

# Some notes on generalized quadrangles of order $s$ with a span of regular points

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## Abstract

Let  $Q$  be a generalized quadrangle of order  $s$  with a regular point  $x$ . The set  $x^\perp$  together with all spans which are contained in  $x^\perp$  define a projective plane of order  $s$ . We introduce a property  $(P_y)$  for every point  $y$  of  $Q$  noncollinear with  $x$  and prove that this property is equivalent with the regularity of the point  $y$ . We will use this to give an elementary proof for the following result: every generalized quadrangle of order  $q$  which has a center of symmetry  $x$  such that  $\pi_x \cong \text{PG}(2, q)$  and a regular point  $y$  not collinear with  $x$  is isomorphic to  $W(q)$ .

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## 1 Introduction

### 1.1 Definitions

A *generalized quadrangle* of order  $(s, t)$ ,  $s, t \in \mathbb{N} \setminus \{0\}$ , or shortly a  $\text{GQ}(s, t)$ , is a point-line incidence structure which satisfies the following properties:

- (GQ1) every two distinct points are incident with at most 1 line;
- (GQ2) every line is incident with precisely  $s + 1$  points and every point is incident with precisely  $t + 1$  lines;
- (GQ3) for every line  $L$  and every point  $p$  not incident with  $L$ , there exists a unique line through  $p$  meeting  $L$ .

The point-line dual of a  $\text{GQ}(s, t)$  is a  $\text{GQ}(t, s)$ . A generalized quadrangle of order  $(s, s)$  is also called a generalized quadrangle of order  $s$  and is denoted as  $\text{GQ}(s)$ . We now recall some basic definitions and properties of  $\text{GQ}$ 's. For more background information on the topic, we refer to [10].

Let  $Q$  be a generalized quadrangle of order  $(s, t)$ . For every point  $x$  of  $Q$ , let  $x^\perp$  denote the set of all points collinear with  $x$  (so  $x \in x^\perp$ ). If  $X$  is a nonempty set of vertices of  $Q$ , then we define  $X^\perp := \bigcap_{x \in X} x^\perp$  and  $X^{\perp\perp} := (X^\perp)^\perp$ . If  $x$  and  $y$  are different points, then  $|\{x, y\}^\perp|$  is equal to either  $s + 1$  or  $t + 1$  depending on whether  $x$  and  $y$  are collinear or not. The set  $\{x, y\}^{\perp\perp}$  is called the *span* of the pair  $(x, y)$ . If  $x$  and  $y$  are collinear, then  $\{x, y\}^{\perp\perp}$  coincides with the point set of the line  $xy$ . If  $x$  and  $y$  are not collinear, then  $\{x, y\}^{\perp\perp}$  is also called the *hyperbolic line* through  $x$  and  $y$ ; since  $\{x, y\}^\perp$  contains two noncollinear points, this hyperbolic line contains at most  $t + 1$  points. If the hyperbolic line through two noncollinear points  $x$  and  $y$  contains precisely  $t + 1$  points, then the pair  $(x, y)$  is called *regular*. A point  $x$  is called *regular* if the pair  $(x, y)$  is regular for every point  $y$  not collinear with  $x$ . If  $x$  is a regular point of a generalized quadrangle of order  $s$ , then by Theorem 1.3.1 of [10], the incidence structure whose points are the elements of  $x^\perp$  and whose lines are all the spans  $\{a, b\}^{\perp\perp}$ ,  $a, b \in x^\perp$  with  $a \neq b$ , is a projective plane of order  $s$ . We will denote this projective plane by  $\pi_x$ .

A *triad*  $\Delta$  of a generalized quadrangle is a set of three mutually non-collinear points. Any point of  $\Delta^\perp$  is called a *center* of  $\Delta$ . If  $\Delta^\perp \neq \emptyset$ , then  $\Delta$  is called *centric*. By Theorem 1.3.6 of [10], the following statements are equivalent for a point  $x$  of a  $\text{GQ}(s)$ : (i)  $x$  is regular, (ii) every triad containing  $x$  is centric, (iii) every triad containing  $x$  has 1 or  $s + 1$  centers.

If  $x$  is a regular point of a generalized quadrangle  $Q$  of order  $s$  with  $s \neq 1$ , then a new generalized quadrangle  $P(Q, x)$  can be derived from it, see [8] or [10]. The points of  $P(Q, x)$  are the points of  $Q$  not collinear with  $x$  and the lines of  $P(Q, x)$  are on the one hand the lines of  $Q$  not containing  $x$  and on the other hand the hyperbolic lines of  $Q$  through  $x$  (natural incidence).  $P(Q, x)$  is a generalized quadrangle of order  $(s - 1, s + 1)$ .

