Majorization-based convergence rate bounds for subspace iterations and the block Lanczos method

Merico Argentati (speaker) and Andrew Knyazev

Department of Mathematical & Statistical Sciences

University of Colorado Denver

Supported by the National Science Foundation

10th Copper Mountain Conference on Iterative Methods, 2008

April 9, 2008

Outline

- The Rayleigh–Ritz method, principal angles between subspaces $\mathcal{X}$ and $\mathcal{Y}$, majorization
- Weak majorization bounds on the change in Ritz values
- Bound on principal angles between $\mathcal{X}$ and $F\mathcal{Y}$ in terms of the principal angles between $\mathcal{X}$ and $\mathcal{Y}$ where F is an operator
- Subspace iteration and weak majorization bounds on Ritz values
- Convergence rate bounds for the block Lanczos method
- Conclusions

The Rayleigh–Ritz Method

- Let A be a Hermitian matrix and $\mathcal{X}$ a subspace
- We define an operator $P_{\mathcal{X}} A|_{\mathcal{X}}$ on $\mathcal{X}$, where $P_{\mathcal{X}}$ is the orthogonal projection onto $\mathcal{X}$ and $P_{\mathcal{X}} A|_{\mathcal{X}}$ denotes the restriction of $P_{\mathcal{X}} A$ to $\mathcal{X}$, as discussed in Parlett [1998]. The eigenvalues $\Lambda(P_{\mathcal{X}} A|_{\mathcal{X}})$ are called Ritz values.
- We have $\Lambda(P_{\mathcal{X}} A|_{\mathcal{X}}) = \Lambda(X^H A X)$ where X is a matrix with orthonormal columns that span $\mathcal{X}$
- Nonzero Ritz values are the nonzero eigenvalues of $P_{\mathcal{X}} A P_{\mathcal{X}}$

Bounding Eigenvalues

If A is a Hermitian matrix, we have Weyl's theorem

$$\max_{j=1,\ldots,n} |\lambda_j(A) - \lambda_j(B)| \le \|A - B\|, \tag{1}$$

and by an analog of the Hoffman–Wielandt theorem, e.g. Stewart and Sun [1990], we have

$$\sqrt{\sum_{j=1}^{n} (\lambda_j(A) - \lambda_j(B))^2} \le \|A - B\|_F \tag{2}$$

How do we bound Ritz values when we vary/perturb the subspace?

Changes in the Trial Subspace in the Rayleigh–Ritz Method

- We can vary a Hermitian matrix and see how the eigenvalues change
- Analogously for a fixed Hermitian matrix, we can vary the subspace and see how the Ritz values change. If a subspace $\mathcal{X}$ is perturbed to give rise to another subspace $\mathcal{Y}$, then we can ask about how to bound

$$|\Lambda(P_{\mathcal{X}}A|_{\mathcal{X}}) - \Lambda(P_{\mathcal{Y}}A|_{\mathcal{Y}})| \ ?$$

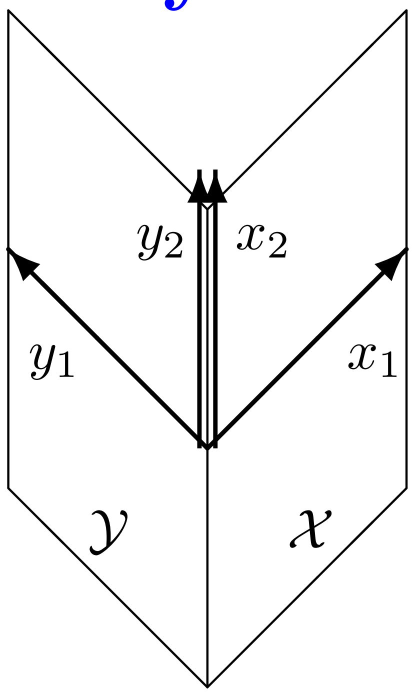
- Answer: We can prove some very flexible and useful bounds using principal angles between subspaces and majorization, also considering the case when $\mathcal{X}$ or $\mathcal{Y}$ may be invariant relative to A.

