Finite Groups that Admit Kantor Families

Stanley E. Payne

Abstract. The basic problem of interest in this note is: What finite groups G admit a Kantor family (i.e., 4-gonal family) of subgroups? We begin with a survey of the known examples (of the groups G, not of the Kantor families) and then pose the question of whether or not two specific examples are isomorphic. The two groups in question have order q^5 and coexist for $q = 3^e \geq 27$. Our conjecture had been that they are not isomorphic. During the conference it was announced that indeed they are *not* isomorphic, but the proof will be published elsewhere.

1. What is a Kantor Family?

In 1980 W. M. Kantor [Ka80] found a new recipe for the construction of finite generalized quadrangles (GQ). The chief ingredient of this recipe is a family of subgroups of a given group satisfying a pair of conditions that are readily translated into necessary and sufficient conditions for a certain coset geometry to be a finite GQ. In this note we are not concerned with the GQ per se, but only with the groups that admit these Kantor families, which we now describe.

Let s and t be positive integers greater than 1, and let G be a group of order s^2t. Let $\mathcal{F}$ be a family $\mathcal{F} = \{A_i : 0 \leq i \leq t\}$ of $t+1$ subgroups of G of order s for which the following condition is satisfied:

K1. $A_i \cdot A_j \cap A_k = \{id\}$ whenever i, j, k are distinct, $0 \leq i, j, k \leq t$.

Put $\Omega = \cup(A_i : 0 \leq i \leq t)$. Then for each i, $0 \leq i \leq t$ define a subset A_i^* of G by

$$A_i^* = A_i \ \cup \ (A_i g : g \in G \text{ and } A_i g \cap \Omega = \emptyset)).$$

Put $\mathcal{F}^* = \{A_i^* : 0 \leq i \leq t\}$. It follows that $|A_i^*| = st$ for each i. If each A_i^* is actually a subgroup of G, then $A_i \leq A_i^*$ for all i and the following second condition of Kantor is also satisfied:

K2. $A_i^* \cap A_j = \{id\}$ whenever $i \neq j$, $0 \leq i, j \leq t$.

In this case we say that $(G, \mathcal{F}, \mathcal{F}^*)$ is a *Kantor family*, or briefly, that $(G, \mathcal{F})$ is a *Kantor family* or that $\mathcal{F}$ is *a Kantor family for G*.

From such a Kantor family there is constructed a GQ $\mathcal{S} = GQ(G, \mathcal{F})$ with parameters (s, t) such that G acts as a group of collineations of $\mathcal{S}$ fixing a certain point (∞), fixing each line incident with (∞), and acting regularly on the set of s^2t points not collinear with (∞). These collineations are called *elations about the point* (∞). If a group G admits a Kantor family it will be called an *elation group*, and the associated generalized quadrangle is called an *elation* GQ (EGQ). All of the so-called classical and dual classical GQ arise in this fashion. An excellent exposition of these classical and dual classical GQ as EGQ is given by Kantor in [Ka86], and we do not repeat this presentation here. However, the theory of (finite and infinite) GQ continues to be developed intensely! We mention especially the books [PT84], [VM98] and [KT04]. Many new examples of GQ have been found since the appearance of [Ka86], essentially all using Kantor's construction technique. In this note we are primarily interested in considering the question of which groups can admit Kantor families.

First, if an elation group G is abelian, it must be elementary abelian (and hence a p-group for some prime p), and the associated GQ is called a *translation* GQ (TGQ). The TGQ are very interesting, but they are the subject of other talks at this conference, and the only *known* elementary abelian groups admitting a Kantor family are those that also give the classical TGQ. Hence we restrict our attention to the non-commutative case.

The basic theory of EGQ (and TGQ) was presented in [PT84], but since the appearance of that monograph a great deal of work has appeared. For every known example of an EGQ with parameters s and t, both s and t are powers of the same prime. Frohardt [Fr88] showed that if G is an elation group with $1 < s \leq t$, then s and t are powers of the same prime. Also, he showed that if s is a power of a prime p, then G is a p-group. And even if s is not a prime power, it must be divisible by at most two distinct primes. In the sequel [CF93], Chen and Frohardt showed that if two members of the Kantor family are normal in G then again G is a p-group for some prime p. Later D. Hachenberger [Ha94] generalized this to the following: Let G be an EGQ for which even one member A of the Kantor family is normal in G. Then s and t are powers of the same prime p, and necessarily one of the following holds: G is elementary abelian, or p is odd and G/A is nonabelian and has exponent p.

