to verify that the result holds for $n \leq 7$. Let $G$ be an $n$-vertex $S(C_5)$-saturated graph of minimum size with $n \geq 8$. Note that $G$ must be connected, but by Lemma 2.14 cannot be 2-connected, as $|E(B_{n-2})| > f(n)$ for $n \geq 8$.
Suppose first that there is a cut-vertex $x$ in $G$ with an $x$-lobe $L$ with at least three vertices such that $|V(G) - (V(L) - x)| \geq 3$. Let $k = |V(L)|$. By choosing $x$ and $L$ so that $|V(L)|$ is minimized, we may assume $|V(G) - (V(L) - x)| \geq 5$. Both $L$ and $G - (L - x)$ are $S(C_3)$-saturated, and by induction $G$ contains at least $\text{sat}(k, S(C_3)) + f(n - k) + f(k) = f(n)$ edges. If $k \in \{3, 4\}$, then $|E(G)| \geq \left(\frac{3}{2}\right) + f(n) \geq f(n)$ (see Table 2). If $k \geq 5$, $|E(G)| \geq f(k) + f(n - k + 1) \geq f(n)$.

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
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<td>$f(n - 3)$</td>
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</table>

Table 2. $f(n - k + 1), k \in \{3, 4\}$, listed by value of $n \pmod{7}$.

If there is no such choice of $x$ and $L$, then by Proposition 2.16 every cut-vertex has exactly two lobes, one of which is $K_2$. Since no two $K_2$-blocks in $G$ are incident, it follows that at most half of the vertices in $G$ are pendant. This implies that $G$ is a book with pendants joined to some of the pages, and thus $|E(G)| \geq \left\lceil \frac{3(n - 1)}{2}\right\rceil + 1 \geq f(n).$ \hfill $\square$

We make the somewhat curious observation that for large $n$, our results demonstrate that $\text{sat}(n, C_t)$ and $\text{sat}(n, S(C_t))$ agree for $t = 3$ and $5$ and differ for all other values of $t$, save perhaps for $t = 6$.

3. Complete Graphs

In their pioneering work on minimal saturated graphs, Erdős, Hajnal, and Moon [12] showed that $\text{sat}(n, K_1) = \left(\binom{n}{2} + (t - 2)(n - t + 2)\right)$ and also demonstrated that the unique $n$-vertex $K_t$-saturated graph of minimum size is $K_{t - 2} \vee \overline{K}_{n - t + 2}$. In this section, we consider $S(K_t)$-saturated graphs. As mentioned in the previous section, $\text{sat}(n, K_3) = \text{sat}(n, S(K_3)) = n - 1$.

A $t$-tree is any graph that can be obtained from $K_t$ by iteratively joining vertices to cliques of size $t$. The following appears as a lemma in [9].

**Theorem 3.1.** A graph $G$ of order at least $2$ is $S(K_4)$-saturated if and only if $G$ is a 2-tree.

As any 2-tree of order $n$ has $2n - 3$ edges, the following corollary is immediate.

**Corollary 3.2.** For $n \geq 2$, $\text{ex}(n, S(K_4)) = \text{sat}(n, S(K_4)) = \text{sat}(n, K_4) = 2n - 3$.

At this stage, it is tempting to conjecture that $\text{sat}(n, S(K_t)) = \text{sat}(n, K_t)$ for all $t$. Our next two constructions demonstrate that this is not the case in general.

**Theorem 3.3.** Let $t \geq 5$ be an odd integer. If $n = d(t - 1) + r$ for $d \geq 2$ and $0 \leq r \leq t - 2$, then $\text{sat}(n, S(K_t)) \leq d\left(\frac{t - 1}{2}\right) \left(\frac{t - 1}{2}\right) + \frac{r - 2}{2} + o(1)\right) n$.