The generalized quadrangle  $W(q)$ ,  $q$  prime power, is the  $\text{GQ}$  of the totally isotropic points and lines of a symplectic polarity in  $\text{PG}(3, q)$ . Every point of  $W(q)$  is regular. So, we can construct a generalized quadrangle  $P(W(q), x)$  of order  $(q - 1, q + 1)$  for every point  $x$  of  $W(q)$ . Since the automorphism group of  $W(q)$  acts transitively on the point set, essentially one  $\text{GQ}$  of order  $(q - 1, q + 1)$  arises this way. If  $q$  is odd, then  $P(W(q), x)$  is isomorphic to

the so-called Ahrens-Szekeres generalized quadrangle  $AS(q)$ , see [1] or [10].

A *spread* of a generalized quadrangle is a set of lines partitioning the point set. If  $S$  is a spread in a  $GQ(s, t)$  with  $t \neq 1$ , then there are at most  $s + 1$  automorphisms of the GQ which fix each line of  $S$ , see [2]. If there are precisely  $s + 1$  such automorphisms, then  $S$  is called a *spread of symmetry*.

If  $x$  is a point of a  $GQ(s, t)$  with  $s \neq 1$ , then there are at most  $t$  automorphisms of the GQ which fix every point of  $x^\perp$ , see Section 8.1 of [10]. If there are precisely  $t$  such automorphisms, then  $x$  is called a *center of symmetry*. Every center of symmetry is a regular point, see [10, 8.1]. Every point of  $W(q)$  is a center of symmetry.

Let  $Q_1$  and  $Q_2$  denote two GQ's. If  $x_i, i \in \{1, 2\}$ , is a point of  $Q_i$ , then we say that  $(Q_1, x_1)$  is equivalent with  $(Q_2, x_2)$  if there exists an isomorphism from  $Q_1$  to  $Q_2$  mapping  $x_1$  to  $x_2$ . If  $S_i, i \in \{1, 2\}$ , is a spread of  $Q_i$ , then we say that  $(Q_1, S_1)$  is equivalent with  $(Q_2, S_2)$  if there exists an isomorphism from  $Q_1$  to  $Q_2$  mapping  $S_1$  to  $S_2$ .

Now, suppose that  $Q$  is a  $GQ(s)$ ,  $s \geq 2$ , with a regular point  $x$ . Then the hyperbolic lines through  $x$  define a spread  $S(Q, x)$  of  $P(Q, x)$ . If  $x$  is a center of symmetry, then the  $s$  automorphisms of  $Q$  which fix each point of  $x^\perp$  induce  $s$  automorphisms of  $P(Q, x)$  fixing each line of  $S(Q, x)$ . Hence,  $S(Q, x)$  is a spread of symmetry of  $P(Q, x)$ . Conversely, if  $S^*$  is a spread of symmetry in a generalized quadrangle  $Q^*$  of order  $(s - 1, s + 1)$ ,  $s \geq 2$ , then by [5], there exists, up to equivalence, a unique pair  $(Q, x)$ , where  $Q$  is a  $GQ(s)$  and  $x$  a regular point in  $Q$ , such that  $(P(Q, x), S(Q, x))$  is equivalent with  $(Q^*, S^*)$ .

## 1.2 Overview

In [3], the first author will prove the following result:

**Proposition 1.1 ([3])** *If  $Q$  is a generalized quadrangle of order  $s \geq 2$  having a hyperbolic line  $\mathcal{H}$  consisting of  $s + 1$  regular points, then all these regular points are also centers of symmetry.*

W. M. Kantor ([7]) and K. Thas ([12]), independently, proved the following result:

**Proposition 1.2** ([7], [12]) *If  $Q$  is a generalized quadrangle of order  $s \geq 2$  having a hyperbolic line  $\mathcal{H}$  consisting of  $s + 1$  centers of symmetry, then  $s$  is a prime power and  $Q \cong W(s)$ .*

Propositions 1.1 and 1.2 have the following corollary.

**Corollary 1.3** *If  $Q$  is a generalized quadrangle of order  $s \geq 2$  having a hyperbolic line  $\mathcal{H}$  consisting of  $s + 1$  regular points, then  $s$  is a prime power and  $Q \cong W(s)$ .*

**Remark.** The interest in the result mentioned in Corollary 1.3 already dates back from the 80's when the second author ([9, III 1], see also [10, 10.7.5]) obtained a short and elementary proof for the following fact.