In the construction of the EGQ $\mathcal{S}$ from the group G with a Kantor family there is a special point usually denoted (∞). When G contains a full group of order t of *symmetries about* (∞) (i.e., collineations that fix each point collinear with (∞)), then $\mathcal{S}$ is called a *skew translation generalized quadrangle* (STGQ). In a preprint circulated around 1990 X. Chen [Ch90] showed that each (STGQ) must have s and t powers of the same prime. We do not know whether or not this preprint was ever published, and we cannot find any record of what happened to X. Chen himself. In the meantime, D. Hachenberger [Ha96] has published a proof of this result. Indeed, D. Hachenberger has obtained a number of other interesting results giving restrictions on the groups that admit Kantor families.

2. The Known Non-abelian Elation Groups

The group G appearing most frequently as an elation group (in the many papers dealing with flock GQ of order (q^2, q)) is the following, where q is any prime power and F is the Galois field with q elements. For $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in F^2$, $\alpha \cdot \beta = a_1b_1 + a_2b_2$.

Example 1. $G = \{(\alpha, c, \beta) : \alpha, \beta \in F^2, c \in F\}$ with binary operation

$$(\alpha, c, \beta) \circ (\alpha', c', \beta') = (\alpha + \beta, c + c' + \beta \cdot \alpha', \beta + \beta').$$

Several examples of Kantor families exist for this group.

By a careful choice of subgroup G' of G having order q^3 we get the usual elation group and the Kantor family yielding an EGQ with parameters (q, q). This appears as follows.

Example 2. Let $G = \{(a, c, b) : a, b, c \in F]\}$ with binary operation

$$(a, c, b) \circ (a', c', b') = (a + a', c + c' + ba', b + b').$$

When q is odd, the only Kantor family known for this group is the classical one giving the symplectic geometry $W(q)$. However, when q is a power of 2, although the only elation group known is the classical one, there are nonclassical Kantor families for it.

Exotic Example 3. When $q = 2^e$ in Example 1 and the Kantor family is the classical one so that the EGQ is isomorphic to the Hermitian geometry $H(3, q^2)$, there is a larger group S of collineations of the EGQ containing $q^2 - 1$ other elation groups of order q^5, each two of which are conjugate in S but which are not isomorphic to the classical group G. Our student Rob Rostermundt has been studying this situation (with inspiration from Tim Penttila) and has shown that although the elation group is nonstandard, the only Kantor family it admits is the classical one. We give his description of this group.

Let $q = 2^e$ and $F_q = GF(q)$. Put $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and for $\alpha, \beta \in F_q^2$ define

$$\alpha \circ \beta := \alpha P \beta^T.$$

Then $(\alpha, \beta) \mapsto \alpha \circ \beta$ is a non-singular, alternating, bilinear form with

$$\alpha \circ \beta = 0 \iff \{\alpha, \beta\} \text{ is } F_q - \text{linearly dependent.}$$

In particular, $\alpha \circ \alpha = 0$ for all $\alpha \in F_q^2$. On the set $G^{\otimes} = F_q^2 \times F_q^2 \times F_q = \{(\alpha, \beta, c) : \alpha, \beta \in F_q^2, c \in F_q\}$ define the binary operation

$$(\alpha, \beta, c) \circ (\alpha', \beta', c') := (\alpha + \alpha', \beta + \beta', c + c' + \beta \circ \alpha').$$

