ENumeration of unlabeled graphs

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Abstract. Using Pólya’s Enumeration Theorem, Harary and Palmer [1] give a function which gives the number of unlabeled graphs on \( p \) vertices with \( q \) edges. We present their work and the necessary background knowledge.

1. Introduction

The problem of counting the number of graphs of a given order \( p \) is perhaps one of the most obvious problems in any study of graphs. When the vertices are labeled, the answer is readily obtained as \( 2^\binom{n}{2} \), since each of the \( \binom{n}{2} \) possible edges may be either present or missing. On the other hand, when the vertices are unlabeled, the problem becomes more interesting. For small values of \( p \) it is easy to determine. For instance, when \( p = 2 \), there are two graphs; when \( p = 3 \), there are four, and when \( p = 4 \), there are eleven distinct unlabeled graphs. The difficulty arises in determining how many graphs are truly distinct.

In 1927, the first solution to the problem appeared in [2]. Subsequently, the problem was successfully solved by other mathematicians, including Pólya, whose enumeration theorem proves valuable in our approach to the problem. Throughout this paper, definitions and theorems are taken from [1], with the exception of the statement of Pólya’s Theorem, which is taken from [3].

2. Pólya’s Theorem

Before examining the enumeration of graphs in detail, it is imperative that we discuss some preliminaries that allow the enumeration. These will be discussed in general, as their application is much broader than graph theory.

Let \( H \) be a permutation group acting on the set \( S \) of order \( n \). Without loss of generality, we may assume that \( S \) consists precisely of the integers from 1 to \( n \). A classic and well known group theory result states that any permutation can be written uniquely as the product of disjoint cycles. For any permutation, then, we can use a set of indexing variables to describe its cycle structure in the following manner. We let \( j_k(a) \) indicate the number of \( k \)-cycles in the disjoint cycle expansion of \( a \) and multiply together terms of the form \( s_k^{j_k(a)} \).

**Definition 2.1.** The cycle index of a permutation \( a \) of a set of order \( n \) is the product \( \prod_{k=1}^{n} s_k^{j_k(a)} \). The cycle index of a permutation group \( H \) acting on a set of order \( n \), denoted \( Z(H) \) is a polynomial in \( s_1, \ldots, s_n \) given by

\[
Z(H) = \frac{1}{|H|} \sum_{a \in H} \prod_{k=1}^{n} s_k^{j_k(a)}
\]

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Lemma 2.2. The cycle index of the symmetric group $S_n$ is given by

$$Z(S_n) = \frac{1}{n!} \prod_{k} k^{j_k} j_k! \prod s_k^{j_k},$$

where the summation is taken over all partitions $(j)$ of $n$.

Proof. Consider some partition $(j)$ of $n$, where $(j) = (j_1, j_2, \ldots, j_n)$. Assume that the cycles of some permutation having cycles given by $(j)$ are ordered from largest to smallest. The $n$ elements of the object set can be ordered in $n!$ different ways. However, for each $k$, the $j_k$ cycles can be ordered in $j_k!$ different ways, and can begin in $k$ different elements. Thus, any permutation is represented $\prod_{k} k^{j_k} j_k!$ times, so that there are a total of

$$\frac{n!}{\prod_{k} k^{j_k} j_k!}$$

permutations with cycle structure given by $(j)$. This allows us to reindex over $(j)$ rather than over the individual permutations, obtaining the cycle structure as stated in the theorem. ∎

Any element of $S$ may be given some weight $w(x)$ from a weight set $W$ in varying ways. An obvious property of weight functions on $S$ is that equivalent weight assignments yield the same total weight. A weight might represent such things as a color with which an edge in a graph is colored, and may be assigned as variables or as actual values. We are interested in determining an inventory of distinct weight assignments for the elements of $S$ from some set of weights. Such an inventory is called the pattern inventory, and determines how many times a given pattern occurs distinctly. The special case below of Pólya’s Enumeration Theorem is the tool that allows us to determine the pattern inventory from the cycle index.

Theorem 2.3. (Pólya’s Enumeration Theorem)\[3\] The pattern inventory is determined by substituting for $s_k$ the sum $\sum w^k$, where the sum is taken over all weights; that is,

$$P = Z \left( \sum_{w \in W} w, \sum_{w \in W} w^2, \ldots, \sum_{w \in W} w^n \right)$$

If, in particular, we assign the weight set as $\{0, 1\}$ we obtain the following important corollary (since $1 + x$ represents a generating function for an item that may appear zero or one times).

Corollary 2.4. The coefficient of $x^r$ in $Z(H, 1 + x)$ is the number of equivalence classes of $r$-sets of $S$.