Principal Angles Between Subspaces

- Let subspaces $\mathcal{X}$ and $\mathcal{Y} \subseteq \mathbf{C^n}$ with dim$\mathcal{X}$ = dim$\mathcal{Y}$, and orthonormal bases given by the columns of the matrices X and Y. The principal angles between $\mathcal{X}$ and $\mathcal{Y}$ arranged in descending order are given by $\Theta(\mathcal{X}, \mathcal{Y}) = \Theta^{\downarrow}(\mathcal{X}, \mathcal{Y})$, and defined using $\cos\Theta(\mathcal{X}, \mathcal{Y}) = S^{\uparrow}(X^H Y)$
- Definition is symmetric: $\Theta(\mathcal{X}, \mathcal{Y}) = \Theta(\mathcal{Y}, \mathcal{X})$.
- Definition of the distance between subspaces: $\text{gap}(\mathcal{X}, \mathcal{Y}) = \|P_{\mathcal{X}} - P_{\mathcal{Y}}\|_2 = \sin(\Theta_{\max}(\mathcal{X}, \mathcal{Y}))$.

Pioneering results using angles between subspaces in the framework of unitarily invariant norms and symmetric gauge functions, equivalent to majorization, appear in Davis and Kahan [1970], which introduces many of the tools that we use here.

The Geometry of Principal Angles



$$\theta_1 \geq \ldots \geq \theta_m$$

The cosines can be defined recursively for $k = m, \ldots, 1$ by
$\cos(\theta_k) = \max_{x\, \in \mathcal{X}} \max_{y\, \in \mathcal{Y}} x^T y \;=\; x_k^T y_k$ subject to
$\|x\| = \|y\| = 1, \quad x^T x_i = 0, \quad y^T y_i = 0, \quad i = m, \ldots, k+1.$

Majorization and Basic Notation

- For vector $x = [x_1, \ldots, x_n]$, we use $x^{\downarrow} \equiv [x_1^{\downarrow}, \ldots, x_n^{\downarrow}]$ to denote x with its elements rearranged in descending order, while $x^{\uparrow} \equiv [x_1^{\uparrow}, \ldots, x_n^{\uparrow}]$ denotes x with its elements rearranged in ascending order. $|x|$ denotes the vector x with the absolute value of its components.

- We say that $x \in \mathcal{R}^n$ is weakly majorized by $y \in \mathcal{R}^n$, written $x \prec_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \qquad 1 \leq k \leq n, \tag{3}$$

while x is (strongly) majorized by y, written $x \prec y$, if (3) holds together with

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{4}$$

Why is Majorization Important?

- The notion of majorization plays an important role in mathematics, statistics and economics
- The following two conditions are equivalent: $x \prec_w y$ and $\sum_{i=1}^n \phi(x_i) \le \sum_{i=1}^n \phi(y_i)$ for all nondecreasing convex functions ϕ. This implies inequalities for unitarily invariant norms and certain p-norms
- For Hermitian matrices A and B let $\Lambda(A)$ and $S(A)$ be, respectively, the vector of eigenvalues and singular values (both nonincreasing), then

$$\text{diag}\,(A) \prec \Lambda(A), \qquad \text{(Schur's Theorem)} \tag{5}$$

$$\Lambda(A) - \Lambda(B) \prec \Lambda(A - B) \quad \text{and} \quad |\Lambda(A) - \Lambda(B)| \prec_w S(A - B) \tag{6}$$

See Lidskii [1950], Mirsky [1960], Marshall and Olkin [1979], Horn and Johnson [1999].

Outline

- The Rayleigh–Ritz method, principal angles between subspaces $\mathcal{X}$ and $\mathcal{Y}$, majorization
- Weak majorization bounds on the change in Ritz values
- Bound on principal angles between $\mathcal{X}$ and $F\mathcal{Y}$ in terms of the principal angles between $\mathcal{X}$ and $\mathcal{Y}$ where F is an operator
- Subspace iteration and weak majorization bounds on Ritz values
- Convergence rate bounds for the block Lanczos method
- Conclusions

Changes in the Trial Subspace in the Rayleigh–Ritz Method

The result of (Argentati [2003], Knyazev and Argentati [2006a]):

Theorem 1 *Let A be a Hermitian matrix and let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces with* $\dim\mathcal{X} = \dim\mathcal{Y} = m$. *Then*

$$\max_{j=1,\dots,m} |\lambda_j(P_{\mathcal{X}} A|_{\mathcal{X}}) - \lambda_j(P_{\mathcal{Y}} A|_{\mathcal{Y}})| \leq (\lambda_{\max} - \lambda_{\min}) \ \sin(\Theta_{\max}(\mathcal{X}, \mathcal{Y}))$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of A.