This makes $G^{\otimes}$ into a group of order q^5 with center

$$Z = \{(\bar{0}, \bar{0}, c) \in G^{\otimes} : c \in F_q\}.$$

Fix a $\delta \in F_q$ such that $tr(\delta) = 1$. Put $A_1 = \begin{pmatrix} \delta & 1 \\ 0 & \delta \end{pmatrix}$, and for general $t \in F_q$ put $A_t = t^{\frac{1}{2}} A_1$. Then define subgroups as follows:

$$\text{For } t \in F_q,\ A(t) := \{(\alpha, t^{1/2}\alpha, \alpha A_t \alpha^T) : \alpha \in F_q^2\},$$

and

$$A^*(t) := \{(\alpha, t^{1/2}\alpha, c) : \alpha \in F_q^2\}.$$

Also put

$$A(\infty) := \{(\bar{0}, \beta, 0) \in G^{\otimes} : \beta \in F_q^2\},$$

and

$$A^*(\infty) := \{(\bar{0}, \beta, c) \in G^{\otimes} : \beta \in F_q^2, c \in F_q\}.$$

Let $\tilde{F} := F_q \cup \{\infty\}$, and put $\mathcal{F} := \{A(t) : t \in \tilde{F}\}$, $\mathcal{F}^* := A^*(t) : t \in \tilde{F}\}$. It is well-known that $(G^{\otimes}, \mathcal{F}, \mathcal{F}^*)$ is a Kantor family and that the associated GQ is isomorphic to $H(3, q^2)$. For a thorough treatment of this general material see Cardinali and Payne [CP02].

Recall that right multiplication by an element of $G^{\otimes}$ is an elation about the point (∞) in $GQ(G^{\otimes}, \mathcal{F}, \mathcal{F}^*)$. Let $\pi(\alpha', \beta', c')$ be the map such that

$$\begin{aligned} (\alpha, \beta, c)^{\pi(\alpha',\beta',c')} &= (\alpha, \beta, c) \cdot (\alpha', \beta', c') \\ &= (\alpha + \alpha', \beta + \beta', c + c' + \beta \circ \alpha'). \end{aligned}$$

So we may define the "usual" group of elations of $H(3, q^2)$ about the point (∞) to be

$$\bar{G} = \{\pi(\alpha, \beta, c) : (\alpha, \beta, c) \in G^{\otimes}\}.$$

Define the map

$$\phi : (\alpha, \beta, c) \mapsto (\alpha P, \beta P, c).$$

Then ϕ is an involutory collineation of $H(3, q^2)$ which is a whorl about (∞). Moreover, the map ϕ normalizes $\bar{G}$ in the group of all whorls about (∞). Indeed,

$$\phi \circ \pi(\alpha, \beta, c) \circ \phi = (\pi(\alpha, \beta, c))^{\phi} = \pi\left((\alpha, \beta, c)^{\phi}\right).$$

Hence

$$\mathcal{S} = \langle \bar{G}, \phi \rangle = \bar{G} \rtimes \langle \phi \rangle.$$

It is worth noting that the binary operation of $\mathcal{S}$ is the following:

$$\begin{aligned} & \left[\pi(\alpha, \beta, c) \circ \phi^i\right] \cdot \left[\pi(\alpha', \beta', c') \circ \phi^j\right] \\ = \;& \pi\left((\alpha, \beta, c) \cdot (\alpha', \beta', c')^{\phi^i}\right) \circ \phi^{i+j} \\ = \;& (\alpha + \alpha' P^i, \beta + \beta' P^i, c + c' + \beta P^{1+i} (\alpha')^T) \circ \phi^{i+j} \quad . \end{aligned} \tag{2.1}$$

Let W be the entire group of whorls about the point (∞) in $H(3,q^2)$. Since there are q^5 points of $H(3,q^2)$ not collinear with (∞), any group of elations acting regularly on these points must be a 2-group and hence contained in some Sylow 2-subgroup. If we look for an elation group contained in some Sylow 2-subgroup, as all Sylow 2-subgroups are conjugate in W, we may choose any particular Sylow 2-subgroup that is convenient. It turns out that the group $\mathcal{S}$ with $|\mathcal{S}| = 2q^5$ is such a group. So we look for a subgroup $E \leq \mathcal{S}$ of elations about (∞). We want $[\mathcal{S} : E] = 2$ and $E \neq \bar{G}$.

Rostermundt shows that there are q^2 possible elation groups E including $\bar{G}$, and that the $q^2 - 1$ others are all isomorphic to each other. He then observes that $\bar{G}$ has nilpotency class 2, since $[\bar{G}, \bar{G}] = Z(G) \neq \{id\}$. On the other hand he shows that each of the other "exotic" choices for E has nilpotency class 3. We will explore this situation a little more.