- (\*) If  $\mathcal{H}$  is a hyperbolic line of a  $\text{GQ}(s)$  consisting of only regular points, then also every point of  $\mathcal{H}^\perp$  is regular.

Attempts to prove that also all points outside  $\mathcal{H} \cup \mathcal{H}^\perp$  had to be regular (i.e. to prove that the  $\text{GQ}$  is isomorphic to  $W(s)$ ) were not successful until very recently.

The proof of Proposition 1.1 is elementary in the sense that only some basic notions from the theory of  $\text{GQ}$ 's are used in the proof. The proof of Proposition 1.2 makes use of quite some group theory. E.g., both authors make use of the classification of the finite groups with a split BN-pair of rank 1 ([6], [11]). The aim of this note is to provide an elementary proof of Proposition 1.2 in the case that at least one of the projective planes  $\pi_x$ ,  $x \in \mathcal{H}$ , is Desarguesian. More precisely, we will prove the following result.

**Theorem 1.4 (Section 4)** *A generalized quadrangle of order  $q$  which has a center of symmetry  $x$  such that  $\pi_x \cong \text{PG}(2, q)$  and a regular point  $y$  not collinear with  $x$  is isomorphic to  $W(q)$ .*

To prove Theorem 1.4, we will make use of the theory of the admissible triples introduced by one of the authors in [2]. We will also make use of a result regarding a certain property  $(P_y)$  in a generalized quadrangle of order  $s$  with a regular point (Theorem 2.1). Property  $(P_y)$ , which we introduce in the following section, not only plays a crucial role in the proof of Theorem 1.4, but also in the proof of Proposition 1.1 in [3].

## 2 Property $(P_y)$

Let  $Q$  be a generalized quadrangle of order  $s \geq 2$  with a regular point  $x$  and let  $y$  be a point of  $Q$  noncollinear with  $x$ . Let  $u, v$  and  $w$  denote three points of  $Q$  noncollinear with  $x$  such that  $y \sim u \sim v \sim w \sim y$ ,  $y \not\sim v$  and  $u \not\sim w$ . The triad  $\{x, y, v\}$  has either 1 or  $s + 1$  centers. Now, the points  $u$  and  $w$  are collinear with  $y$  and  $v$ , but not with  $x$ . So, the triad  $\{x, y, v\}$  has a unique center  $a_1$ . In a similar way, one shows that the triad  $\{x, u, w\}$  has a unique center  $a_2$ . Obviously,  $a_1 \neq a_2$ . If the points  $x, a_1$  and  $a_2$  are collinear for all possible choices for  $u, v$  and  $w$  such that  $y \sim u \sim v \sim w \sim y$ ,  $x \not\sim u$ ,  $x \not\sim v$ ,  $x \not\sim w$ ,  $y \not\sim v$  and  $u \not\sim w$ , then we say that the pair  $(Q, x)$  satisfies property  $(P_y)$ .

**Theorem 2.1** *Let  $Q$  be a generalized quadrangle of order  $s \geq 2$  with a regular point  $x$  and let  $y$  be a point of  $Q$  noncollinear with  $x$ . Then the pair  $(Q, x)$  satisfies property  $(P_y)$  if and only if  $y$  is a regular point.*

**Proof.** (1) Suppose that  $y$  is a regular point and choose arbitrary points  $u, v$  and  $w$  not collinear with  $x$  such that  $y \sim u \sim v \sim w \sim y$ ,  $y \not\sim v$  and  $u \not\sim w$ . Let  $a_1$  denote the unique center of the triad  $\{x, y, v\}$  and let  $a_2$  denote the unique center of the triad  $\{x, u, w\}$ . If  $a_2$  were collinear with  $y$ , then by the nonexistence of triangles in  $Q$ , we would have that  $a_2$  is contained in the lines  $yu$  and  $yw$ , a contradiction. So,  $a_2$  and  $y$  are not collinear. In a similar way one shows that  $a_2$  and  $v$  are not collinear. So,  $\{a_2, y, v\}$  is a triad with at least 2 centers (namely  $u$  and  $w$ ). Since  $y$  is a regular point, this triad must have precisely  $s + 1$  centers. So, the point  $a_1$  which is collinear with  $y$  and  $v$  must also be collinear with  $a_2$ . So,  $x, a_1$  and  $a_2$  are on a line and  $(Q, x)$  satisfies property  $(P_y)$ .

(2) Conversely, suppose that  $(Q, x)$  satisfies property  $(P_y)$  for a certain point  $y$  not collinear with  $x$  and let  $y'$  denote a point of  $Q$  not collinear with  $y$ . We will prove that the pair  $(y, y')$  is regular. Obviously, this holds if  $y' = x$ .