**Proof.** We construct an $n$-vertex $S(K_t)$-saturated graph $G_{t,n}$. Let $A$ and $B$ be two copies of $K_{t-1}$ with vertex sets $V(A) = \{a_1, \ldots, a_{t-1}\}$ and $V(B) = \{b_1, \ldots, b_{t-1}\}$, respectively, and then remove the perfect matching consisting of the edges $a_i a_{i+1}$ for odd $i$ from $A$. Now, for each $i$, join $a_i$ and $b_i$ with a path $P_i$ of length $d - 1$ having vertices $\{a_i, a_i^1, \ldots, a_i^{d+1}, b_i\}$. For each $k \in [d - 2]$, let the vertex set $\{a_i^k : 1 \leq i \leq d - 1\}$ induce $K_{t-1}$ minus the edges $a_i^1 a_i^2, a_i^2 a_i^3, \ldots, a_i^{d-2} a_i^{d-1}, a_i^{d-1} a_i^d$. Finally, add vertices $v_1, \ldots, v_r$ such that $v_i$ is adjacent to $a_i$ and $a_i^1$ (or $b_i$, if $d = 2$) for $i \in [r]$. For an example when $t = 5$ see Figure 5.

If $r = 0$, then $G_{t,n}$ has only $t - 1$ vertices of degree at least $t - 1$, implying that $G_{t,n}$ does not contain a subdivision of $K_t$. If $r > 0$ and $G_{t,n}$ contains a $K_t$-subdivision, then either $a_i$ or $a_i^1$ is a branch vertex and $a_i a_i^1$ and $a_i v_i a_i^1$ lie within distinct edge-paths, an impossibility. Thus, for all $n$, $G_{t,n}$ contains no member of $S(K_t)$ as a subgraph.

Given any pair of nonadjacent vertices $u$ and $v$ in $V(G_{t,n})$, it remains to show that $G_{t,n} + uv$ contains a subdivision of $K_t$. Assume that $r = 0$ and suppose without loss of generality that $u$ lies on $P_1$, so $u$ is $a_1, b_1$ or $a_1^1$ for some $i$. For each choice of $v$ that follows, $G_{t,n} + uv$ contains a $K_t$-subdivision with
Theorem 3.4. Let $u, v, \ldots, b_{t-1},$ and $u,$ that uses each edge between the vertices of $B$ as an edge-path. Consequently, we will only describe the edge-paths incident to $u.$ By symmetry we may assume that if $u = a_1,$ then either $v = b_j$ or $v = a_i^k$ where $k \geq i.$ In other words, referring to Figure 5, we may assume that $v$ lies “below” $u.$ Note that $u \neq b_1$ as $B$ is complete.

If $v$ lies in $P_1,$ then $v \neq a_i^{i+1}$ (if $u = a_1$ use the convention $a_1 = a_0$) and the edge-paths incident to $u$ are $uvP_1b_1, ua_i^1P_1a_2P_2b_2, ua_i^{i+1}P_1a_3, uP_1a_1a_{i-1}P_{i-1}b_{i-1},$ and $ua_i^1P_jb_j$ for $4 \leq j \leq t - 2.$ If $v$ lies in $P_2,$ then the edge-paths incident to $u$ are $uP_1b_1, uvP_2b_2, uP_1a_1a_{i-1}P_{i-1}b_{i-1},$ and $ua_i^1P_jb_j$ for $3 \leq j \leq t - 2.$ For $v$ lying on $P_{t-1},$ the edge-paths from $u$ are $uP_1b_1, uP_1a_1a_{i-1}a_2P_2b_2, uvP_{t-1}b_{t-1},$ and $ua_i^1P_jb_j$ for $3 \leq j \leq t - 2.$ Finally, when $v$ lies in $P_t$ for $3 \leq \ell \leq t - 2,$ the edge-paths incident to $u$ are $uP_1b_1, uvP_tb_t, uP_1a_1a_{i-1}a_2P_2b_2, uP_1a_1a_{i-2}P_{t-2}b_{t-2},$ and $ua_i^1P_jb_j$ for $4 \leq j \leq t - 2.$ Some examples of the subdivisions described above can be found in Figure 6.