Main Weak Majorization Theorem for the Rayleigh–Ritz Method

Theorem 2 *Let A be a Hermitian matrix and let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces with* $\dim\mathcal{X} = \dim\mathcal{Y}$, *then (see Knyazev and Argentati [2006b])*

$$|\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) - \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)| \prec_w \left(\lambda_{\max} - \lambda_{\min}\right) \sin\Theta(\mathcal{X},\mathcal{Y}), \tag{7}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of the A, respectively. If in addition one of the subspaces is A-invariant then (see conjecture in Argentati, Knyazev, Paige, and Panayotov [2008])

$$|\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) - \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)| \prec_w \left(\lambda_{\max} - \lambda_{\min}\right) \sin^2\Theta(\mathcal{X},\mathcal{Y}). \tag{8}$$

Theorem 2 is Sharp

Consider the following example for (8) where $\mathcal{X}$ is A-invariant (see Argentati, Knyazev, Paige, and Panayotov [2008]). Let I be an $m \times m$ identity matrix and let $A = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Let Σ be a diagonal of given cosines and let Γ be the diagonal of sines, so $\Sigma^2 + \Gamma^2 = I$. Let $X = \begin{bmatrix} I \\ 0 \end{bmatrix}$, and $Y = \begin{bmatrix} \Sigma \\ \Gamma \end{bmatrix}$. Then $X^H AX = I$ and $Y^H AY = 2\Sigma^2 - I$ and

$$|\Lambda(X^H AX) - \Lambda(Y^H AY)|^{\downarrow} = 2[\sin^2 \Theta(\mathcal{X}, \mathcal{Y})] = (\lambda_{\max} - \lambda_{\min})[\sin^2 \Theta(\mathcal{X}, \mathcal{Y})].$$

Weak Majorization Implications of

$$|\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) - \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)| \prec_w \left(\lambda_{\max} - \lambda_{\min}\right) \; \sin^2\Theta(\mathcal{X},\mathcal{Y})$$

The implications of weak majorization may not be obvious. Let $\dim\mathcal{X} = \dim\mathcal{Y} = m$, and let $\alpha_1 \geq \cdots \geq \alpha_m$ be the Ritz values of A with respect to $\mathcal{X}$ and $\beta_1 \geq \cdots \geq \beta_m$ be the Ritz values of A with respect to $\mathcal{Y}$. Then using (8) we have

$$\sum_{i=1}^{k} |\alpha_i - \beta_i|^{\downarrow} \leq (\lambda_{\max} - \lambda_{\min}) \sum_{i=1}^{k} \sin^2(\Theta_i(\mathcal{X},\mathcal{Y})), \qquad k = 1, \ldots, m.$$

For $k = 1$ we have

$$\begin{aligned} \max_{j=1,\ldots,m}|\alpha_j - \beta_j| &\leq (\lambda_{\max} - \lambda_{\min}) \sin^2(\Theta_{\max}(\mathcal{X},\mathcal{Y})) \\ &= (\lambda_{\max} - \lambda_{\min}) \mathrm{gap}^2(\mathcal{X},\mathcal{Y}), \end{aligned}$$

Weak Majorization Implications (Cont.)

$$\left|\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right)-\Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)\right|\prec_w\left(\lambda_{\max}-\lambda_{\min}\right)\ \sin^2\Theta(\mathcal{X},\mathcal{Y}),$$

and for $k=m$ we obtain

$$\sum_{i=1}^{m}|\alpha_i-\beta_i|\leq(\lambda_{\max}-\lambda_{\min})\sum_{i=1}^{m}\sin^2(\Theta_i(\mathcal{X},\mathcal{Y}))$$

We have in general

$$\left(\sum_{i=1}^{m}|\alpha_i-\beta_i|^p\right)^{\frac{1}{p}}\leq(\lambda_{\max}-\lambda_{\min})\left(\sum_{i=1}^{m}\sin^{2p}(\Theta_i(\mathcal{X},\mathcal{Y}))\right)^{\frac{1}{p}},\quad 1\leq p<\infty.$$

Bounding Eigenvalues (Cont.)