Suppose that  $y' \in x^\perp \setminus \{x\}$ . Let  $a_1$  denote the unique point of  $xy'$  collinear with  $y$  and let  $u$  denote a common neighbour of  $y$  and  $y'$  different from  $a_1$ . We must show that every common neighbour  $v$  of  $a_1$  and  $u$  is collinear with every common neighbour  $w$  of  $y$  and  $y'$ . We may suppose that  $y \neq v \neq y'$  and  $a_1 \neq w \neq u$ . If  $v'$  denotes the unique point of  $uv$  collinear with  $w$ , then by property  $(P_y)$  it follows that either  $v' \sim x$  or  $v' \sim a_1$ . If  $v' \sim x$ , then  $v'$  and  $y'$  are two centers of the triad  $\{x, u, w\}$ . So,  $\{x, u, w\}$  has  $s + 1$  centers

and the neighbour  $y$  of  $u$  and  $w$  would be collinear with  $x$ , a contradiction. So,  $v' \sim a_1$ . But then  $v' = v$  and  $w \sim v$ . This proves that  $(y, y')$  is regular.

Finally, suppose that  $y'$  is not collinear with  $x$ . If  $y' \in \{x, y\}^{\perp\perp}$ , then  $(y, y')$  is regular. Suppose therefore that  $y' \notin \{x, y\}^{\perp\perp}$ , then the triad  $\{x, y, y'\}$  has a unique center  $a_1$ . Let  $u$  and  $w$  denote two neighbours of  $y$  and  $y'$  different from  $a_1$ . Then the unique center  $a_2$  of the triad  $\{x, u, w\}$  lies on the line  $xa_1$ . Now,  $(a_2, y)$  is regular. Since  $y'$  is collinear with two points of  $\{a_2, y\}^\perp$ , it belongs to  $\{a_2, y\}^{\perp\perp}$ , proving that also  $(y, y')$  is regular.  $\square$

### 3 Admissible triples

**Definition.** An *admissible triple* is a triple  $T = (\mathcal{L}, G, \Delta)$ , where:

- $G$  is a nontrivial group. We put  $s := |G| - 1 \geq 1$ .
- $\mathcal{L}$  is a linear space, different from a point, in which each line is incident with exactly  $s + 1$  points. We denote the point set of  $\mathcal{L}$  by  $P$ . Then  $\mathcal{L}$  has order  $(s, t - 1)$ , where  $t := \frac{|P|-1}{s}$ .
- $\Delta$  is a map from  $P \times P$  to  $G$  such that the following holds for all points  $x, y$  and  $z$  of  $\mathcal{L}$ :

$$(AT) \quad x, y \text{ and } z \text{ are collinear} \Leftrightarrow \Delta(x, y) + \Delta(y, z) = \Delta(x, z).$$

If  $T$  is an admissible triple, then  $\Delta(x, x) = 0$  and  $\Delta(y, x) = -\Delta(x, y)$  for all points  $x$  and  $y$  of  $\mathcal{L}$ . Notice that we have used the additive notation for the group  $G$ .

**Proposition 3.1 ([2])** *Suppose that  $T = (\mathcal{L}, G, \Delta)$  is an admissible triple and let  $P$  denote the point set of  $\mathcal{L}$ . Let  $\Gamma$  be the graph with vertex set  $G \times P$ , with two vertices  $(g_1, x_1)$  and  $(g_2, x_2)$  adjacent whenever either  $(x_1 = x_2 \text{ and } g_1 \neq g_2)$  or  $(x_1 \neq x_2 \text{ and } g_2 = g_1 + \Delta(x_1, x_2))$ . Then  $\Gamma$  is the collinearity graph of a generalized quadrangle  $Q$  of order  $(s, t)$ . Moreover, the set  $L_x := \{(g, x) \mid g \in G\}$  is a line of  $Q$  for every point  $x$  of  $\mathcal{L}$  and the lines  $L_x, x \in P$ , form a spread of symmetry  $S$  of  $Q$ .*

For every admissible triple  $T$ , we put  $\Omega(T) := (Q, S)$  where  $Q$  and  $S$  are as in Proposition 3.1.