![Figure 5. An $S(K_5)$-saturated graph of order 18. ($d = 4, r = 2$)](image)

![Figure 6. Examples of $K_r$-subdivisions in $G(5, 18) + uv.$ Open circles represent the branch vertices of the subdivisions formed.](image)

We have verified that $G_{t,n}$ is $S(K_t)$-saturated when $r = 0.$ For $r \geq 1,$ we proceed by induction. It remains to show that if $r \geq 1,$ then $G + uv$ contains a $K_r$-subdivision. Let $G' = G - v_r$ and consider the $K_r$-subdivision $X$ formed by adding the edge $ua_r$ to $G'$ (which exists by induction). Replacing the edge $a_ru$ in $X$ with $a_rv_ud$ yields a $K_r$-subdivision in $G + uv_r.$

Theorem 3.4. Let $t \geq 6$ be an even integer. If $n = d(t - 1) + r$ for $d \geq 2$ and $0 \leq r \leq t - 2,$ then

$$\text{sat}(n, S(K_t)) \leq d \left( \frac{t-1}{2} \right) + \frac{t}{2} + 2r - 2 = \left( \frac{t-2}{2} + o(1) \right)n.$$ 

Proof. As in the proof of Theorem 3.3, we construct an $n$-vertex $S(K_t)$-saturated graph $G_{t,n}.$ Let $A$ and $B$ be two copies of $K_{t-1}$ with vertex sets $V(A) = \{a_1, \ldots, a_{t-1}\}$ and $V(B) = \{b_1, \ldots, b_{t-1}\}$ respectively,
and then remove the edges $a_1a_2, a_2a_3$, and $a_3a_1$ as well as the edges $a_i a_{i+1}$ for all even $i \geq 4$. Then add the edges $a_1b_2, a_2b_3$ and $a_3b_1$. Now, for each $i$, join $a_i$ and $b_i$ with a path $P_i$ of length $d - 1$ having vertices $\{a_i, a_i^1, \ldots, a_i^{d-2}, b_i\}$. For each $k \in [d-2]$, let the vertex set $\{a_i^k : 1 \leq i \leq d - 1\}$ induce $K_{i-1}$ minus the edges $a_i^k a_i^{k+1}, a_i^{k+1} a_i^{k+2}, \ldots, a_i^{d-2} a_i^{d-1}, a_i^{d-1} a_i^1$. Finally, add vertices $v_1, \ldots, v_r$ such that $v_i$ is adjacent to $a_i$ and $a_i^1$ (or $b_i$, if $d = 2$) for $i \in [r]$. For an example when $t = 6$ see Figure 7.

![Figure 7. $G(6, 21)$, an $S(K_6)$-saturated graph of order 21. ($d = 4, r = 1$)](image)

The proof that $G(t, n)$ contains no subdivision of $K_t$ is identical to the argument when $t$ is odd, and hence we do not repeat it here.

It remains to verify that $G_{t,n} + uv$ contains a $K_t$-subdivision for any nonadjacent $u$ and $v$ in $V(G_{t,n})$. As in the proof of Theorem 3.3, we assume that $v$ lies at or below the same “vertical” level as $u$ when $G_{t,n}$ is drawn as in Figure 7. We also begin by supposing that $r = 0$. In the interest of concision, we consider only the case where $u$ lies on $P_1$, as the remaining possibilities are handled similarly. Some examples of the subdivisions created for various choices of $u$ and $v$ are depicted in Figure 8.

![Figure 8. Examples of $K_t$-subdivisions in $G(6, 21) + uv$. Open circles represent the branch vertices of the subdivisions formed.](image)

For each choice of $u$ and $v$, the vertices of $B$ will be branch vertices of the $K_t$-subdivision in $G_{t,n} + uv$, with the edges in $B$ serving as edge-paths. The remaining branch vertex will be either $u$ or $v$, depending on the situation, and hence we only describe the edge-paths incident to this vertex. For the remainder of the proof, given $u$ (respectively $v$) on some $P_k$, we let $u_k$ (resp. $v_k$), $k \neq i$, denote the neighbor of $u$ ($v$) on $P_k$, provided such a neighbor exists.
Suppose that both $u$ and $v$ lie on $P_1$. First assume that $u = a_1^i$ for some $1 \leq i \leq d - 2$. It follows that $v \neq a_{i+1}$ and $u$ is a branch vertex of the newly formed subdivision in $G_{t,n} + uv$. The edge-paths incident to $u$ are $uP_1b_1, uP_1a_2b_2, uu_1a_3a_2b_3, uu_1a_{i+1}a_{i+1}P_1b_4, uu_3u_{t-1}P_1b_{t-1}$, and $uu_1P_kb_k$ for $5 \leq k \leq t - 2$. If $u = u_1$, then $u$ is still a branch vertex in the newly formed subdivision and the edge-paths incident to $u$ are $uP_1b_1, uP_1b_2, uu_1a_3P_3b_3$, and $uu_1P_kb_k$ for $4 \leq k \leq t - 1$.

