Some Problems on Graphic Sequences

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Dedicated to Charles Suffel on the Occasion of his 70th Birthday

Abstract

A nonnegative integer sequence $\pi$ is graphic if there is some simple graph $G$ having degree sequence $\pi$. In that case, $G$ is said to realize or be a realization of $\pi$. A given degree sequence may have many realizations, and it has been of interest to examine the spectrum of properties and parameters that occur across these realizations. In this survey, we present five areas of recent research on graphic sequences and present a number of approachable open problems.

Keywords: Degree sequence, potential, forcible

1 Introduction

We refer the reader to [1] for any undefined terminology and notation. A nonnegative integer sequence $\pi$ is graphic if there is some simple graph $G$ having degree sequence $\pi$. In that case, $G$ is said to realize or be a realization of $\pi$. As there are a number of necessary and sufficient conditions for a sequence to be graphic (c.f [2, 3, 4, 5]), considerable attention has been given to the study of when a graphic sequence has a realization with a given property. In general, the literature is divided into two categories: forcible degree sequence problems, in which all realizations of a sequence $\pi$ have a given property, and potential degree sequence problems, in which at least one realization of $\pi$ has the desired property. In this paper, we focus on potential problems, in part because many forcible degree sequence problems are equivalent to well-studied problems in extremal graph theory.

These two broad classes of problems are the genesis of many fascinating results and problems on graphic sequences. Several older, but extremely thorough surveys [6, 7, 8, 9, 10] give a most excellent background on this work, together containing several hundred references. The goal of this survey, in the spirit of these works, is to discuss five areas of recent research on graphic sequences and highlight some interesting, and in the author’s opinion approachable, open problems.

\textsuperscript{1}Research supported in part by Simons Foundation Collaboration Grant # 206692.
2 Packing Graphic Sequences

Two $n$-vertex graphs $G_1$ and $G_2$ pack if they can be expressed as edge-disjoint subgraphs of $K_n$, the complete graph on $n$ vertices. Graph packing has received a great deal of attention, with interesting results and challenging open problems appearing throughout the literature ([11], [12] and [13] are detailed and useful surveys).

In this section, we discuss an extension of graph packing to the setting of graphic sequences. Let $\pi_1$ and $\pi_2$ be graphic sequences, with $\pi_1 = (d^{(1)}_1, \ldots, d^{(1)}_n)$ and $\pi_2 = (d^{(2)}_1, \ldots, d^{(2)}_n)$ (they need not be monotone). Then $\pi_1$ and $\pi_2$ pack if there exist edge-disjoint graphs $G_1$ and $G_2$, both with vertex set $\{v_1, \ldots, v_n\}$, such that
\[
d_G(v_i) = d^{(1)}_i, \quad d_G(v_i) = d^{(2)}_i
\]
and
\[
d_{G_1 \cup G_2}(v_i) = d^{(1)}_i + d^{(2)}_i.
\]

Interestingly, the problem of packing graphic sequences has concrete applications to imaging science. Of particular interest here is so-called discrete tomography, which uses low-dimensional projections to reconstruct discrete objects, such as the atomic structure of crystalline lattices and other polyatomic structures.

Numerous papers (c.f. [14, 15, 16]) study the $k$-color Tomography Problem, in which the goal is to color the entries of an $m \times n$ matrix using $k$ colors so that each entry receives at most one color and each row and column receives a prescribed number of entries of each color. The colors represent different types of atoms appearing in a crystal, and the number of times an atom appears in a given row or column is generally obtained using high resolution transmission electron microscopes [17, 18]. The given number of molecules of a given color in each row and column is equivalent to the degree sequence of a $m \times n$ bipartite graph. Thus, the $k$-color Tomography Problem is precisely that of packing the degree sequences of $k$ bipartite graphs with partite sets of size $m$ and $n$.

An exploration of the $k$-color Tomography Problem has led to proofs that packing as few as two graphic sequences is NP-complete [14, 19]. These computational difficulties give rise to the following problem which has been considered in [14, 15, 16, 20] and elsewhere.