**Proposition 3.2** ([2]) *If  $S$  is a spread of symmetry of a generalized quadrangle  $Q$ , then there exists an admissible triple  $T$  such that  $\Omega(T)$  is equivalent with  $(Q, S)$ .*

**Example.** Let  $\mathcal{L}$  be the Desarguesian affine plane  $\text{AG}(2, q)$  coordinatized in the natural way by the finite field  $\mathbb{F}_q$ . Let  $G$  be the additive group of  $\mathbb{F}_q$ . For all points  $(x_1, y_1)$  and  $(x_2, y_2)$  of  $\mathcal{L}$ , we define  $\Delta[(x_1, y_1), (x_2, y_2)] := x_1y_2 - y_1x_2$ . By [2],  $(\mathcal{L}, G, \Delta)$  is an admissible triple and the associated generalized quadrangle is isomorphic to  $P(W(q), x)$ , where  $x$  is a point of  $W(q)$ .

**Proposition 3.3** ([2], [4]) *Suppose that  $T = (\mathcal{L}, G, \Delta)$  is an admissible triple and let  $P$  denote the point set of  $\mathcal{L}$ . Let  $\mathcal{L}'$  be a linear space isomorphic to  $\mathcal{L}$  and let  $G'$  denote a group isomorphic to  $G$ . Let  $\alpha$  denote an isomorphism from  $\mathcal{L}$  to  $\mathcal{L}'$ , let  $\theta$  denote an isomorphism from  $G$  to  $G'$  and let  $f$  denote an arbitrary map from  $P$  to  $G$ . For all points  $x$  and  $y$  of  $\mathcal{L}$ , we define*

$$\Delta'(\alpha(x), \alpha(y)) := [f(x) + \Delta(x, y) - f(y)]^\theta.$$

*Then  $T' := (\mathcal{L}', G', \Delta')$  is an admissible triple and  $\Omega(T')$  is equivalent with  $\Omega(T)$ .*

**Definition.** Let  $T_1 = (\mathcal{L}_1, G_1, \Delta_1)$  and  $T_2 = (\mathcal{L}_2, G_2, \Delta_2)$  be two admissible triples. Let  $P_i, i \in \{1, 2\}$ , denote the point set of  $\mathcal{L}_i$ .

If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are lines, then we say that  $T_1$  and  $T_2$  are *equivalent* if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  contain the same number of points.

If  $\mathcal{L}_1$  or  $\mathcal{L}_2$  is not a line, then we say that  $T_1$  and  $T_2$  are *equivalent* if there exists an isomorphism  $\alpha$  from  $\mathcal{L}_1$  to  $\mathcal{L}_2$ , an isomorphism  $\theta$  from  $G_1$  to  $G_2$  and a map  $f : P_1 \rightarrow G_1$ , such that

$$\Delta_2(\alpha(x), \alpha(y)) = [f(x) + \Delta_1(x, y) - f(y)]^\theta$$

for all points  $x$  and  $y$  of  $\mathcal{L}_1$ .

**Proposition 3.4** ([4]) *Two admissible triples  $T_1$  and  $T_2$  are equivalent if and only if  $\Omega(T_1)$  and  $\Omega(T_2)$  are equivalent.*

**Lemma 3.5** *If  $T = (\mathcal{L}, G, \Delta)$  is an admissible triple and if  $o$  is an arbitrary point of  $\mathcal{L}$ , then there exists an admissible triple  $T' = (\mathcal{L}, G, \Delta')$  equivalent with  $T$  such that  $\Delta'(o, x) = 0$  for every point  $x$  of  $\mathcal{L}$ .*

**Proof.** In the above definition, we put  $\alpha$  equal to the trivial automorphism of  $\mathcal{L}$ ,  $\theta$  equal to the trivial automorphism of  $G$  and we define  $f(x) := \Delta(o, x)$  for every point  $x$  of  $\mathcal{L}$ . We then have  $\Delta'(o, x) = \Delta(o, o) + \Delta(o, x) - \Delta(o, x) = 0$  for every point  $x$  of  $\mathcal{L}$ .  $\square$

## 4 Proof of Theorem 1.4

In this section, we suppose that  $Q$  is a generalized quadrangle of order  $q$ , that  $x$  is a center of symmetry of  $Q$  such that  $\pi_x \cong \text{PG}(2, q)$  and that  $y$  is a regular point not collinear with  $x$ . Consider the generalized quadrangle  $P(Q, x)$  of order  $(q-1, q+1)$ . The hyperbolic lines through  $x$  define a spread  $S(Q, x)$  of  $P(Q, x)$ . The  $q$  symmetries about  $x$  define  $q$  automorphisms of  $P(Q, x)$  fixing each line of  $S(Q, x)$ . So,  $S(Q, x)$  is a spread of symmetry of  $P(Q, x)$ . Consider the following linear space  $\mathcal{L}'$ :

- the points of  $\mathcal{L}'$  are the elements of the spread  $S(Q, x)$ ;
- the lines of  $\mathcal{L}'$  are the spans  $\{A, B\}^{\perp\perp}$  where  $A$  and  $B$  are two different elements of  $S(Q, x)$ ;
- incidence is containment.