2.1. The exotic E.

Let $tr : F_q \to F_2$ be the absolute trace function. Then for $\alpha = (a_1, a_2) \in F_q^2$, put

$$T(\alpha) = tr(a_1) + tr(a_2) \in \{0, 1\}.$$

Rostermundt chose the following exotic elation group for study.

$$E = \left\{ \pi(\alpha, \beta, c) \circ \phi^{T(\alpha)} : \alpha, \beta \in F_q^2, c \in F_q \right\}. \tag{2.2}$$

If $g = \pi(\alpha, \beta, c) \circ \phi^{T(\alpha)}$ and $h = \pi(\alpha', \beta', c') \circ \phi^{T(\alpha')}$, their group product is

$$g \cdot h = \pi \left(\alpha + \alpha' P^{T(\alpha)}, \beta + \beta' P^{T(\alpha)}, c + c' + \beta P^{1+T(\alpha)} (\alpha')^T \right) \circ \phi^{T(\alpha+\alpha')}. \tag{2.3}$$

$$g^{-1} = (\alpha P^{T(\alpha)}, \beta P^{T(\alpha)}, c + \alpha \circ \beta) \circ \phi^{T(\alpha)}. \tag{2.4}$$

$$\begin{aligned}(hg)^{-1} \;&=\; (\alpha P^{T(\alpha)} + \alpha' P^{T(\alpha+\alpha')}, \beta P^{T(\alpha)} + \beta' P^{T(\alpha+\alpha')}, \\ &\quad c + c' + \alpha \circ \beta + \alpha' \circ \beta' + \alpha' P^{1+T(\alpha')} \beta^T) \circ \phi^{T(\alpha+\alpha')}.\end{aligned} \tag{2.5}$$

$$\begin{aligned} &[g, h] = (gh)(hg)^{-1} = \\ &(\alpha(I + P^{T(\alpha')}) + \alpha'(I + P^{T(\alpha)}), \beta(I + P^{T(\alpha')}) + \beta'(I + P^{T(\alpha)}), \\ &\alpha(P + P^{1+T(\alpha')})\beta^T + \alpha(P^{1+T(\alpha+\alpha')})\beta'^T + \\ &+\alpha'(P + P^{1+T(\alpha)} + P^{1+T(\alpha')})\beta^T + \alpha'(P + P^{1+T(\alpha)})\beta'^T). \end{aligned} \tag{2.6}$$

At this stage it is easy to check that the product of two commutators is a commutator, so $[E, E]$ is just the set of commutators. Moreover, the commutators are just the elements of E of the form $((a, a), (b, b), c)$ for which $a, b, c \in F_q$ with

$tr(a) = 0$. Hence

$$|[E, E]| = q^3/2. \qquad (2.7)$$

Moreover, it now follows easily that if h is a commutator, then $[g, h]$ is in the center. If there is such an element that is not zero, then E must have nilpotency class 3. So put $g = ((a, 0), (0, 0), 0)$ and $h = ((0, 0), (b, b), 0)$ with $tr(a) = 1$ and $b \neq 0$. Then h is a commutator and $[g, h] = (\overline{0}, \overline{0}, ab)$, which is not zero. By varying b over the nonzero elements of F_q we see that $[E, [E, E]] = Z$.

Rostermundt goes on to show that any Kantor family in E must be the classical one.

2.2. The Point-Line Dual of $T_3(\Omega)$.

Example 4.

The GQ $T_3(\Omega)$ where Ω is the Tits-ovoid is naturally viewed as a TGQ of order (q, q^2), where q is a power of 2 but not a square. In this section we view it as an EGQ with parameters (q^2, q). The fact that the elation group G admits this one Kantor family at least suggests the possibility that it may admit another one. We give an explicit description of this group G with the hope that this view of it may suggest some other way to construct a Kantor family. However, this approach has not led to anything new so far.