This corollary is the powerful tool we apply to counting graphs.

3. COUNTING GRAPHS

To effectively count graphs, we must define what it means for two graphs to be distinct. The natural definition would be that two graphs $H$ and $H'$ are distinct if there is no permutation of the vertices mapping the edges of $H$ to the edges of $H'$. The group of permutations on the vertices of a graph of order $n$ is precisely
transitive. Thus, this cycle within \( \alpha \) represents the greatest common divisor of where \( \alpha \). Distinct graphs, then, are represented by distinct equivalence classes under the action of \( S_n^{(2)} \).

**Theorem 3.1.** The number of graphs of order \( n \) having \( x \) edges is given by the polynomial

\[
g_n(x) = Z\left(S_n^{(2)}, 1 + x\right).
\]

The cycle index \( Z\left(S_n^{(2)}\right) \) is given by

\[
Z\left(S_n^{(2)}\right) = \frac{1}{n!} \prod_{k} s_k^{k} \prod_{r<t} s_{r}^{(r,t) j_r j_t},
\]

where \( j_k \) again represents the number of \( k \)-cycles in a given permutation \( \alpha \), \((r, t)\) represents the greatest common divisor of \( r \) and \( t \), and where \([r, t]\) represents the least common multiple of \( r \) and \( t \).

**Proof.** Corollary 2.4 states that the coefficient of \( x^r \) in \( Z\left(S_n^{(2)}, 1 + x\right) \) counts the number of equivalence classes of \( r \)-sets from our base set being acted upon by \( S_n^{(2)} \). But this is the set of possible edges in a graph, so that we are counting the number of equivalence classes of graphs with \( r \) edges, as desired. As such, it remains only to show that the cycle index is as determined.

We would like to derive a connection between the cycle structure of vertices in a graph and the induced cycle structure of edges. That is, if the cycle structure on the vertices is \( s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \), what is the cycle structure on the edges. To determine this, we consider a permutation \( \alpha \) from \( S_n \) acting upon the vertices. Such a permutation contributes some \( \prod s_k^{j_k} \) to \( Z\left(S_n\right) \).

There are two ways in which we can describe edges under the permutation \( \alpha' \) of \( S_n^{(2)} \) induced by \( \alpha \). An edge may exist between two vertices of a common cycle under \( \alpha \), or it may exist between two vertices of distinct cycles of \( \alpha \). Consider edges between two vertices of a common cycle of \( \alpha \). If this cycle has odd length, the edges between members of this cycle can be classified based on the lowest power of \( \alpha \) mapping one end of the edge to the other. If the cycle has length \( k \), there are \( k-1 \) such classes. An edge in one class can only be mapped to another edge within the same class by \( \alpha \), and the action of \( \alpha \) on the edges within such a class is transitive. Thus, this cycle within \( \alpha \) induces \( k-1 \) cycles of the same length within \( \alpha' \). On the other hand, if \( k \) is even, edges between opposite vertices on the cycle, where \( \alpha^k \) simply interchanges the end-vertices, complete a single cycle of length \( k/2 \), while the remaining edges comprise \( k-2 \) cycles of length \( k \).

Thus, every \( s_k \) in the cycle index of \( \alpha \) contributes \( s_k^{(k-1)/2} \) to the cycle index of \( \alpha' \) if \( k \) is odd, and \( s_k s_k^{(k-2)/2} \) to the cycle index of \( \alpha' \) if \( k \) is even. We raise each of these to the number of such cycles \( j_k \) to obtain terms of \( s_k^{j_k(k-1)/2} \) and \( s_k^{j_k(k-2)/2} \) when \( k \) is odd or even respectively. We can express these equivalently as \( s_{2k+1} \) and \( s_k s_k^{(k-1)/2} \) and simply take the product of these two over all values of \( k \) to obtain the entire contribution of edges between vertices of a common
cycle of $\alpha$ to $\alpha'$ as

$$\prod_k s_{2k+1}^{j_{2k+1}} (s_k s_{2k}^{k-1})^{j_{2k}}$$

We now consider the edges between vertices in distinct cycles. These can be considered in two cases: whether or not the cycles have the same length. If the cycles have the same length, an edge between vertices of the cycles in $\alpha$ will lie in an induced cycle of the same length in $\alpha'$. Every pair of $k$-cycles of vertices has $k^2$ edges between vertices of the distinct cycles, which are grouped in cycles of length $k$, for a total of $k$ such cycles. There are $\binom{k}{2}$ such pairs, for a total of $k \binom{k}{2}$ cycles induced under $\alpha'$. Thus, the following term is introduced into the cycle index of $\alpha'$.