Since  $S(Q, x)$  is a spread of symmetry,  $(A, B)$  is a regular pair of lines for all  $A, B \in S(Q, x)$  with  $A \neq B$ , see e.g. [2]. So, every line of  $\mathcal{L}'$  is incident with  $s$  points. Since each hyperbolic line  $\mathcal{H}$  through  $x$  is uniquely determined by the set  $\mathcal{H}^\perp$ , the linear space  $\mathcal{L}'$  is isomorphic to the following linear space  $\mathcal{L}''$ :

- the points of  $\mathcal{L}''$  are the spans  $\{a, b\}^{\perp\perp}$ , where  $a, b \in x^\perp \setminus \{x\}$  such that  $b \notin xa$ ;
- the lines of  $\mathcal{L}''$  are all the points of  $x^\perp \setminus \{x\}$ ;
- incidence is reverse containment.

So,  $\mathcal{L}' \cong \mathcal{L}'' \cong \text{AG}(2, q)$ .

Let  $L_y$  denote the line of  $P(Q, x)$  corresponding with the hyperbolic line  $\{x, y\}^{\perp\perp}$ . For every two different lines  $L$  and  $L'$  of  $S(Q, x)$  and for every point  $z$  of  $L$ , let  $p_{L \rightarrow L'}(z)$  denote the unique point of  $L'$  collinear with  $z$ . For every line  $L$  of  $S(Q, x)$  and every point  $z$  of  $L$ , we put  $p_{L \rightarrow L}(z) = z$ . The group  $\Pi := \langle p_{L' \rightarrow L_y} \circ p_{L \rightarrow L'} \circ p_{L_y \rightarrow L} \mid L, L' \in S(Q, x) \rangle$  of permutations of  $L_y$  is called the *group of projectivities* of  $L_y$  with respect to  $S(Q, x)$ . If an element of  $\Pi$  fixes one point of  $L_y$ , then it is trivial, see [2]. For all lines  $L_1, L_2, L_3$  of  $S(Q, x)$ , let  $\phi(L_1, L_2, L_3)$  denote the following element of  $\Pi$ :  $p_{L_3 \rightarrow L_y} \circ p_{L_2 \rightarrow L_3} \circ p_{L_1 \rightarrow L_2} \circ p_{L_y \rightarrow L_1}$ .

**Lemma 4.1** *If  $L_1, L_2$  and  $L_3$  are lines of  $S(Q, x)$  such that  $L_y, L_1, L_2, L_3$  are mutually different and  $\phi(L_1, L_2, L_3)$  is trivial, then the lines  $\{L_y, L_2\}^{\perp\perp}$  and  $\{L_1, L_3\}^{\perp\perp}$  of  $\mathcal{L}'$  are parallel.*

**Proof.** Clearly, this is the case when  $L_y, L_1, L_2$  and  $L_3$  are contained in the same grid. So, suppose that  $L_y, L_1, L_2$  and  $L_3$  are not contained in the same grid. Let  $u$  denote the unique point of  $L_1$  collinear with  $y$ , let  $v$  denote the unique point of  $L_2$  collinear with  $u$  and let  $w$  denote the unique point of  $L_3$  collinear with  $v$ . Since  $\phi(L_1, L_2, L_3) = 0$ ,  $w \sim y$ . Since  $L_y, L_1, L_2$  and  $L_3$  are not contained in the same grid,  $y \not\sim v$  and  $u \not\sim w$ . Now, we can regard  $y, u, v$  and  $w$  as points of  $Q$ . Applying Theorem 2.1, we find that  $a_1 \sim a_2$  where  $a_1$  is the unique center of the triad  $\{x, y, v\}$  and where  $a_2$  is the unique center of the triad  $\{x, u, w\}$ . It follows that  $\{L_y, L_2\}^{\perp\perp}$  and  $\{L_1, L_3\}^{\perp\perp}$  are parallel lines of  $\mathcal{L}'$ .  $\square$