**Problem 1.** Determine meaningful sufficient conditions that ensure $k \geq 2$ graphic sequences pack.

One strategy for approaching Problem 1 is to look to the graph packing literature for inspiration. For instance, the classical 1978 graph packing theorem of Sauer and Spencer [21] states that if $G_1$ and $G_2$ are $n$-vertex graphs with $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack. This gives rise to the following conjecture from [20], a weaker version of which, concerned with termwise sums as opposed to products, was proved there.

**Conjecture 2.** Let $\pi_1$ and $\pi_2$ be $n$-term graphic sequences with $\delta$ the least entry in $\pi_1 + \pi_2$. If $\delta \geq 1$ and the product of corresponding entries in $\pi_1$ and $\pi_2$ is always less than $n/2$, then $\pi_1$ and $\pi_2$ pack.

In the spirit of Conjecture 2, nearly any theorem on graph packing could be reformulated as a problem on packing degree sequences. In the author’s opinion, exploration of such problems would be a particularly interesting way in which to approach both Problem 1 and the $k$-color Tomography Problem.
3 Disjoint 1-factors

Let $G$ be a graph and $f$ be a function from $V(G)$ to the nonnegative integers. A spanning subgraph $H$ of $G$ is an $f$-factor of $G$ if $d_H(v) = f(v)$ for every vertex $v$ in $G$. If $f(v) = k$ for all $v$, then $H$ is said to be a $k$-factor of $G$. In the interest of clarity, for the remainder of this section assume that all realizations $G$ of a graphic sequence $(d_1, \ldots, d_n)$ have $V(G) = \{v_1, \ldots, v_n\}$ with $d(v_i) = d_i$.

If a graphic sequence $(d_1, \ldots, d_n)$ has a realization $G$ containing an $f$-factor $F$, then necessarily the sequence $(d_1 - f(v_1), \ldots, d_n - f(v_n))$ is also graphic, as this is the degree sequence of $G - E(F)$. Kundu’s $k$-factor theorem [22] states that if the sequence $(f(v_1), \ldots, f(v_n))$ is nearly regular, then this necessary condition is also sufficient.

Theorem 1 (Kundu’s $k$-factor Theorem 1973). Let $k \geq 1$ be an integer and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $(d_1 - x_1, \ldots, d_n - x_n)$ is also graphic, where each $x_i$ is either $k$ or $k - 1$. Then there is some realization $G$ of $\pi$ that contains an $f$-factor with $f(v_i) = x_i$ for all $i$.

A.R. Rao and S.B. Rao [23] obtained the following interesting result during their attempts to prove the (then) $k$-factor conjecture.

Theorem 2. Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi_k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$. Then for any nonnegative integer $r \leq k$ such that $rn$ is even, $\pi_r = (d_1 - r, \ldots, d_n - r)$ is also graphic.

Together with Kundu’s theorem, Theorem 2 implies that if $\pi$ has a realization containing a $k$-factor, then $\pi$ has a realization containing an $r$-factor for each $r < k$ satisfying the necessary parity conditions. In [24], Brualdi conjectured the following extension of both Kundu’s Theorem and Theorem 2.

Conjecture 3. Let $n$ be an even integer and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $(d_1 - k, \ldots, d_n - k)$ is graphic, for some $k > 0$. Then there exists a realization $G$ of $\pi$ that contains $k$ edge-disjoint 1-factors.

Unaware of Brualdi’s prior statement, Conjecture 3 was also posed in [20] where the following partial results appear.

Theorem 3. Let $k \geq 1$ be an integer and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence such that $n$ is even. If $\pi_k = (d_1 - k, \ldots, d_n - k)$ is graphic and any of the following hold

1. $k \leq 3$,
2. $d_1 \leq \frac{n}{2} + 1$, or
3. $d_n \geq \frac{n}{2} + k - 2$,

then there exists a realization of $\pi$ that contains $k$ edge-disjoint 1-factors.
Hartke and Seacrest subsequently improved upon conditions (2) and (3) in [25]. Aside from these results, however, Conjecture 3 remains open. In [20], the proof of the \( k = 3 \) case utilized the Gallai-Edmonds Decomposition [26, 27, 28], and it seems quite feasible that other powerful tools from matching theory could prove useful when approaching this tantalizing conjecture.