$$\prod_k s_k^{\binom{k}{2}}$$

Finally, we consider edges between cycles of different lengths $r$ and $t$, with $r < t$. There are $rt$ such edges, and each edge lies in an induced cycle of length $[r, t]$. Each vertex must be mapped to itself, so that $r$ and $t$ must both divide the length of this cycle, so that $[r, t]$ is a lower bound on the length of such a cycle. On the other hand, at $[r, t]$ applications of $\alpha$, such an edge is mapped to itself, forcing equality. Since each cycle has length $[r, t]$, there must be exactly $(r, t)$ such cycles for each choice of $r$ and $t$. Clearly, there are $j_r j_t$ choices of $r$ and $t$, so that the contribution for given values of $r$ and $t$ is $s_{[r,t]}^{(r,t)j_r j_t}$. Thus, the total term contributed to the cycle index of $\alpha'$ by these edges is

$$\prod_{r < t} s_{[r,t]}^{(r,t)j_r j_t}.$$  

It follows then, that the cycle index is given by

$$Z \left( S_n^{(2)} \right) = \frac{1}{n!} \sum_{\alpha \in S_n} \prod_k s_{2k+1}^{j_{2k+1}} (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{\binom{k}{2}} \prod_{r < t} s_{[r,t]}^{(r,t)j_r j_t}.$$

We now have only to re-index the summation to obtain the final result. But this is completed by Lemma 2.2, yielding the statement of the theorem. □

4. Graphs on Five Vertices

We complete this discussion with an illustration for graphs on five vertices. Given five vertices, we obtain the following options for $(j) = (j_1, j_2, j_3, j_4, j_5)$.

- $(0, 0, 0, 0, 1)$
- $(0, 1, 1, 0, 0)$
- $(1, 2, 0, 0, 0)$
- $(5, 0, 0, 0, 0)$

These give the following summands.

For $j = (0, 0, 0, 0, 1)$:

$$\frac{1}{5!} \left( s_5 \right)^2 \left( s_2 \right) \left( s_1 \right) = \frac{1}{5} s_5^2.$$

For $j = (1, 0, 0, 1, 0)$:

$$\frac{1}{4!} \left( s_2 s_4 \right) \left( s_1 \right) \left( s_1 \right) = \frac{1}{4} s_2 s_4^2.$$
For \( j = (0, 1, 1, 0, 0) \):
\[
\frac{1}{3!1!2!1!} (s_3) (s_1 s_2^0) (s_6) = \frac{1}{6} s_3 s_6.
\]

For \( j = (2, 0, 1, 0, 0) \):
\[
\frac{1}{3!1!1!2!} (s_3) (1) (s_1) (s_3^2) = \frac{1}{6} s_3^3.
\]

For \( j = (1, 2, 0, 0, 0) \):
\[
\frac{1}{1!1!2!2!} (1) (s_1 s_2^0) (s_2) (s_2) = \frac{1}{8} s_1^2 s_2^2.
\]

For \( j = (3, 1, 0, 0, 0) \):
\[
\frac{1}{1^3!2!1!} (1) (s_1 s_2^0) (s_3^2) (s_2^2) = \frac{1}{12} s_1^4 s_2^3.
\]

For \( j = (5, 0, 0, 0, 0) \):
\[
\frac{1}{1^5!} (1) (1) (s_1^{10}) (1) = \frac{1}{120} s_1^{10}.
\]

Summing and substituting \((1 + x^k)\) for \(s_k\) yields
\[
\frac{1}{5} (1 + x^5)^2 + \frac{1}{4} (1 + x^2) (1 + x^4)^2 + \frac{1}{6} (1 + x) (1 + x^3) (1 + x^6) + \frac{1}{6} (1 + x) (1 + x^3)^3 + \frac{1}{8} (1 + x)^2 (1 + x^3)^4 + \frac{1}{12} (1 + x)^4 (1 + x^2)^3 + \frac{1}{120} (1 + x)^{10}.
\]

Expanding and combining like terms yields
\[
1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + 2x^8 + x^9 + x^{10}
\]
so that there is one graph on five vertices with each of zero, one, nine, or ten edges, two graphs with each of two or eight edges, four graphs with each of three or seven edges, and six graphs with each of five, six, or seven edges, for a total of thirty-four graphs on five vertices.

Although not formulated specifically as a single function, this approach can be used to determine the coefficients on a two-variable generating function \(g(x, y)\), where the coefficient on \(x^m y^n\) gives the number of graphs on \(m\) vertices and \(n\) edges.

**References**