Let $F_q = GF(q)$, $q = 2^e$, e odd. Let $\sigma \in Aut(F_q)$ be chosen so that $\sigma^2 = 2$. Define

$$f : F_q^2 \to F_q : (a, b) \mapsto a^{\sigma+2} + ab + b^{\sigma}.$$

The Tits-ovoid Ω of $\Sigma = PG(3, q)$ is given by

$$\Omega = \{(0, 1, 0, 0)\} \cup \{(1, f(a, b), a, b) : a, b \in F_q\}.$$

The GQ $T_3(\Omega)$ with parameters (q, q^2) is constructed as follows. First embed Σ into $PG(4, q)$ by $(x, y, z, w) \mapsto (0, x, y, z, w)$.

Points of $T_3(\Omega)$ are of three types:

(i) points of $PG(4, q) \setminus \Sigma$;
(ii) solids of $PG(4, q) \setminus \Sigma$ meeting Σ in a plane tangent to Ω;
(iii) a symbol (∞).

Lines of $T_3(\Omega)$ are of two types:

(a) lines of $PG(4, q)$ meeting Σ in a point of Ω.
(b) points of Ω.

Incidence in $T_3(\Omega)$ is defined by the following:

The point (∞) is incident with the $1 + q^2$ lines of type (b). Suppose $\triangle$ is a solid of $PG(4, q)$ meeting Σ in the plane T_p tangent to Ω at the point p. Then $\triangle$

is incident with p (as a line of type (b)) and with the q^2 lines of $\triangle$ not in Σ. The point x of $PG(4,q) \setminus \Sigma$ is incident with the $1+q^2$ lines px, $p \in \Omega$.

This construction gives $T_3(\Omega)$ as a translation GQ (TGQ) of order (q, q^2) whose point-line dual $T_3(\Omega)\hat{}$ is an elation GQ (EGQ) of order (q^2, q).

Define $\theta(a,b,c,d,e) : (u,x,y,z,w) \mapsto (u,x,y,z,w)[a,b,c,d,e]$, where

$$[a,b,c,d,e] = \begin{pmatrix} 1 & 0 & c & d & e \\ 0 & 1 & f(a,b) & a & b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a^{\sigma+1}+b & 1 & a^{\sigma} \\ 0 & 0 & a & 0 & 1 \end{pmatrix}; \quad \text{for } a,b,c,d,e \in F_q.$$

A routine check shows that $G = \{\theta(a,b,c,d,e) : a,b,c,d,e \in F_q\}$ is a group of order q^5 with binary operation

$$[a,b,c,d,e] \cdot [a',b',c',d',e'] =$$
$$[a+a', b+b'+a\cdot a'^{\sigma}, c+c'+d(a'^{\sigma+1}+b')+ea', d+d', e+e'+da'^{\sigma}].$$

Moreover, G leaves Ω invariant. In fact, G fixes the "line" $P = (0,0,1,0,0) \in \Omega$ and is sharply transitive on the q^5 lines of $T_3(\Omega)$ not concurrent with P (i.e., the q^5 lines of $PG(4,q)$ meting Σ at a point of Ω different from P).

Let Q be the "line" $Q = (0,1,0,0,0) \in \Omega$. The plane $T_Q = \{(0,z_1,0,z_3,z_4) : z_1,z_3,z_4 \in F_q\}$ is the plane of Σ tangent to Ω at Q. Let L_∞ be the line joining the point $(1,0,0,0,0)$ with the "point" T_Q, literally the line $L_\infty = \langle (1,0,0,0,0),(0,1,0,0,0)\rangle$. So L_∞ is an "arbitrary" line of $T_3(\Omega)$ not meeting P. The points of $T_3(\Omega)$ with which it is incident are the points of $PG(4,q) \setminus \Sigma$ incident in Σ with L_∞ and the solid $\langle Q, T_Q\rangle$. The subgroups of G fixing the various points of L form a 4-gonal family for G. We are about to calculate these groups.

First note that the center $Z(G) = \{[0,0,c,0,0] : c \in F_q\}$ of G acts as a group of symmetries about the "line" P (i.e., fixing all lines meeting P).