4 Degree sequences of uniform hypergraphs

A hypergraph \( H \) is \( k \)-uniform, or is a \( k \)-graph, if every edge contains \( k \) vertices. A \( k \)-uniform hypergraph is simple if there are no repeated edges. Thus, a simple 2-uniform hypergraph is a simple graph. If \( \pi = (d_1, \ldots, d_n) \) is the degree sequence of a simple \( k \)-graph \( H \), we say \( \pi \) is \( k \)-graphic, and that \( H \) is a \( k \)-realization of \( \pi \). When \( k = 2 \), we will retain the terminology from elsewhere in this paper and say that \( \pi \) is graphic and that \( H \) is a realization of \( \pi \).

While the body of knowledge pertaining to degree sequences of 2-graphs is extensive, relatively little is known about the degree sequences of uniform hypergraphs. As an example, the only known characterization of \( k \)-graphic sequences is due to Dewdney [29].

**Theorem 4** (Dewdney 1975). Let \( \pi = (d_1, \ldots, d_n) \) be a nonincreasing sequence of non-negative integers. \( \pi \) is \( k \)-graphic if and only if there exists a nonincreasing sequence \( \pi' = (d'_2, \ldots, d'_n) \) of nonnegative integers such that

1. \( \pi' \) is \((k-1)\)-graphic,
2. \( \sum_{i=2}^n d'_i = (k-1)d_1 \), and
3. \( \pi'' = (d_2 - d'_2, d_3 - d'_3, \ldots, d_n - d'_n) \) is \( k \)-graphic.

This characterization, which is similar in most respects to the Havel-Hakimi [3, 4] characterization of graphic sequences, requires one to find a \((k-1)\)-graphic sequence that satisfies condition (3) of the theorem. Unfortunately, there is no apparent way to efficiently search the collection of \((k-1)\)-graphic sequences for an appropriate representative. This observation gives rise to the following general problem.

**Problem 4.** Determine an efficient characterization of \( k \)-graphic sequences for all \( k \geq 3 \).

Of course, the less optimistic (or perhaps more realistic?) reader may posit that no such characterization exists. He or she may then strive to demonstrate that the decision problem associated with Problem 4 is NP-complete.

In addition to Theorem 4, some partial progress has been made towards Problem 4. Bhave, Bam and Deshpande [30] gave an Erdős-Gallai-type characterization of degree sequences of loopless linear hypergraphs (hypergraphs in which a pair of edges share at most one vertex). While interesting in its own right, their result does not directly generalize to Problem 4. Several necessary conditions for a sequence to be 3-graphic have been found (see Achuthan, Achuthan, and Simanihuruk [31], Billington [32], and Choudum [33] for some), although it was shown in [31] that none of these conditions are sufficient.

One feasible direction of study is to determine meaningful sufficient conditions for a sequence to be \( k \)-graphic. Behrens et al. give several sufficient conditions in [34], but it seems that few others exist at this time.
Problem 5. Determine sharp sufficient conditions for a sequence to be $k$-graphic when $k \geq 3$.

In addition to finding meaningful sufficient conditions, one might also explore which types of results on graphic sequences translate to the setting of hypergraphic sequences. For instance, we make the following conjecture that would extend Theorem 1. As with 2-graphs, a $t$-factor of a $k$-uniform hypergraph is a spanning $t$-regular subgraph.

Conjecture 6 (The $t$-factor conjecture for hypergraphic sequences). Let $k \geq 3, t \geq 1$, and let $\pi = (d_1, \ldots, d_n)$ be a $k$-graphic sequence. If $\pi' = (d_1 - t, \ldots, d_n - t)$ is also graphic, then there exists a realization of $\pi$ containing a $t$-factor.