If A is a Hermitian matrix, we have Weyl's theorem

$$\max_{j=1,\ldots,n}|\lambda_j(A) - \lambda_j(B)| \le \|A - B\|, \tag{9}$$

and by an analog of the Hoffman–Wielandt theorem, e.g. Stewart and Sun [1990], we have

$$\sqrt{\sum_{j=1}^{n}(\lambda_j(A) - \lambda_j(B))^2} \le \|A - B\|_F \tag{10}$$

Note that the above formulas are two of a large class of inequalities that follow from

$$|\Lambda(A) - \Lambda(B)| \prec_w S(A - B)$$

Matlab Example for Many Zero Angles

Let $n = 5$ and $\dim\mathcal{X} = \dim\mathcal{Y} = 4$

$$A = \mathrm{diag}([1\ \ .9\ \ .7\ \ .5\ \ 0]) \quad X = \mathrm{eye}(n, m) \quad Y = \mathrm{orth}(\mathrm{rand}(n, m)).$$

Then $\mathcal{X}$ is A-invariant and

$$\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) = [1.0000\ \ 0.9000\ \ 0.7000\ \ 0.5000],$$

$$\Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right) = [0.9452\ \ 0.7484\ \ 0.6572\ \ 0.0330].$$

Then

$$|\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) - \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)|^{\downarrow} = [0.4670\ \ 0.1516\ \ 0.0548\ \ 0.0428]$$

$$\prec_w \left(\lambda_{\max} - \lambda_{\min}\right)\ \sin^2\Theta(\mathcal{X}, \mathcal{Y}) = [0.9513\ \ 0\ \ 0\ \ 0].$$

Conclusion: Zero angles do not imply that $\Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)$ includes eigenvalues of A.

Outline

- The Rayleigh–Ritz method, principal angles between subspaces $\mathcal{X}$ and $\mathcal{Y}$, majorization
- Weak majorization bounds on the change in Ritz values
- Bound on principal angles between $\mathcal{X}$ and $F\mathcal{Y}$ in terms of the principal angles between $\mathcal{X}$ and $\mathcal{Y}$ where F is an operator
- Subspace iteration and weak majorization bounds on Ritz values
- Convergence rate bounds for the block Lanczos method
- Conclusions

Subspace Iterations: FY as a Possibly Improved Approximation of X

Theorem 3 *Let* $\dim\mathcal{X} = \dim\mathcal{Y}$ *and assume* $\Theta(\mathcal{X},\mathcal{Y}) < \pi/2$. *Let* F *be invariant on both* $\mathcal{X}$ *and on its orthogonal complement* $\mathcal{X}^\perp$, *and assume that* $(P_{\mathcal{X}}F)\,|_{\mathcal{X}}$ *is invertible. Then* $\dim(F\mathcal{Y}) = \dim\mathcal{Y}$ *and*

$$\tan\Theta(\mathcal{X},F\mathcal{Y}) \prec_w S\left(((P_{\mathcal{X}}F)\,|_{\mathcal{X}})^{-1}\right) S\left((P_{\mathcal{X}^\perp}F)\,|_{\mathcal{X}^\perp}\right)\tan\Theta(\mathcal{X},\mathcal{Y}).$$

We can also prove the stronger form

$$\log\left(\frac{\tan\Theta(\mathcal{X},F\mathcal{Y})}{\tan\Theta(\mathcal{X},\mathcal{Y})}\right) \prec_w \log\left(S\left(((P_{\mathcal{X}}F)\,|_{\mathcal{X}})^{-1}\right) S\left((P_{\mathcal{X}^\perp}F)\,|_{\mathcal{X}^\perp}\right)\right).$$