Now, since  $S(Q, x)$  is a spread of symmetry of  $P(Q, x)$ , there exists, up to equivalence, a unique admissible triple  $T = (\mathcal{L}, G, \Delta)$  such that  $\Omega(T)$  is equivalent with  $(P(Q, x), S(Q, x))$ . Here  $G$  is a group of order  $q$  (we will use the additive notation for  $G$ ) and  $\mathcal{L}$  denotes a linear space isomorphic to  $\mathcal{L}'$ , see [2]. So,  $\mathcal{L} \cong \text{AG}(2, q)$ . We will coordinatize  $\mathcal{L}$  using the finite field  $\mathbb{F}_q$  in such a way that the point  $o = (0, 0)$  corresponds with the line  $L_y$  (given a certain isomorphism from  $\mathcal{L}'$  to  $\mathcal{L}$ ). Since  $T$  is an admissible triple, we have the following property:

- (I) Three points  $u, v$  and  $w$  of  $\mathcal{L}$  are collinear if and only if  $\Delta(u, v) + \Delta(v, w) = \Delta(u, w)$ .

We want to determine  $T$  up to equivalence. So, by Lemma 3.5, we may also suppose that

(II)  $\Delta(o, u) = 0$  for every point  $u$  of  $\mathcal{L}$ .

From Property (II), Proposition 3.1 and Lemma 4.1, the following condition follows:

(III) If  $o, u, v$  and  $w$  are mutually different points of  $\mathcal{L}$  such that  $\Delta(v, u) = \Delta(v, w)$ , then the line  $ov$  is parallel with  $uw$ .

We will now use Properties (I), (II) and (III) to determine  $T$ .

**Lemma 4.2** (a)  $\Delta(u, v) = 0$  if and only if  $o, u$  and  $v$  are collinear.

(b) If  $L$  is a line through  $o$  and if  $u$  is a point of  $\mathcal{L}$  not contained in  $L$ , then  $G = \{\Delta(u, l) \mid l \in L\}$ .

(c) If  $L$  is a line of  $\mathcal{L}$  not containing  $o$  and if  $u$  is a point of  $L$ , then  $G = \{\Delta(u, l) \mid l \in L\}$ .

(d) For every point  $u$  of  $\mathcal{L}$  different from  $o$  and for every  $g \in G$ , the points  $v$  satisfying  $\Delta(u, v) = g$  form a line parallel with  $ou$ .

**Proof.** (a) This follows from Properties (I) and (II).

(b) If  $l_1$  and  $l_2$  are two different points of  $L$ , then  $\Delta(u, l_2) \neq \Delta(u, l_1) + \Delta(l_1, l_2) = \Delta(u, l_1)$ . Hence,  $G = \{\Delta(u, l) \mid l \in L\}$  since both sets contain  $q$  elements.

(c) If  $l_1$  and  $l_2$  are two different points of  $L$ , then  $\Delta(u, l_2) = \Delta(u, l_1) + \Delta(l_1, l_2) \neq \Delta(u, l_1)$ . Hence,  $G = \{\Delta(u, l) \mid l \in L\}$  since both sets contain  $q$  elements.

(d) If  $g = 0$ , then the required points are the points on the line  $ou$ . If  $g \neq 0$ , then by (c), there exists a unique point  $v$  with  $\Delta(u, v) = g$  on every line through  $u$  different from  $ou$ . Hence, there are precisely  $q$  points  $v$  satisfying  $\Delta(u, v) = g$ . If  $v_1$  and  $v_2$  are two such points, then by Property (III),  $v_1v_2$  must be parallel with  $ou$ . The property now easily follows.  $\square$

We are now ready to determine  $\Delta$ . For all  $i, j \in \mathbb{F}_q$ , we define  $f(i, j) := \Delta[(i, 0), (0, j)]$ . Then  $f(0, i) = f(i, 0) = 0$  for every  $i \in \mathbb{F}_q$ . By Lemma 4.2, we have  $\Delta[(i, 0), (j, k)] = \Delta[(i, 0), (0, k)] = f(i, k)$  and  $\Delta[(j, k), (0, i)] = \Delta[(j, 0), (0, i)] = f(j, i)$  for all  $i, j, k \in \mathbb{F}_q$ . Now, choose points  $(x_1, y_1)$  and  $(x_2, y_2)$  with  $x_1 \neq 0 \neq y_1$ . The line through  $(x_2, y_2)$  parallel with the line through  $(0, 0)$  and  $(x_1, y_1)$  contains the points  $(\frac{x_2y_1 - y_2x_1}{y_1}, 0)$  and  $(0, \frac{y_2x_1 - x_2y_1}{x_1})$ . By Lemma 4.2,  $\Delta[(x_1, y_1), (x_2, y_2)] = \Delta[(x_1, y_1), (0, \frac{y_2x_1 - x_2y_1}{x_1})]$