For $t \in F_q$, put

$$\begin{aligned} A(t) &= \text{the stabilizer in } G \text{ of } (1,t,0,0,0) \\ &= \{[a,b,tf(a,b),ta,tb] : a,b \in F_q\}; \\ A^*(t) &= \{[a,b,c,ta,tb] : a,b,c \in F_q\}; \\ A(\infty) &= \text{the stabilizer of the remaining point of } L_\infty \\ &= \{[0,0,0,d,e] : d,e \in F_q\}; \\ A^*(\infty) &= \{[0,0,c,d,e] : c,d,e \in F+q\}. \end{aligned}$$

We have shown directly that the groups $A(t)$, $t \in F_q \cup \{\infty\}$ actually give a Kantor family for G. The details can be found in [Pa04].

3. Groups Defined by Pairings

The main problem posed in this section is whether or not the elation group of the Roman EGQ with parameters (q^2, q), $q = 3^e \geq 27$ is isomorphic to the usual elation group of the GQ $H(3, q^2)$. Along the way we study the automorphisms of the two groups, but the main problem remains unresolved.

3.1. Definitions and Basic Observations.

Let $F = GF(q)$, $q = p^e$, p an odd prime. Let $f : F^2 \times F^2 \to F$ be a symmetric, biadditive map. Further, we suppose that if $\overline{0} \neq \alpha \in F^2$, then $\{\beta \in F^2 : f(\alpha, \beta) = 0\}$ (is an additive subgroup of F^2 which) has order q. This means that for a fixed nonzero α, $|\{f(\alpha, \beta) : \beta \in F^2\}| = q$ also. We call such an f a *nonsingular pairing.*

Let $G = \{(\alpha, \beta, c) : \alpha = (a_1, a_2) \in F^2, \beta = (b_1, b_2) \in F^2, c \in F\}$. Clearly G has q^5 elements. We want to form G into a group using a given pairing.

Let $f : F^2 \times F^2 \to F$ be a given nonsingular pairing. Define a binary operation on G by

$$(\alpha, \beta, c) \circ (\alpha', \beta', c') \;=\; (\alpha + \alpha', \beta + \beta', c + c' + f(\beta, \alpha')). \qquad (3.1)$$

This makes G into a group (G, f) that we also denote by G_f. The center (and commutator and Frattini subgroup) of this group is

$$Z(G_f) = \{(\overline{0}, \overline{0}, c) \in G_f : c \in F\}.$$

Let $\overline{f} : F^2 \times F^2 \to F$ be a second (not necessarily distinct) nonsingular pairing, so we also have a group $G_{\overline{f}}$ of order q^5, etc. We want to determine as explicitly as possible all isomorphisms $\theta : G_f \to G_{\overline{f}}$. First, however, we examine the two examples of special interest.

3.2. The classical examples.

The classical examples have

$$f(\alpha, \beta) = \alpha \cdot \beta^T.$$

Let A and B be invertible 2×2 matrices over F. Consider the mapping $\theta : (\alpha, \beta, c) \mapsto (\alpha A^T, \beta B, c)$. Then $\theta : G_f \to G_{\overline{f}}$ where

$$\overline{f}(\alpha, \beta) = \alpha (AB)^{-1} \beta^T. \qquad (3.2)$$

Let n be a nonsquare in F and let

$$Q^{-1} = B = \begin{pmatrix} -1 & 0 \\ 0 & n^{-1} \end{pmatrix}.$$

Then $\theta : (\alpha, \beta, c) \mapsto (\alpha, \beta B, c)$ maps G_f to $G_{\overline{f}}$ where

$$\overline{f}(\alpha, \beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & n \end{pmatrix} \beta^T = \alpha Q \beta^T. \qquad (3.3)$$

Until further notice we assume that the group G_f has the pairing given in Eq. 3.3, but we continue to use the notation $\alpha \circ \beta = \alpha \cdot \beta^T$ and introduce the new notation $f(\alpha, \beta) = \alpha \hat{\circ} \beta = \alpha Q \beta^T$ for the new pairing.

Theorem 3.1. *Let D and K be 2×2 matrices over F with D invertible and K symmetric. Then the map*

$$\varphi(D, K) : (\alpha, \beta, c) \mapsto (\alpha D,\ \beta Q D^{-T} Q^{-1} + \alpha D K Q^{-1},\ c - \alpha D K D^T \alpha^T) \qquad (3.4)$$

is an automorphism of G_f.