Subspace Iterations and Ritz Values

Theorem 4 *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be subspaces of* $\mathcal{H}$ *such that* $\dim\mathcal{X} = \dim\mathcal{Y}$ *and* $\Theta(\mathcal{X}, \mathcal{Y}) < \pi/2$*. Let the operator* A *be Hermitian, and let* $\mathcal{X}$ *be an* A*-invariant subspace. Let* $F = f(A)$ *and* $f\left(\Lambda\left((P_{\mathcal{X}}A)\,|_{\mathcal{X}}\right)\right) \neq 0$*. Then* $\dim(F\mathcal{Y}) = \dim\mathcal{Y}$ *and*

$$
\begin{aligned}
&\left|\Lambda\left((P_{\mathcal{X}}A)\,|_{\mathcal{X}}\right) - \Lambda\left((P_{F\mathcal{Y}}A)\,|_{F\mathcal{Y}}\right)\right| \\
&\qquad\qquad \prec_w \left(\lambda_{\max} - \lambda_{\min}\right) \left(\frac{\left|f\left(\Lambda\left((P_{\mathcal{X}^\perp}A)\,|_{\mathcal{X}^\perp}\right)\right)\right|^{\downarrow}}{\left|f\left(\Lambda\left((P_{\mathcal{X}}A)\,|_{\mathcal{X}}\right)\right)\right|^{\uparrow}}\right)^2 \tan^2\Theta(\mathcal{X}, \mathcal{Y}).
\end{aligned}
$$

Subspace Iterations and Ritz Values

Example: $f(A) = A^k$

Let the A-invariant subspace X correspond to the contiguous set of the largest eigenvalues of A. Let

$$\begin{aligned}\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) &= [\lambda_1 \geq \ldots \geq \lambda_m] \quad \text{top } m \text{ eigenvalues of } A \\ \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right) &= [\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_m].\end{aligned}$$

Then $\lambda_i \geq \tilde{\lambda}_i,\ i = 1, \ldots, m$ and

$$\left[\lambda_1 - \tilde{\lambda}_1, \ldots, \lambda_m - \tilde{\lambda}_m\right]$$

$$\prec_w (\lambda_{\max} - \lambda_{\min}) \left[\left(\frac{\lambda_{m+1}}{\lambda_m}\right)^{2k} \tan^2\theta_1(\mathcal{X}, \mathcal{Y}), \ldots, \left(\frac{\lambda_{2m}}{\lambda_1}\right)^{2k} \tan^2\theta_m(\mathcal{X}, \mathcal{Y})\right]$$

Outline

- The Rayleigh–Ritz method, principal angles between subspaces $\mathcal{X}$ and $\mathcal{Y}$, majorization
- Weak majorization bounds on the change in Ritz values
- Bound on principal angles between $\mathcal{X}$ and $F\mathcal{Y}$ in terms of the principal angles between $\mathcal{X}$ and $\mathcal{Y}$ where F is an operator
- Subspace iteration and weak majorization bounds on Ritz values
- Convergence rate bounds for the block Lanczos method
- Conclusions

Convergence Rate Bounds of the Block Lanczos Method

Theorem 5 *Let* $\dim\mathcal{X} = \dim\mathcal{X}_0 = m$*, operator* A *be Hermitian, and let the* A*-invariant subspace* X *correspond to the contiguous set of the largest eigenvalues of* A*. Let* $\mathcal{Y} = \mathcal{X}_0 + A\mathcal{X}_0 + \cdots + A^k\mathcal{X}_0$*, then*

$$0 \leq \frac{\Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) - \Lambda_{\dim\mathcal{X}}\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right)}{\Lambda_{\dim\mathcal{X}}\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right) - \lambda_{\min}} \prec_w [\sigma_m^*, \ldots, \sigma_1^*] \tan^2\Theta(\mathcal{X}, \mathcal{X}_0).$$

where

$$\sigma_i^* = \left| T_k\left(\frac{\lambda_{m+1} + \lambda_{\min} - 2\lambda_i}{\lambda_{m+1} - \lambda_{\min}}\right)\right|^{-2} \quad i = 1, \ldots, m,$$

and where T_k *is the* k*th Chebyshev polynomial of the first kind.*

See, e.g., Saad [1980], Knyazev [1987].

Convergence Rate Bounds of the Block Lanczos Method (Cont.)