In 1988, Chen [35] gave a short, elegant proof of Kundu’s theorem. While the method employed there might provide guidance towards a proof of Conjecture 6, it is not clear that this is the case at this time.

5 Realizations with Large Complete Minors

Posed by Hugo Hadwiger in 1943, Hadwiger’s conjecture is amongst the most significant and challenging open problems in modern graph theory. This is in part because it represents a deep extension of the 4-color Theorem and other results.

Conjecture 7 (Hadwiger’s Conjecture). Every $k$-chromatic graph contains a $K_k$-minor.

For a graphic sequence $\pi$, let $h(\pi)$ denote the largest $k$ such that some realization of $\pi$ has a $K_k$-minor, and let $\chi(\pi)$ denote the largest chromatic number amongst all realizations of $\pi$. In 2009, Robertson and Song [36] posed the following potential degree sequence variant of Hadwiger’s conjecture.

Conjecture 8. For any graphic sequence $\pi$,

$$h(\pi) \geq \chi(\pi).$$

A stronger form of this conjecture was proved by Dvořák and Mohar in 2011 [37]. Despite this, Conjecture 8 suggests a line of inquiry that we highlight here.

Problem 9. Given a graphic sequence $\pi$,

1. compute $h(\pi)$ exactly, or find reasonable estimates, or

2. determine meaningful sufficient conditions that ensure $h(\pi) \geq k$.

These are clearly broad problems, so we suggest two more specific directions here. Given a fixed graph $H$, there is an $O(n^3)$ algorithm to determine if a graph $G$ of order $n$ contains $H$ as a minor [38]. As a given graphic sequence may have exponentially many realizations, it is not clear that $h(D)$ can be bounded as efficiently.

Problem 10. Does there exist a polynomial-time algorithm to determine if $h(\pi) \geq k$?
To remain in line with [38], we mean an algorithm that is polynomial in \( n \), where \( n \) is the length of the input sequence \( \pi \).

Rao gave the following Erdős-Gallai-type characterization of graphic sequences with a realization containing \( K_k \) as a subgraph. While Rao’s original proof remains unpublished, this result was reproved using network flows by Kézdy and Lehel [39] and algorithmically by Yin [40].

**Theorem 5.** A degree sequence \( d_1 \geq \cdots \geq d_n \) has a realization containing \( K_k \) as a subgraph if and only if \( \sum_{i=1}^{n} d_i \) is even, \( d_k \geq k - 1 \) and, for all \( p \in \{0, \ldots, k\} \) and \( q \in \{0, \ldots, n - k\} \),

\[
\sum_{i=1}^{p} d_i + \sum_{i=p+1}^{k+q} d_i \leq (p+q)(p+q-1) + \sum_{i=p+1}^{k} \min\{p+q,d_i-k+1+p\} + \sum_{i=k+q+1}^{n} \min\{p+q,d_i\}.
\]

Rao’s result inspires the following problem.

**Problem 11.** Find a Rao-type characterization of degree sequences \( \pi \) with \( h(\pi) \geq k \).

# 6 Ramsey-type Numbers for Degree Sequences

Given graphs \( G_1 \) and \( G_2 \), the **graph Ramsey number** \( r(G_1, G_2) \) is the minimum integer \( n \) such that for every graph \( G \) of order \( n \), either \( G_1 \subseteq G \) or \( G_2 \subseteq G^c \). Exact values of \( r(G_1, G_2) \) are known for limited collections of graphs (for a thorough survey see [41]).

In this section, we consider a Ramsey-type parameter for degree sequences that was first introduced in [42]. For a graphic sequence \( \pi = (d_1, \ldots, d_n) \), write \( \pi \to (G_1, G_2) \) if either \( \pi \) is potentially \( G_1 \)-graphic or \( \pi \) is potentially \( G_2 \)-graphic, where \( \pi = ((n-1)-d_1, \ldots, (n-1)-d_n) \).