Proof. Start with

$$(\alpha, \beta, c) \circ (\alpha', \beta', c') = (\alpha + \alpha', \beta + \beta', c + c' + \beta Q \alpha').$$

The image under φ of the product is

$$((\alpha + \alpha')D,\ (\beta + \beta')QD^{T}Q^{-1} + (\alpha + \alpha')DKQ^{-1},\ c + c' +$$

$$\beta Q \alpha' - (\alpha + \alpha')DKD^T(\alpha + \alpha')^T)$$

$$= ((\alpha + \alpha')D,\ (\beta + \beta')QD^{T}Q^{-1} + (\alpha + \alpha')DKQ^{-1},\ c + c' + \beta Q\alpha' - \alpha DKD^T\alpha^T$$

$$-\alpha' DKD^T(\alpha')^T + (\alpha DKD^T(\alpha')^T \quad (\text{ because } K \text{ is symmetric and } -2 = 1.$$

The product of the images under φ is

$$((\alpha{+}\alpha')D, (\beta{+}\beta')QD^TQ^{-1}{+}(\alpha{+}\alpha')DKDQ^{-1}, c{+}c'{-}\alpha DKD^T\alpha^T{-}\alpha' DKD^T(\alpha')^T{+}$$

$$+(\beta Q D^{-T}Q^{-1} + \alpha DKQ^{-1})Q(D^T(\alpha')^T$$

$$= (--, --, c + c' - \alpha DKD^T\alpha^T - \alpha' DKD^T(\alpha')^T + \beta Q(\alpha')^T + \alpha DKD^T(\alpha')^T).$$

Comparing these two results shows that φ preserves the group product. □

Lemma 3.2. *Let $\gamma = (g_1, g_2)$ and $\delta = (d_1, d_2)$ be any two elements of F^2 with $\gamma \neq \overline{0}$. Then there is a symmetric matrix K with $\gamma K = \delta$.*

Proof. If $g_1 \neq 0$, put $a = \frac{g_2^2}{g_1} + \frac{d_1}{g_1} - \frac{g_2 d_2}{g_1^2}$; $b = -g_2 + \frac{d_2}{g_1}$, $c = g_1$. Then $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has the desired property. If $g_1 = 0 \neq g_2$, put $a = 0$; $b = \frac{d_1}{g_2}$; $c = \frac{d_2}{g_2}$. Then $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has the desired property. □

For $0 \neq s,\ t \in F$, define

$$\rho_{s,t} : (\alpha, \beta, c) \mapsto (s\alpha, t\beta, stc). \tag{3.5}$$

Then $\rho_{s,t}$ is an automorphism of G_f.

The map

$$\varphi : (\alpha, \beta, c) \mapsto (\beta, -\alpha, c - f(\alpha, \beta)) \tag{3.6}$$

is an automorphism of G_f.

Note: This automorphism φ works for any nonsingular pairing f.

The cosets of the center have as distinct coset representatives group elements of the form $(\alpha, \beta, -)$, where the third element may be chosen to be anything and the (α, β) range over all pairs of elements from F^2. We want to show that the automorphisms of G_f act transitively on these cosets different from C itself. To do this we show that for any $(\alpha, \beta), (\gamma, \delta) \in F^2 \times F^2$ with $(\alpha, \beta) \neq (\overline{0}, \overline{0}) \neq (\gamma, \delta)$ there is a collineation mapping $(\alpha, \beta, --)$ to $(\gamma, \delta, --)$ where we do not have to keep track of the third entries.

Theorem 3.3. *The automorphisms of G_f act transitively on the cosets of the center of G_f.*

Proof. We show that there is an automorphism $\varphi(D, K)$ mapping $(\alpha, \beta, --)$ to $(\gamma, \delta, --)$. This means we need to find an invertible matrix D and a symmetric matrix K for which

$$(\alpha D,\ \beta Q D^{-T} Q^{-1} + \alpha D K Q^{-1}) = (\gamma, \delta).$$

Since the φ of Eq. 3.6 maps $(\overline{0}, \delta, c)$ to $(\delta, \overline{0})$, without loss of generality we may assume that $\alpha \neq \overline{0} \neq \gamma$, and we may choose any invertible matrix D with $\alpha D = \gamma$. Then we need to find a symmetric K for which $\beta Q D^{-T} Q^{-1} + \gamma K Q^{-1}) = \delta$, i.e., for which

$$\gamma K = \delta Q - \beta Q D^{-T}.$$

By Lemma 3.2 there is a symmetric K (which may be the zero matrix) for which $\gamma K = \delta Q - \beta Q D^{-T}$, completing the proof. □