Here is a more concrete interpretation of Theorem 5. Let

$$\begin{array}{rcll} \Lambda\left((P_{\mathcal{X}}A)|_{\mathcal{X}}\right) & = & [\lambda_1 \geq \ldots \geq \lambda_m] & \text{top } m \text{ eigenvalues of } A \\ \Lambda_{\dim \mathcal{X}}\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right) & = & [\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_m] & \text{top } m \text{ Ritz values of } \Lambda\left((P_{\mathcal{Y}}A)|_{\mathcal{Y}}\right) \end{array}$$

Then

$$\left[\left(\frac{\lambda_1 - \tilde{\lambda}_1}{\tilde{\lambda}_1 - \lambda_{\min}}\right), \ldots, \left(\frac{\lambda_m - \tilde{\lambda}_m}{\tilde{\lambda}_m - \lambda_{\min}}\right)\right]$$

$$\prec_w \left[\sigma_m^* \tan^2 \theta_1(\mathcal{X}, \mathcal{X}_0), \ldots, \sigma_1^* \tan^2 \theta_m(\mathcal{X}, \mathcal{X}_0)\right]$$

Conclusions

- We can prove some very flexible and useful bounds for Ritz values using principal angles and majorization
- The absolute value of the difference of Ritz values for a Hermitian matrix are majorized by $\lambda_{\text{max}} - \lambda_{\text{min}}$ times the sines of the angles between the perturbed subspaces, with majorization by the squares of the sines (see conjecture in [2008]) if one of the subspaces is A-invariant
- We can characterize a possible improved approximation to X by applying a function F to Y, in terms of singular values and the tangent of principal angles
- Using this general approach we obtain a useful characterization of the eigenvalue error for the Block Lanczos Method

References

M. E. Argentati, A. V. Knyazev, C. C. Paige, and I. Panayotov. Bounds on changes in Ritz values for a perturbed invariant subspace of a Hermitian matrix. *SIAM J. Matrix Anal. Appl. Accepted*, 2008.

Merico E. Argentati. *Principal Angles Between Subspaces as Related to Rayleigh Quotient and Rayleigh Ritz Inequalities with Applications to Eigenvalue Accuracy and an Eigenvalue Solver*. PhD thesis, University of Colorado at Denver, 2003.

C. Davis and W. M. Kahan. The rotation of eigenvectors by a perturbation. III. *SIAM Journal on Numerical Analysis*, 7(1):1–46, 1970.

R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, New York, NY, 1999.

A. V. Knyazev. Convergence rate estimates for iterative methods for mesh symmetric eigenvalue problem. *Soviet J. Numerical Analysis and Math. Modelling*, 2(5):371–396, 1987.

A. V. Knyazev and M. E. Argentati. On proximity of Rayleigh quotients for different vectors and Ritz values generated by different trial subspaces. *Linear Algebra and its Applications*, 415:82–95, 2006a.

A. V. Knyazev and M. E. Argentati. Majorization for changes in angles between subspaces, Ritz values, and graph Laplacian spectra. *SIAM J. Matr. Anal. Appl.*, 29:15–32, 2006b.

V. B. Lidskii. On the proper values of a sum and product of symmetric matrices. *Dokl. Akad. Nauk SSSR*, 75:769–772, 1950.

A. W. Marshall and I. Olkin. *Inequalities: theory of majorization and its applications*, volume 143 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979. ISBN 0-12-473750-1.

L. Mirsky. Symmetric gauge functions and unitarily invariant norms. *Quarterly Journal of Mathematics. Oxford Series(2)*, 11:50–59, 1960.

Beresford N. Parlett. *The Symmetric Eigenvalue Problem*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1998. ISBN 0-89871-402-8. Corrected reprint of the 1980 original.

Y. Saad. On the rates of convergence of the lanczos and the block–lanczos methods. *SIAM Journal on Numerical Analysis*, 17(5):687–706, 1980.

G. W. Stewart and J. G. Sun. *Matrix perturbation theory*. Academic Press Inc., Boston, MA, 1990. ISBN 0-12-670230-6.

Backup Slides

We Can Replace $\lambda_{\max} - \lambda_{\min}$ With a Reduced Constant

- As in Knyazev and Argentati [2006a, Remark 7], the constant $\lambda_{\max} - \lambda_{\min}$ can be replaced with

$$\max_{x \in \mathcal{X}+\mathcal{Y},\, \|x\|=1} (x, Ax) - \min_{x \in \mathcal{X}+\mathcal{Y},\, \|x\|=1} (x, Ax),$$

  which for some subspaces $\mathcal{X}$ and $\mathcal{Y}$ can provide a significant improvement.

- This effectively replaces A with $P_{(\mathcal{X}+\mathcal{Y})} A P_{(\mathcal{X}+\mathcal{Y})}$, which has the same action on $\mathcal{X}$ and $\mathcal{Y}$.