$= f(x_1, \frac{y_2x_1 - x_2y_1}{x_1})$  and  $\Delta[(x_1, y_1), (x_2, y_2)] = \Delta[(x_1, y_1), (\frac{x_2y_1 - y_2x_1}{y_1}, 0)] = -f(\frac{x_2y_1 - y_2x_1}{y_1}, y_1)$ . Hence,

$$f(x_1, \frac{y_2x_1 - x_2y_1}{x_1}) = -f(\frac{x_2y_1 - y_2x_1}{y_1}, y_1)$$

for all  $x_1, y_1, x_2, y_2 \in \mathbb{F}_q$  with  $x_1 \neq 0 \neq y_1$ . If we choose  $x_1 = y_1 \neq 0$ , then  $f(x_1, y_2 - x_2) = -f(x_2 - y_2, x_1)$ . As a consequence,  $f(a, b) = -f(-b, a) = f(-a, -b) = -f(b, -a)$  for all  $a, b \in \mathbb{F}_q$ . Hence,

$$f(x_1, \frac{y_2x_1 - x_2y_1}{x_1}) = f(y_1, \frac{y_2x_1 - x_2y_1}{y_1})$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{F}_q$  with  $x_1 \neq 0 \neq y_1$ . Now, put  $x_1 = a, y_1 = 1, x_2 = a - ab$  and  $y_2 = 1$  for certain  $a, b \in \mathbb{F}_q$  with  $a \neq 0$ . Then we find that  $f(a, b) = f(1, ab)$ . This formula still holds if  $a = 0$ . Now, put  $g(c) := f(1, c)$  for all  $c \in \mathbb{F}_q$ . Then  $g(0) = 0$  and  $f(a, b) = g(ab)$  for all  $a, b \in \mathbb{F}_q$ . By Lemma 4.2 and the definition of  $f$ ,  $f(1, b) \neq f(1, b')$  for all  $b, b' \in \mathbb{F}_q$  with  $b \neq b'$ . So,  $g$  is a bijection between  $\mathbb{F}_q$  and  $G$ . Also,  $\Delta[(x_1, y_1), (x_2, y_2)] = f(x_1, \frac{y_2x_1 - x_2y_1}{x_1}) = g(x_1y_2 - x_2y_1)$  for all  $x_1, y_1, x_2, y_2 \in \mathbb{F}_q$  with  $x_1 \neq 0 \neq y_1$ . If  $x_1 = 0$ , then  $\Delta[(x_1, y_1), (x_2, y_2)] = \Delta[(0, y_1), (x_2, y_2)] = \Delta[(0, y_1), (x_2, 0)] = -\Delta[(x_2, 0), (0, y_1)] = -f(x_2, y_1) = f(y_1, -x_2) = g(-x_2y_1) = g(x_1y_2 - x_2y_1)$ . If  $y_1 = 0$ , then  $\Delta[(x_1, y_1), (x_2, y_2)] = \Delta[(x_1, 0), (x_2, y_2)] = \Delta[(x_1, 0), (0, y_2)] = f(x_1, y_2) = g(x_1y_2) = g(x_1y_2 - x_2y_1)$ . As a consequence,

$$\Delta[(x_1, y_1), (x_2, y_2)] = g(x_1y_2 - y_1x_2)$$

for all points  $(x_1, y_1)$  and  $(x_2, y_2)$  of  $\mathcal{L}$ . Since the points  $(1, 0), (1, a)$  and  $(1, a+b)$  are collinear for all  $a, b \in \mathbb{F}_q$ , we have  $\Delta[(1, 0), (1, a)] + \Delta[(1, a), (1, a+b)] = \Delta[(1, 0), (1, a+b)]$  or equivalently  $g(a+b) = g(a) + g(b)$  for all  $a, b \in \mathbb{F}_q$ . Together with  $g(0) = 0$  this implies that the bijection  $g$  is an isomorphism from the additive group of  $\mathbb{F}_q$  to the group  $G$ . So, we find that our admissible triple  $T$  is equivalent with the admissible triple  $T' = (AG(2, q), \mathbb{F}_q, \Delta')$ , where  $\Delta'[(x_1, y_1), (x_2, y_2)] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ . Now,  $\Omega(T')$  is equivalent with  $(P(W(q), u), S(W(q), u))$  where  $u$  is any point of  $W(q)$ . So,  $(P(Q, x), S(Q, x))$  is equivalent with  $(P(W(q), u), S(W(q), u))$ . This proves that  $Q$  is isomorphic to  $W(q)$ .

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