The **potential-Ramsey number**, \( r_{\text{pot}}(G_1, G_2) \), is the smallest integer \( n \) such that for every \( n \)-term graphic sequence \( \pi \), we have \( \pi \to (G_1, G_2) \). Observe that for \( G_1 \) and \( G_2 \), \( r_{\text{pot}}(G_1, G_2) \leq r(G_1, G_2) \) as the realizations of any \( n \)-term graphic sequence implicitly determine 2-colorings of \( E(K_n) \). This bound is sharp in a number of instances (c.f. [43]), in particular when there exists some graph \( G \) of order \( r(G_1, G_2) - 1 \) such that \( G_1 \not\subseteq G \), \( G_2 \not\subseteq G^c \) and \( G \) is the unique realization of its degree sequence.

The potential-Ramsey numbers \( r_{\text{pot}}(K_t, K_n) \), \( r_{\text{pot}}(K_t, C_n) \) and \( r_{\text{pot}}(K_t, P_n) \) were calculated in [42]. More recently, Erbes has calculated \( r_{\text{pot}}(G_1, H) \) when \( |G_1| \leq 4 \) and \( H \) is arbitrary [44]. It would be interesting to obtain more specific results, as a pathway to building a more robust theory.

**Problem 12.** Determine \( r_{\text{pot}}(G_1, G_2) \) for “interesting” choices of \( G_1 \) and \( G_2 \).

Interestingly, while the problem of calculating \( r(K_n, K_t) \) is widely accepted as one of the most difficult problems in all of combinatorics, determining \( r_{\text{pot}}(K_n, K_t) \) is relatively straightforward. On the other hand, determining \( r_{\text{pot}}(K_t, C_n) \) is comparatively complex.

In line with this observation, it seems to be much more difficult to develop tools for approaching \( r_{\text{pot}}(G_1, G_2) \) when one of \( G_1 \) or \( G_2 \) is sparse. This is even the case when the sparse graph has high uniformity of structure, as is the case with \( C_n \). As such, we pose the following specific case of Problem 12, which is inspired by Chvátal’s classical determination of clique-tree Ramsey numbers [45].
Problem 13. Determine \( r_{pot}(T_m, K_t) \) for an arbitrary tree \( T_m \) of order \( m \) and any \( t \geq 3 \).

6.1 Multicolor Potential-Ramsey Theory

The work described earlier in this section can be thought of as 2-color potential-Ramsey theory. However, as in the classical setting it is possible to formulate a multicolor version of potential-Ramsey numbers. This formulation is tied to the notion of graphic sequence packing discussed in Section 2.

Define \( r_{pot}(G_1, \ldots, G_k) \) to be the smallest integer \( n \) such that for any collection of \( n \)-term graphic sequences \( \pi_1, \ldots, \pi_k \) that sum termwise to \( n - 1 \) and pack, there exist edge disjoint graphs \( F_1, \ldots, F_k \), all with vertex set \( \{v_1, \ldots, v_n\} \), such that \( d_{F_i}(v_j) = d_j^{(i)} \) for all \( i, j \) and also that \( F_i \) contains \( G_i \) as a subgraph for some \( i \). As is the case with the two-color version, the multicolor potential-Ramsey numbers are bounded from above by the traditional multicolor graph Ramsey numbers, but the added challenges inherent in degree sequence packing would seem to make the determination of \( r_{pot}(G_1, \ldots, G_k) \) for \( k \geq 3 \) both interesting and significantly more difficult than the \( k = 2 \) case. While any progress towards understanding multicolor potential-Ramsey numbers would be notable, the following problem is a feasible starting point.

Problem 14. Determine \( r_{pot}(K_3, K_3, K_3) \).

It is worthwhile to note that by [46], \( r_{pot}(K_3, K_3, K_3) \leq r(K_3, K_3, K_3) = 17 \). As is frequently the case with Ramsey-type problems, this tantalizingly small range of possible values for \( r_{pot}(K_3, K_3, K_3) \) does not necessarily imply that Problem 14 is nontrivial.

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