If $v$ lies on $P_3$, we handle the case where $v = b_2$ separately from the other possible choices of $v$. If $v = b_2$, then $u \neq a_1$ and $u$ is the additional branch vertex in the $K_t$-subdivision including $uv$, with the edge-paths incident to $u$ being $uP_1b_1, uvP_1a_1b_2, uP_1a_2b_2a_3$, and $uu_3u_{t-1}P_1b_{t-1}$, and $uu_1P_kb_k$ for $3 \leq k \leq t - 2$. If $v \neq b_2$, then $v$ is a branch vertex in a $K_t$-subdivision, and the edge-paths incident to $v$ are $vvP_1b_1, vvP_2b_2, vv_1a_3b_3$, and $vv_1P_kb_k$ for $4 \leq k \leq t - 1$. If $v$ lies on $P_3$ and $v \neq b_3$, then $v$ is the branch vertex of the $K_t$-subdivision in $G_{t,n} + uv$, with $vP_3a_1b_1, vP_1a_1b_2, vP_1a_2b_3, vP_1a_3b_4$, and $vP_kb_k$, for $5 \leq k \leq t - 1$ as the edge-paths incident to $v$. If $v = b_3$, we first consider when $u = a_1$. In this case, $a_1$ is the branch vertex with edge-paths $P_1, a_1b_1, a_1a_2, a_1a_3P_3b_3$, and $a_1a_4P_4$ for $j \geq 4$. Otherwise $u \neq a_1$ and $v = b_3$. In this case, $u$ is the branch vertex with edge-paths $uP_1b_1, uP_1a_1b_2, vP_1a_3b_3$, and $vP_kb_k$ for $4 \leq k \leq t - 2$, and $uu_3u_{t-1}P_1b_{t-1}$.

When $v \in P_j$ with $j \in \{4, \ldots, t-1\}$, then $u$ is the additional branch vertex, and the $u - b_j$ edge-paths for $i \neq 3$ are $uP_1b_1, uP_1a_1b_2, uu_1a_3b_3$, and $uu_3u_{t-1}P_1b_{t-1}$, and $uu_1P_kb_k$ for $5 \leq k \leq t - 2$. There are two cases for the $u - b_3$ edge-path. If $u = a_1$, then the desired edge-path is $uu_1a_3P_3b_3$. If $u \neq a_1$, then the desired edge path is simply $uu_1P_3b_3$.

If $r \geq 1$, the argument mirrors the argument in the proof of Theorem 3.3 and is omitted for concision.

In 1998 Mader [19] proved that $\text{ex}(n, S(K_5)) = 3n - 5$, resolving a long-standing conjecture of Dirac [10]. Next, we improve upon the construction in Theorem 3.3 in order to answer an analogous result for the saturation function.

**Theorem 3.5.** For $n \geq 10$,

\[
\text{sat}(n, S(K_5)) = \left\lceil \frac{3n + 4}{2} \right\rceil.
\]

**Proof.** We modify the graph $G(5,n)$ from Theorem 3.3 based upon the value of $n$ modulo 4 to obtain a new $S(K_5)$-saturated graph $G^*(5,n)$. Suppose that $n = 4d + r$ with $d \geq 2$ and $0 \leq r \leq 3$. If $r = 0$ or $1$, then $G^*(5,n) = G(5,n)$. If $r = 2$ then we obtain $G^*(5,n)$ from $G(5,n - 2)$ by subdividing the edges $a_1a_2$ and $a_2a_3$ and then connecting the two new vertices resulting from these subdivisions (for an example, see Figure 9). If $r = 3$, we add a vertex $v_0$ to $G^*(5,n - 1)$ and connect $v_0$ to both $a_3$ and $a_4^1$. The proof that $G^*(5,n)$ is $S(K_5)$-saturated is nearly identical to that for $G(n,5)$, with the exception of the addition of an edge with any of the new vertices as endpoints. Due to these similarities, we leave it to the reader to verify that the addition of these edges creates a subdivision of $K_5$.