3.3. The Roman $G_{\overline{f}}$.

In this section $q = 3^e \geq 27$ and n will be any fixed nonsquare of F. Finally, $\overline{f}(\alpha, \beta)$ will be defined by

$$\overline{f}(\alpha, \beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & n \end{pmatrix} \beta^T + \left\{ \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta^T \right\}^{\frac{1}{3}} + \left\{ \alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \beta^T \right\}^{\frac{1}{9}}. \tag{3.7}$$

Then $G_{\overline{f}}$ is the elation group of the Roman GQ with parameters (q^2, q). (See [Pa89] for details.) In [Pa89] it was claimed that we had shown that the group G_f associated with $H(3, q^2)$ is not isomorphic to the group $G_{\overline{f}}$ associated with the Roman GQ. Unfortunately, we no longer have those computations, and it seems quite likely that we did not actually have a valid proof. So we present this as an open

Problem: Determine whether the group G_f associated with $H(3, q^2)$ is isomorphic to the group $G_{\overline{f}}$ associated with the Roman GQ.

3.4. Addendum.

When this paper was presented at the Pingree Park Conference, a few of the mathematicians in attendance (including at least George Havas, Michael Slattery, Eamonn O'Brien, Charles Leedham-Green, Alexander Hulpke) immediately began to attempt to use computational methods to solve this problem. Before the conference was over George Havas announced that indeed they had showed that the two groups are not isomorphic.

References

[CP02] I. Cardinali and S. E. Payne, The q-Clan Geometries with $q = 2^e$, available at **http://www-math.cudenver.edu/∼ spayne/**.

[Ch90] X. Chen, Skew-Translation GQ have s and t powers of the same prime, preprint, circa 1990.

[CF93] X. Chen and D. Frohardt, Normality in a Kantor Family, *Dour. Combin. Theory* (A), 64(1993), 130 – 136.

[Fr88] D. Frohardt, Groups which produce generalized quadrangles, *Jour. Combin. Theory* (A), 48(1988), 139 – 145.

[Ha96] D. Hachenberger, Groups admitting a Kantor family and a factorized normal subgroup, *Designs, Codes and Cryptography*, 8(1996), 135 – 143.

[Ha94] D. Hachenberger, On finite elation generalized quadrangles with symmetries, (1994)*Jour. London Math. Soc.*, 53(1996), 397 – 406.

[Ka80] W. M. Kantor, Generalized quadrangles associated with $G_2(q)$, *Jour. Comb. Theory (A)*, 29(1980), 212-219.

[Ka86] W. M. Kantor, Generalized Polygons, SCABS and GABS, in *Lecture Notes in Mathematics (Buildings and the Geometry of Diagrams, Como 1984)*, Springer-Verlag, 1986, 79 – 158.

[Pa89] S. E. Payne, An essay on skew translation generalized quadrangles, *Geom. Ded.*, 32(1989), 93 – 118.

[Pa04] S. E. Payne, Ten Lectures in Perth (Elation Generalized Quadrangles, Etc.), available as a PDF file on the home page of the School of Mathematics and Statistics at the University of Western Australia, Research Report 2004/11, 127 pages.

[PT84] S. E. Payne and J. A. Thas, *Finite generalized quadrangles*, Pitman, 1984.

[KT04] K. Thas, *Symmetry in Finite Generalized Quadrangles*, Frontiers in Mathematics, Birkhäuser-Verlag, 2004.

[VM98] H. Van Maldeghem, *Generalized Polygons*, Monographs in Mathematics 93, Birkhäuser-Verlag, 1998.

Stanley E. Payne, Department of Mathematics, University of Colorado at Denver and Health Sciences Center, CB 170, P.O. Box 173364, Denver, CO 80217-3364, U.S.A.

Email: spayne@math.cudenver.edu