![Figure 9. The graphs $G^*(5,22)$ and $G^*(5,23)$.](image)

To show that this construction is best possible, consider an $n$-vertex $S(K_5)$-saturated graph $G$ with less than $\left\lceil \frac{3n + 4}{2} \right\rceil$ edges. Suppose that $G$ has a vertex $v$ with degree 1 and neighbor $v'$. If $v$ is another neighbor of $v'$, then the $K_5$-subdivision in $G + uv$ uses the path $uvv'$. Replacing $uvv'$ with $uv$ produces a $K_5$-subdivision in $G$, so we conclude that $\delta(G) \geq 2$. If $\delta(G) \geq 3$, then $G$ has at most three vertices with degree at least 4. Because $G'$ is $S(K_5)$-saturated, the addition of any edge to $G'$ must produce a
graph with at least five vertices with degree at least 4, and thus each vertex with degree greater than 3 in $G'$ is adjacent to every vertex in $G'$. It follows that $|E(G)| \geq 3|V(G)| - 6$, a contradiction. Therefore $\delta(G) = 2$.

As in the proof of Theorem 3.3, we observe that the neighbors of any vertex $v$ of degree 2 must be adjacent and that $G - v$ is $\mathcal{S}(K_5)$-saturated. Iteratively delete all vertices of degree 2 in $G$ to obtain the $\mathcal{S}(K_5)$-saturated graph $G'$. Because $|E(G)| < \left\lceil \frac{3n+4}{2} \right\rceil$ and $n \geq 10$, it follows that $G'$ is a noncomplete graph with at least five vertices and $\delta(G') \geq 3$. Thus $|E(G')| \geq 3|V(G')| - 6$. Furthermore, $|E(G')| < \left\lceil \frac{3|V(G')|+4}{2} \right\rceil$. We conclude that $G'$ has five vertices and $|E(G')| = 9$ (there is only one 5-vertex $\mathcal{S}(K_5)$-saturated graph). Therefore, $G$ has more than $\left\lceil \frac{3n+4}{2} \right\rceil$ edges. □

While attempting to give a non-topological proof of the (then) 4-Color Conjecture, Wagner [23] characterized the family of $\mathcal{M}(K_5)$-saturated graphs, where $\mathcal{M}(H)$ denotes the collection of graphs containing $H$ as a minor. Let $W$ denote the graph obtained from $C_6$ by connecting the four pairs of antipodal vertices.

Theorem 3.6 (Wagner 1935). If $G$ is an $\mathcal{M}(K_5)$-saturated graph of order $n \geq 4$, then $G$ can be constructed recursively from maximally planar graphs and $W$ by associating copies of $K_2$ or $K_3$.

Since a maximal planar graph of order $n$ has $3n - 6$ edges, it is not difficult to see that a minimal $\mathcal{M}(K_5)$-saturated graph can be obtained by associating multiple copies of $W$ at a single edge and then utilizing appropriate maximally planar graphs to obtain a graph of the desired order. Therefore $	ext{sat}(n, \mathcal{M}(K_5)) \approx \frac{3}{2}n$. It is well documented (see, for instance, [18]) that the saturation function is not necessarily decreasing with respect to the inclusion relation. Thus we have another example of this phenomenon, as $\mathcal{S}(K_5) \subset \mathcal{M}(K_5)$, but for $n$ large, $\text{sat}(n, \mathcal{S}(K_5)) \geq \text{sat}(n, \mathcal{M}(K_5))$.

4. Conclusion

It is unclear at this time if, for $t \geq 6$, the constructions in Theorems 3.3 and 3.4 are even asymptotically sharp. In fact, given that an arbitrarily large $K_t$-subdivision has average degree approaching 2, it seems feasible that there is some absolute constant $c$ such that $\text{sat}(n, \mathcal{S}(K_t)) < cn$ for all $t$ and large enough $n$. At this time, we do not conjecture the existence of such a constant, but feel that further investigation would be of interest.

References